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# Spectral theory of elliptic differential operators with indefinite weights

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The spectral properties of a class of non-self-adjoint second-order elliptic operators with indefinite weight functions on unbounded domains $\Omega$ are investigated. It is shown, under an abstract regularity assumption, that the non-real spectrum of the associated elliptic operators in $L^2(\Omega)$ is bounded. In the special case where $\Omega = \mathbb{R}^n$ decomposes into subdomains $\Omega_+$ and $\Omega_-$ with smooth compact boundaries and the weight function is positive on $\Omega_+$ and negative on $\Omega_-$, it turns out that the non-real spectrum consists only of normal eigenvalues that can be characterized with a Dirichlet-to-Neumann map.

## 1. Introduction

This paper studies the spectral properties of partial differential operators associated to second-order elliptic differential expressions of the form

$$\mathcal{L}f = \frac{1}{r}\ell(f), \qquad \ell(f) = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} f + af, \tag{1.1}$$

with variable coefficients $a_{jk}$, $a$ and a weight function $r$ defined on some bounded or unbounded domain $\Omega \subset \mathbb{R}^n$, $n > 1$. It is assumed that the differential expression $\ell$ is formally symmetric and uniformly elliptic. The peculiarity here is that the function $r$ is allowed to have different signs on subsets of positive Lebesgue measure of $\Omega$. For this reason $\mathcal{L}$ is said to be an *indefinite* elliptic differential expression.

The differential expression $\ell$ in (1.1) gives rise to a self-adjoint unbounded operator $A$ in the Hilbert space $L^2(\Omega)$ which is defined on the dense linear subspace $\operatorname{dom} A = \{f \in H_0^1(\Omega)\colon \ell(f) \in L^2(\Omega)\}$. The spectral properties of the elliptic differential operator $A$ depend on the geometry of $\Omega$ and the coefficients $a_{jk}$ and $a$, and are, from a qualitative point of view at least, well understood. The self-adjointness and ellipticity of $A$ imply that the spectrum $\sigma(A)$ is contained in $\mathbb{R}$ and that it is semi-bounded from below. If the domain $\Omega$ is bounded or 'thin' at $\infty$, then the resolvent of $A$ is compact and, hence, $\sigma(A)$ consists of a sequence of eigenvalues with finite-dimensional eigenspaces which accumulates to $+\infty$ (see, for example, [19]). For general unbounded domains, $\sigma(A)$ may also contain continuous and essential spectrum of rather arbitrary form. However, if, for example, the coefficients $a_{jk}$ and $a$ converge to a limit for $|x| \to \infty$, then the essential spectrum of $A$ consists of a single unbounded interval.

In contrast with the self-adjoint case, the spectral properties of the non-self-adjoint indefinite elliptic operator

$$T = \frac{1}{r}A, \qquad \operatorname{dom} T = \operatorname{dom} A, \tag{1.2}$$

associated to the differential expression in (1.2) are much less well understood, particularly if the domain $\Omega$ is unbounded. The case of a bounded domain $\Omega$ is discussed in, for example, [21,22], where the point of view is similar to ours. Further properties of indefinite elliptic operators on bounded domains such as asymptotical behaviour of eigenvalues or Riesz basis properties of eigenfunctions have been studied (also for more general elliptic problems involving indefinite weights) in various works. We mention here in particular the work of Faierman [20, 23–27], Pyatkov [39–42] and others [2, 18, 28].

The main objective of this paper is to study the spectral properties of non-self-adjoint indefinite elliptic operators of the form (1.2) on unbounded domains. Such problems are more difficult to investigate, and a purely abstract operator-theoretic and functional-analytic approach is insufficient in this situation (since, for example, the essential spectrum of $A$ is in general non-empty, it is difficult to conclude that the spectrum of $T$ does not cover the whole complex plane). Therefore, in this paper we combine methods from the classical theory of elliptic differential equations with modern spectral and perturbation techniques for unbounded operators that are symmetric with respect to an indefinite inner product. Our investigations lead to new insights and results on the spectral properties of indefinite elliptic operators on unbounded domains. For example, we prove that, under an abstract regularity assumption, the non-real spectrum of $T$ is bounded. Furthermore, in the special case where $\Omega = \mathbb{R}^n$ decomposes into subdomains $\Omega_+$ and $\Omega_-$ with smooth compact boundaries such that the weight function $r$ is positive (respectively, negative) on $\Omega_+$ (respectively, $\Omega_-$), it is shown that the non-real spectrum of $T$ consists only of normal eigenvalues which can be characterized with Dirichlet-to-Neumann maps acting on interior and exterior domains.

The paper is organized as follows. After the precise assumptions and basic facts explained in § 2, the known case of a bounded domain $\Omega$ is discussed in § 3 for completeness (see, for example, [16, 22, 23, 39]). As one might expect, it turns out that, in this case, the resolvent of $T$ is compact and, hence, $\sigma(T)$ consists only of eigenvalues with finite multiplicity. Some additional facts on self-adjoint operators with finitely many negative squares in indefinite inner product spaces from [15, 34, 35] imply that the non-real spectrum of $T$ consists of at most finitely many eigenvalues. Section 4 deals with general unbounded domains. If the spectrum or essential spectrum of $A$ is positive, then abstract methods again ensure that the non-real spectrum of $T$ is bounded and consists of at most finitely many eigenvalues (see [9, 15, 16, 32, 34, 35] and theorems 4.2 and 4.3). One of our main results states that, without further assumptions on the operator $A$, the non-real spectrum of $T$ remains bounded if a certain isomorphism $W$ which ensures the regularity of the critical point $\infty$ exists (see condition (i) of theorem 4.4). In § 5 the special case $\Omega = \mathbb{R}^n$ with $r$ having negative sign outside a bounded set is studied. A sufficient condition in terms of the weight function $r$ is given such that the non-real spectrum of $T$ is bounded. A more detailed analysis is provided in theorem 5.4, where a multidimensional variant of Glazman's decomposition method is used to show that the non-real spectrum of $T$ consists only of eigenvalues with finite multiplicity which may accumulate to certain subsets of the real line. Finally, in theorem 5.6 it is shown how the non-real spectrum of $T$ can be characterized with the help of Dirichlet-to-Neumann maps acting on interior and exterior domains. Furthermore, a variant of Krein's resolvent formula for indefinite elliptic differential operators is obtained in theorem 5.7.

## 2. Elliptic differential operators in $L^2(\Omega)$

In this preliminary section we define an elliptic differential expression $\mathcal{L}$ with an indefinite weight function on some domain $\Omega$ and we associate an unbounded differential operator in $L^2(\Omega)$ to $\mathcal{L}$ which is self-adjoint with respect to an indefinite metric on $L^2(\Omega)$ (see theorem 2.1).

### 2.1. The elliptic differential expression

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\ell$ be the 'formally self-adjoint' uniformly elliptic second-order differential expression

$$(\ell f)(x) := -\sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_j} a_{jk} \frac{\partial f}{\partial x_k}\right)(x) + (af)(x), \quad x \in \Omega, \tag{2.1}$$

with bounded coefficients $a_{jk} \in C^\infty(\Omega)$ satisfying $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x \in \Omega$ and $j, k = 1, \ldots, n$, the function $a \in L^\infty(\Omega)$ is real valued and

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geqslant C \sum_{k=1}^{n} \xi_k^2$$

holds for some $C > 0$, all $\xi = (\xi_1, \ldots, \xi_n)^\top \in \mathbb{R}^n$ and $x \in \Omega$.

In what follows we investigate operators induced by the second-order elliptic differential expression $\mathcal{L}$ with the indefinite weight $r$ defined as

$$(\mathcal{L}f)(x) := \frac{1}{r(x)}(\ell f)(x), \quad x \in \Omega.$$

Throughout this paper it is assumed that $r$ is a real-valued function such that $r, r^{-1} \in L^\infty(\Omega)$ and each of the sets

$$\Omega_+ := \{x \in \Omega \colon r(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega \colon r(x) < 0\} \tag{2.2}$$

has positive Lebesgue measure. Observe that $\Omega \setminus (\Omega_+ \cup \Omega_-)$ is a Lebesgue null set. The restriction of the weight function $r$ onto $\Omega_\pm$ is denoted by $r_\pm$. Similarly, for a function $f$ defined on $\Omega$, the restriction onto $\Omega_\pm$ is denoted by $f_\pm$. Moreover, $\ell_\pm$ and $\mathcal{L}_\pm$ stand for the restrictions of the differential expressions $\ell$ and $\mathcal{L}$ onto $\Omega_\pm$.

### 2.2. Differential operators in $L^2(\Omega)$ associated to $\ell$ and $\mathcal{L}$

We associate the elliptic differential operator

$$Af := \ell(f), \qquad \operatorname{dom} A = \{f \in H_0^1(\Omega) \colon \ell(f) \in L^2(\Omega)\}, \tag{2.3}$$

to the differential expression $\ell$, where $H^1_0(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega)$. It is well known that $A$ is an unbounded self-adjoint operator in the Hilbert space $(L^2(\Omega), (\cdot,\cdot))$ with spectrum semi-bounded from below by $\operatorname{ess\,inf} a$. This can be seen, for example, with the help of the sesquilinear form associated to $\ell$ and the first representation theorem from [31].

Besides the Hilbert space inner product $(\cdot,\cdot)$ in $L^2(\Omega)$, we will make use of the indefinite inner product

$$[f,g] := \int_\Omega f(x)\overline{g(x)}r(x)\,\mathrm{d}x, \quad f,g \in L^2(\Omega). \tag{2.4}$$

The space $(L^2(\Omega), [\cdot,\cdot])$ is a so-called Krein space (see [5, 13, 33–35]). Observe that $[\cdot,\cdot]$ is non-positive on functions with support in $\Omega_-$ and non-negative on functions with support in $\Omega_+$. Note also that the assumptions $r \in L^\infty(\Omega)$ and $r^{-1} \in L^\infty(\Omega)$ imply that the multiplication operator $Rf = rf$, $f \in L^2(\Omega)$, is an isomorphism in $L^2(\Omega)$ with inverse $R^{-1}f = r^{-1}f$, $f \in L^2(\Omega)$. In particular, $\ell(f) \in L^2(\Omega)$ if and only if $\mathcal{L}(f) \in L^2(\Omega)$. Furthermore, the inner products $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are connected via

$$[f,g] = (Rf,g), \quad (f,g) = [R^{-1}f,g] \quad \text{for } f,g \in L^2(\Omega). \tag{2.5}$$

Next we introduce the differential operator $T$ associated to the indefinite elliptic expression $\mathcal{L}$ and we summarize some of its properties. The following theorem is a direct consequence of (2.5) and the self-adjointness of $A$.

Theorem 2.1. *The differential operator*

$$Tf := \mathcal{L}(f), \qquad \operatorname{dom} T = \{f \in H^1_0(\Omega) \colon \mathcal{L}(f) \in L^2(\Omega)\}, \tag{2.6}$$

*is self-adjoint with respect to the Krein space inner product $[\cdot,\cdot]$ in $L^2(\Omega)$, and $T$ is connected with the elliptic differential operator $A$ in (2.3) via*

$$T = R^{-1}A \quad \textit{and} \quad A = RT.$$

We remark that the adjoint of an (unbounded) operator with respect to a Krein space inner product is defined in the same way as with respect to a usual scalar product. Here the adjoint $T^+$ of $T$ with respect to $[\cdot,\cdot]$ can be equivalently defined as $T^+ := R^{-1}A^* = R^{-1}A$, where '$*$' denotes the adjoint with respect to $(\cdot,\cdot)$. In particular, this implies $[Tf,g] = [f,Tg]$ for all $f,g \in \operatorname{dom} T$.

We also point out that the spectrum of an operator which is self-adjoint in the Krein space $(L^2(\Omega), [\cdot,\cdot])$ can be quite arbitrary. In particular, the spectrum is in general not a subset of $\mathbb{R}$, and simple examples show that the spectrum can be empty or cover the whole complex plane. However, the non-real spectrum is necessarily symmetric with respect to the real line.

### 2.3. Spectral points of closed operators

Let $S$ be a closed operator in a Hilbert space. The *resolvent set* $\rho(S)$ of $S$ consists of all $\lambda \in \mathbb{C}$ such that $S-\lambda$ is bijective. The complement of $\rho(S)$ in $\mathbb{C}$ is the *spectrum* $\sigma(S)$ of $S$. The *point spectrum* $\sigma_\mathrm{p}(S)$ is the set of eigenvalues of $S$, i.e. those $\lambda \in \mathbb{C}$ for which $S - \lambda$ is not injective. An eigenvalue $\lambda$ is said to be *normal* if $\lambda$ is an isolated point of $\sigma(S)$ and its (algebraic) multiplicity is finite. The *essential* *spectrum* $\sigma_{\mathrm{ess}}(S)$ consists of those points $\lambda \in \mathbb{C}$ for which $S - \lambda$ is not a semi-Fredholm operator. Recall that the essential spectrum is stable under compact and relative compact perturbations (see [19, 31]). If $S$ is a self-adjoint operator, then $\sigma_{\mathrm{ess}}(S)$ consists of the accumulation points of $\sigma(S)$ and the isolated eigenvalues of infinite multiplicity; the set of normal eigenvalues is the complement of $\sigma_{\mathrm{ess}}(S)$ in $\sigma(S)$. Recall that the eigenvalues of a self-adjoint operator are semi-simple. We say that the *positive* (respectively, *negative*) *spectrum* of (a not necessarily self-adjoint operator) $S$ has *infinite multiplicity* if $\sigma(S) \cap (0, +\infty)$ (respectively, $\sigma(S) \cap (-\infty, 0)$) contains infinitely many eigenvalues or points of the essential spectrum of $S$.

## 3. Spectral properties of indefinite elliptic operators on bounded domains

In this section we study the spectral properties of the indefinite elliptic operator $T$ in theorem 2.1 in the case where $\Omega$ is a bounded domain in $\mathbb{R}^n$. Throughout this section it will be tacitly assumed that $\Omega$ is bounded, but no further (regularity) assumptions on the boundary are imposed.

Let us first recall the following well-known theorem on the qualitative spectral properties of the self-adjoint elliptic operator $A$ which is essentially a consequence of the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ (see, for example, [44, theorem 7.1]), the ellipticity of $\ell$ and the boundedness of the coefficient $a$.

Theorem 3.1. *The spectrum of $A$ is bounded from below and consists of normal semi-simple eigenvalues that accumulate to $+\infty$.*

The main result in this section is the following, which is well known and follows from the more general and abstract considerations in [22, 23, 39] and [15, 34, 35]. A short proof is included for the reader's convenience.

Theorem 3.2. *The spectrum of $T$ consists of normal eigenvalues that accumulate to $+\infty$ and $-\infty$. The non-real spectrum of $T$ is bounded and consists of at most finitely many normal eigenvalues which are symmetric with respect to the real line.*

Before we prove this theorem, a preparatory lemma on the resolvent set of $T$ will be proved.

Lemma 3.3. *The set $\rho(T)$ is non-empty.*

*Proof.* If 0 is not a normal eigenvalue of $A$, then $0 \in \rho(A)$ and it follows from $T = R^{-1}A$ that $T^{-1} = A^{-1}R$ is a bounded and everywhere defined operator in $L^2(\Omega)$, i.e. $0 \in \rho(T)$. Therefore, assume that $0 \in \sigma(A)$, that is, 0 is an isolated eigenvalue of finite multiplicity of $A$ by theorem 3.1. The restriction

$$B := A \upharpoonright (\operatorname{dom} A \cap (\ker A)^{\perp})$$

of $A$ on the orthogonal complement of $\ker A$ in $L^2(\Omega)$ is regarded as a non-densely defined symmetric operator in $L^2(\Omega)$ with finite equal defect numbers. Note that $B$ is injective and that $\operatorname{ran} B = (\ker A)^{\perp}$ is closed and has finite codimension. Hence, there exists a self-adjoint operator $\tilde{A}$ in $L^2(\Omega)$ which is an extension of $B$ such that

$0 \in \rho(\tilde{A})$. Furthermore, since $B$ is a finite-dimensional restriction of both $A$ and $\tilde{A}$, it follows that

$$\dim \operatorname{ran}((A-\lambda)^{-1} - (\tilde{A}-\lambda)^{-1}) \leqslant \dim \ker A < \infty$$

holds for all $\lambda \in \rho(A) \cap \rho(\tilde{A})$ and, hence, $\sigma(\tilde{A})$ is semi-bounded from below and consists of normal eigenvalues.

The operator $\tilde{T} := R^{-1}\tilde{A}$ is a self-adjoint operator in the Krein space $(L^2(\Omega), [\cdot,\cdot])$ and from $0 \in \rho(\tilde{A})$ we conclude $0 \in \rho(\tilde{T})$. Furthermore, since $\tilde{A}$ is semi-bounded from below and $[\tilde{T}f, g] = (\tilde{A}f, g)$ holds for all $f, g \in \operatorname{dom} \tilde{T} = \operatorname{dom} \tilde{A}$, it follows that the form $[\tilde{T}\cdot,\cdot]$ has finitely many negative squares. It is easy to see that $\tilde{T}$ and $T$ are both finite-dimensional extensions of the non-densely defined operator $S := R^{-1}B$. Now $\rho(T) \neq \emptyset$ follows from a slight modification of [15, proposition 1.1] (see also [6, corollary 2.5]). □

*Proof of theorem 3.2.* Observe that, by theorem 3.1, the resolvent $(A-\lambda)^{-1}$ is compact for all $\lambda \in \rho(A)$ and that lemma 3.3 implies $\rho(T) \cap \rho(A) \neq \emptyset$. A simple computation shows that the relation

$$(T-\lambda)^{-1} = (A-\lambda)^{-1}R - \lambda(A-\lambda)^{-1}(I-R)(T-\lambda)^{-1}$$

holds for all $\lambda \in \rho(A) \cap \rho(T)$, and since the right-hand side is a compact operator the same holds for the left-hand side. Hence, $\sigma(T)$ consists of normal eigenvalues. As the negative spectrum of $A$ consists of at most finitely many normal eigenvalues the form $[T\cdot,\cdot] = (A\cdot,\cdot)$ has finitely many negative squares, and it follows from the general results in [15, 34, 35] that the non-real spectrum of $T$ consists of at most finitely many normal eigenvalues which are symmetric with respect to the real line. Finally, the assumption that the sets $\Omega_+$ and $\Omega_-$ in (2.2) have positive Lebesgue measure imply that the indefinite inner product $[\cdot,\cdot]$ in (2.4) has infinitely many positive and negative squares. The reasoning in [15, proof of proposition 1.8] shows that the positive spectrum and the negative spectrum of $T$ are both of infinite multiplicity. Hence, the real eigenvalues of $T$ accumulate to $+\infty$ and $-\infty$. □

## 4. Spectral properties of indefinite elliptic operators on unbounded domains

In this section we study the spectral properties of the indefinite elliptic operator $T$ in (2.6) on an unbounded domain $\Omega \subset \mathbb{R}^n$. Since, for an unbounded domain, the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is in general not compact, the resolvent of the self-adjoint operator $A$ in (2.3) is also in general not compact and, hence, the essential spectrum $\sigma_{\mathrm{ess}}(A)$ of $A$ may be non-empty. Only the following weaker variant of theorem 3.1 holds.

Theorem 4.1. *The spectrum of $A$ is bounded from below and accumulates to $+\infty$.*

If the lower bound $\min \sigma(A)$ of the spectrum of $A$ or the lower bound $\min \sigma_{\mathrm{ess}}(A)$ of the essential spectrum of $A$ is positive, then it is known that $T = R^{-1}A$ is positive in the Krein space $(L^2(\Omega), [\cdot,\cdot])$ or has a finite number of negative squares, respectively. We recall these and some other facts in theorem 4.2 and theorem 4.3 for the reader's convenience. The proofs of the statements are essentially contained in [13,15,34,35] (see also [32, theorem 3.3], [10, theorem 3.1] and [17, proposition 1.6]).

THEOREM 4.2. *If $\min \sigma(A) > 0$, then the spectrum of $T$ is real, $0 \in \rho(T)$, and $T$ has positive and negative spectrum, both of infinite multiplicities.*

THEOREM 4.3. *If $\min \sigma_{\mathrm{ess}}(A) > 0$, then the essential spectrum of $T$ is real and $T$ has positive and negative spectrum, both of infinite multiplicities. The non-real spectrum of $T$ is bounded and consists of at most finitely many normal eigenvalues which are symmetric with respect to the real line.*

In the next step, the assumption $\min \sigma_{\mathrm{ess}}(A) > 0$ will be dropped. The following considerations and theorem 4.4 are partly inspired by general results on self-adjoint operators in Krein spaces and the regularity of the critical point $\infty$ from [14, 16, 34] and [9, proof of theorem 5.4]. Fix some $\nu < \min \sigma(A)$ and define the space $\mathcal{H}_s$, $s \in [0, 2]$, as the domains of the $\frac{1}{2}s$th powers of the positive operator $A - \nu$,

$$\mathcal{H}_s := \mathrm{dom}((A-\nu)^{s/2}), \quad s \in [0, 2].$$

Note that $\mathcal{H} = \mathcal{H}_0$, $\mathrm{dom}\, A = \mathcal{H}_2$, and the form domain of $A$ is $\mathcal{H}_1$. The spaces $\mathcal{H}_s$ become Hilbert spaces when they are equipped with the usual inner products, the induced topologies do not depend on the particular choice of $\nu < \min \sigma(A)$ (see [31]).

The following theorem is one of the main results in this paper. Under an additional abstract condition from [16], it will be shown that the non-real spectrum of the indefinite elliptic operator is bounded. Roughly speaking, this condition is satisfied in special situations when choosing $W = R$ (see lemma 5.1).

THEOREM 4.4. *Assume that $\min \sigma_{\mathrm{ess}}(A) \leqslant 0$ and that the following condition holds:*

(i) *there exists an isomorphism $W$ in $L^2(\Omega)$ such that $RW$ is positive in $L^2(\Omega)$ and $W\mathcal{H}_s \subset \mathcal{H}_s$ holds for some $s \in (0, 2]$.*

*Then the non-real spectrum of $T$ is bounded.*

*Proof.* 1. In this step of the proof we construct an indefinite elliptic operator $T_\eta$ which is a bounded perturbation of the indefinite elliptic operator $T$ and which induces (via its spectral decomposition) a new equivalent norm $\|\cdot\|_\sim$ on $L^2(\Omega)$.

For this, fix some $\eta < \min \sigma(A)$ and consider the elliptic differential operator $A_\eta$ defined as

$$A_\eta f := (A - \eta) f = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial f}{\partial x_k} + (a - \eta) f, \quad f \in \mathrm{dom}\, A_\eta = \mathrm{dom}\, A.$$

Clearly, $A_\eta$ is a positive self-adjoint operator in the Hilbert space $L^2(\Omega)$ and, hence, the indefinite elliptic operator

$$T_\eta f := \frac{1}{r}\left(-\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial f}{\partial x_k} + (a-\eta) f\right), \quad f \in \mathrm{dom}\, T_\eta = \mathrm{dom}\, A_\eta,$$

is non-negative in the Krein space $(L^2(\Omega), [\cdot,\cdot])$, the spectrum $\sigma(T_\eta)$ is a subset of $\mathbb{R}$ and $0 \in \rho(T_\eta)$ (see theorem 4.2). Note that $T_\eta$ and $T$ are connected via

$$T_\eta = R^{-1} A_\eta = R^{-1} A - \eta R^{-1} = T - V, \qquad V := \eta R^{-1}, \tag{4.1}$$

and that the perturbation term $V$ in (4.1) is bounded.

By [34, 35], $T_\eta$ possesses a spectral function defined for all bounded subintervals of the real line. As a consequence of condition (i) and [16, part (iii) of theorem 2.1] (see also [14]), it follows that $\infty$ is not a singular critical point of the operator $T_\eta$ and, therefore, the spectral projections $E_+$ and $E_-$ corresponding to the intervals $(0, +\infty)$ and $(-\infty, 0)$ exist. Moreover, as $T_\eta$ is non-negative in $(L^2(\Omega), [\cdot,\cdot])$, the spectral subspaces $(E_\pm L^2(\Omega), \pm[\cdot,\cdot])$ are both Hilbert spaces and $L^2(\Omega)$ can be decomposed in

$$L^2(\Omega) = E_+L^2(\Omega)[\dot{+}]E_-L^2(\Omega). \tag{4.2}$$

We point out that the subspaces $E_\pm L^2(\Omega)$ differ from $L^2(\Omega_\pm)$ and that, within this proof, the subscripts $\pm$ are used in the sense of (4.2). From the properties of the spectral function it follows that $T_\eta$ has diagonal form with respect to the space decomposition (4.2),

$$T_\eta = \begin{pmatrix} T_{\eta,+} & 0 \\ 0 & T_{\eta-} \end{pmatrix},$$

and that the spectrum of $T_{\eta,\pm}$ is contained in $\mathbb{R}^\pm$. The perturbation term $V = \eta R^{-1}$ in (4.1) admits the matrix representation

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

with respect to the decomposition (4.2). Together with (4.1), we then have

$$T = T_\eta + V = T_\eta + \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}. \tag{4.3}$$

In the following, we write functions $x, y \in L^2(\Omega)$ in the form $x = x_+ + x_-$ and $y = y_+ + y_-$, where $x_\pm, y_\pm \in E_\pm L^2(\Omega)$ (see (4.2)). We emphasize that $x_\pm$ are the components of $x$ with respect to the space decomposition (4.2) and that $x_\pm$ do not coincide with the restrictions of the function $x$ onto $\Omega_\pm$. Since the spectral subspaces $(E_\pm L^2(\Omega), \pm[\cdot,\cdot])$ are Hilbert spaces, the inner product $(\cdot,\cdot)_\sim$ defined as

$$(x, y)_\sim := [x_+, y_+] - [x_-, y_-], \quad x, y \in L^2(\Omega),$$

is positive definite. Furthermore, this scalar product is connected with the usual scalar product $(\cdot,\cdot)$ on $L^2(\Omega)$ via

$$\begin{aligned}(x, y)_\sim &= [E_+x, E_+y] - [E_-x, E_-y] \\ &= [(E_+ - E_-)x, y] \\ &= (R(E_+ - E_-)x, y).\end{aligned}$$

Therefore, as $R(E_+ - E_-)$ is an isomorphism, the norms $\|\cdot\|$ and $\|\cdot\|_\sim$ induced by the scalar products $(\cdot,\cdot)$ and $(\cdot,\cdot)_\sim$, respectively, are equivalent. In particular, with $\nu := \|R(E_+ - E_-)\|^{-1}$, we have

$$\|x\| \leqslant \sqrt{\nu}\|x\|_\sim \quad \text{for all } x \in L^2(\Omega). \tag{4.4}$$

2. In this step, it will be shown that, for sufficiently large $|\mu|$, $\mu \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{Re}\mu \leqslant 0$, the operator $T_{\eta,+} + V_{11} - \mu$ is invertible and the estimates

$$\|(T_{\eta,+} + V_{11} - \mu)^{-1}\|_\sim < \tfrac{1}{2} \quad \text{and} \quad \|(T_{\eta,+} + V_{11} - \mu)^{-1}V_{12}\|_\sim < \tfrac{1}{2} \tag{4.5}$$

hold. By replacing $V_{11}$ and $V_{12}$ in the reasoning below with $V_{22}$ and $V_{21}$, respectively, it follows that, for sufficiently large $|\mu|$, $\mu \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{Re}\mu \geqslant 0$, the operator $T_{\eta,-} + V_{22} - \mu$ is invertible and the estimates

$$\|(T_{\eta,-} + V_{22} - \mu)^{-1}\|_\sim < \tfrac{1}{2} \quad \text{and} \quad \|(T_{\eta,-} + V_{22} - \mu)^{-1} V_{21}\|_\sim < \tfrac{1}{2} \tag{4.6}$$

are valid.

In the following, we assume that the entry $V_{12}$ in the perturbation term $V$ is non-zero (otherwise the first estimate in (4.6) follows with $\delta > 2$ and $\tau = \delta + \|V_{11}\|_\sim$ in the argument below, and the second estimate is trivial). Choose $\delta > 0$ such that

$$\delta + \|V_{11}\|_\sim > \max\left\{\frac{2 + \|V_{11}\|_\sim}{\|V_{12}\|_\sim}, 2 + \frac{\|V_{11}\|_\sim}{\|V_{12}\|_\sim}\right\} \tag{4.7}$$

and define the constant $\tau$ by

$$\tau := (\delta + \|V_{11}\|_\sim)\|V_{12}\|_\sim. \tag{4.8}$$

Let $\mu \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{Re}\mu \leqslant 0$ and $|\mu| > \tau$. Since $\sigma(T_{\eta,+}) \subset \mathbb{R}^+$, it is clear that

$$\operatorname{dist}(\mu, \sigma(T_{\eta+})) > \tau$$

holds and, therefore, we have $\|(T_{\eta,+} - \mu)^{-1}\|_\sim < \tau^{-1}$. This implies

$$\|V_{11}(T_{\eta,+} - \mu)^{-1}\|_\sim < \frac{1}{\tau}\|V_{11}\|_\sim$$

and it follows from (4.7) and (4.8) that $\|V_{11}(T_{\eta,+} - \mu)^{-1}\|_\sim < 1$. Therefore, the operator $I + V_{11}(T_{\eta,+} - \mu)^{-1}$ is boundedly invertible and the norm of the inverse can be estimated by

$$\|(I + V_{11}(T_{\eta,+} - \mu)^{-1})^{-1}\|_\sim < \left(1 - \frac{1}{\tau}\|V_{11}\|_\sim\right)^{-1}.$$

It also follows that the operator

$$T_{\eta,+} + V_{11} - \mu = (I + V_{11}(T_{\eta,+} - \mu)^{-1})(T_{\eta,+} - \mu)$$

is boundedly invertible, and we conclude that

$$\begin{aligned} \|(T_{\eta,+} + V_{11} - \mu)^{-1}\|_\sim &< \frac{1}{\tau}\left(1 - \frac{1}{\tau}\|V_{11}\|_\sim\right)^{-1} \\ &= \frac{1}{\tau - \|V_{11}\|_\sim}. \end{aligned} \tag{4.9}$$

Since, by (4.7) and (4.8), $\tau - \|V_{11}\|_\sim > 2$, we obtain the first estimate in (4.6). Furthermore, as a consequence of (4.7) and (4.8) we have

$$\frac{\|V_{12}\|_\sim}{\tau - \|V_{11}\|_\sim} = \frac{\|V_{12}\|_\sim}{(\delta + \|V_{11}\|_\sim)\|V_{12}\|_\sim - \|V_{11}\|_\sim} < \tfrac{1}{2}$$

and therefore (4.9) yields the second estimate in (4.6),

$$\|(T_{\eta,+} + V_{11} - \mu)^{-1} V_{12}\|_\sim < \frac{\|V_{12}\|_\sim}{\tau - \|V_{11}\|_\sim} < \tfrac{1}{2}.$$

3. Next, we verify the inequality

$$\|(T-\mu)x\|_\sim \geqslant \left(\sqrt{\left(1+\frac{\nu}{|\operatorname{Im}\mu|}\right)^2+1}-\left(1+\frac{\nu}{|\operatorname{Im}\mu|}\right)\right)\|x\|_\sim \tag{4.10}$$

for $x \in \operatorname{dom} T$ and all sufficiently large $|\mu|$, $\mu \in \mathbb{C}\setminus\mathbb{R}$. Observe first that, for $x \in \operatorname{dom} T$, we have

$$[(T-\mu)x,x] = [(T-\operatorname{Re}\mu)x,x] - \mathrm{i}\operatorname{Im}\mu[x,x],$$

which, together with (4.4), implies

$$|\operatorname{Im}\mu|\,|[x,x]| \leqslant |[(T-\mu)x,x]| \leqslant \|(T-\mu)x\|\|x\| \leqslant \nu\|(T-\mu)x\|_\sim\|x\|_\sim$$

and, hence,

$$\frac{\nu}{|\operatorname{Im}\mu|}\|(T-\mu)x\|_\sim\|x\|_\sim \geqslant \pm[x,x]. \tag{4.11}$$

On the other hand, when we consider the equation $(T-\mu)x = y$ with $x = x_+ + x_-$, $y = y_+ + y_-$, $x_\pm, y_\pm \in E_\pm L^2(\Omega)$, that is (see (4.3)),

$$\begin{aligned}(T_{\eta,+}+V_{11}-\mu)x_+ + V_{12}x_- &= y_+,\\ V_{21}x_+ + (T_{\eta,-}+V_{22}-\mu)x_- &= y_-,\end{aligned}$$

then we conclude, with the help of the estimates from step 2 that, for sufficiently large $|\mu|$, $\mu \in \mathbb{C}\setminus\mathbb{R}$ with $\operatorname{Re}\mu \leqslant 0$,

$$\begin{aligned}\|x_+\|_\sim &\leqslant \|(T_{\eta_+}+V_{11}-\mu)^{-1}y_+\|_\sim + \|(T_{\eta_+}+V_{11}-\mu)^{-1}V_{12}x_-\|_\sim\\ &\leqslant \tfrac12\|y_+\|_\sim + \tfrac12\|x_-\|_\sim\\ &\leqslant \tfrac12\|y_+\|_\sim + \tfrac12\|x\|_\sim\end{aligned}$$

holds and that, for sufficiently large $|\mu|$, $\mu \in \mathbb{C}\setminus\mathbb{R}$ with $\operatorname{Re}\mu \geqslant 0$,

$$\begin{aligned}\|x_-\|_\sim &\leqslant \|(T_{\eta_-}+V_{22}-\mu)^{-1}y_-\|_\sim + \|(T_{\eta_-}+V_{22}-\mu)^{-1}V_{21}x_+\|_\sim\\ &\leqslant \tfrac12\|y_-\|_\sim + \tfrac12\|x_+\|_\sim\\ &\leqslant \tfrac12\|y_-\|_\sim + \tfrac12\|x\|_\sim\end{aligned}$$

holds. Since $\|y_\pm\|_\sim \leqslant \|y\|_\sim = \|(T-\mu)x\|_\sim$, we have

$$\|x_\pm\|_\sim^2 \leqslant \tfrac14\|(T-\mu)x\|_\sim^2 + \tfrac14\|x\|_\sim^2 + \tfrac12\|(T-\mu)x\|_\sim\|x\|_\sim \tag{4.12}$$

for sufficiently large $|\mu|$, $\mu \in \mathbb{C}\setminus\mathbb{R}$ with $\operatorname{Re}\mu \leqslant 0$ and $\operatorname{Re}\mu \geqslant 0$, respectively. From $\|x_+\|_\sim^2 + \|x_-\|_\sim^2 = \|x\|_\sim^2$, we obtain

$$\pm[x,x] = \pm\|x_+\|_\sim^2 \mp \|x_-\|_\sim^2 = \|x\|_\sim^2 - 2\|x_\mp\|_\sim^2$$

and, together with (4.12), we conclude that

$$\pm[x,x] \geqslant \tfrac12\|x\|_\sim^2 - \tfrac12\|(T-\mu)x\|_\sim^2 - \|(T-\mu)x\|_\sim\|x\|_\sim$$

for sufficiently large $|\mu|$, $\mu \in \mathbb{C}\setminus\mathbb{R}$ with $\operatorname{Re}\mu \geqslant 0$ and $\operatorname{Re}\mu \leqslant 0$, respectively. Together with (4.11), this leads to

$$\frac{\nu}{|\operatorname{Im}\mu|}\|(T-\mu)x\|_\sim\|x\|_\sim \geqslant \tfrac12\|x\|_\sim^2 - \tfrac12\|(T-\mu)x\|_\sim^2 - \|(T-\mu)x\|_\sim\|x\|_\sim,$$

for all sufficiently large $|\mu|$, $\mu \in \mathbb{C} \setminus \mathbb{R}$. In other words, $\|(T-\mu)x\|_\sim$ satisfies the quadratic inequality

$$\|(T-\mu)x\|_\sim^2 + 2\left(1 + \frac{\nu}{|\operatorname{Im}\mu|}\right)\|x\|_\sim \|(T-\mu)x\|_\sim - \|x\|_\sim^2 \geqslant 0.$$

Hence, it follows that (4.10) holds for all $x \in \operatorname{dom} T$ and all $\mu \in \mathbb{C} \setminus \mathbb{R}$ with sufficiently large $|\mu|$.

4. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that (4.10) is satisfied with $\mu = \lambda$ and $\mu = \bar{\lambda}$. Then we have $\ker(T-\lambda) = \{0\}$ and $\operatorname{ran}(T-\lambda)$ is closed as $T$ is closed. Furthermore, since $T$ is self-adjoint in the Krein space $(L^2(\Omega), [\cdot,\cdot])$, it is clear that $\operatorname{ran}(T-\lambda)^{[\perp]} = \ker(T-\bar{\lambda})$ holds. As $\|(T-\bar{\lambda})x\|_\sim$, $x \in \operatorname{dom} T$, satisfies the same estimate as $\|(T-\lambda)x\|_\sim$ in (4.10), this also implies that $\ker(T-\bar{\lambda})$ is trivial. Therefore, $T-\lambda$ is bijective, i.e. $\lambda \in \rho(T)$. Since this is true for every $\lambda = \mu$ that satisfies (4.10), we conclude that the non-real spectrum of $T$ is bounded. □

## 5. Spectral properties of indefinite elliptic operators on $\mathbb{R}^n$

In this section we consider the case where $\Omega = \mathbb{R}^n$ and we assume that the subsets $\Omega_\pm = \{x \in \mathbb{R}^n\colon \pm r(x) > 0\}$ consist of finitely many connected components with compact smooth boundaries. In particular, this implies that one of the sets $\Omega_\pm$ is bounded and one is unbounded, and that the boundaries $\partial\Omega_+$ and $\partial\Omega_-$ coincide. Here, and in what follows, we discuss the case where $\Omega_-$ is unbounded and $\Omega_+$ is bounded and we denote the boundary $\partial\Omega_\pm$ by $\mathcal{C}$. The simple modifications of the results to the other case are left to the reader. Since the weight function satisfies $r, r^{-1} \in L^\infty(\mathbb{R}^n)$, the restrictions $r_\pm, r_\pm^{-1}$ belong to $L^\infty(\Omega_\pm)$ and, hence, the multiplication operators $R_\pm f_\pm = r_\pm f_\pm$ are isomorphisms in $L^2(\Omega_\pm)$ with inverses $R_\pm^{-1} f_\pm = r_\pm^{-1} f_\pm$, $f_\pm \in L^2(\Omega_\pm)$.

Let us now assume that the coefficients $a_{jk} \in C^\infty(\mathbb{R}^n)$ in (2.1) and their derivatives are uniformly continuous and bounded, and that (as before) $a \in L^\infty(\mathbb{R}^n)$ is real valued. An essential ingredient for the following considerations is that, by elliptic regularity and interpolation,

$$\operatorname{dom} A = \operatorname{dom} T = H^2(\mathbb{R}^n), \quad \mathcal{H}_s = H^s(\mathbb{R}^n), \quad s \in [0,2],$$

holds (see [1, 7, 36, 44], [37, condition 3.1], (2.3) and (2.6)). Here, $H^s(\mathbb{R}^n)$ is the Sobolev space or order $s$. The spaces consisting of restrictions of functions from $H^s(\mathbb{R}^n)$ onto $\Omega_\pm$ are denoted by $H^s(\Omega_\pm)$. In the next lemma and remark we give simple sufficient conditions for the weight function $r$ such that condition (i) of theorem 4.4 holds.

Lemma 5.1. *Assume that, for some $s \in (0, \frac{1}{2})$ the spaces $H^s(\Omega_+)$ and $H^s(\Omega_-)$ are invariant subspaces of the multiplication operators $R_+$ and $R_-$, respectively. Then $H^s(\mathbb{R}^n)$ is an invariant subspace of the multiplication operator $R$, and condition (i) of theorem 4.4 is satisfied with $W$ replaced by $R$.*

*Proof.* Let $s$ be as in the assumptions of the lemma and let $f \in H^s(\mathbb{R}^n)$. Then the restrictions $f_\pm$ of $f$ onto $\Omega_\pm$ are functions in $H^s(\Omega_\pm)$ and, therefore, by assumption, the functions $g_\pm := r_\pm f_\pm$ also belong to $H^s(\Omega_\pm)$. As $0 < s < \frac{1}{2}$, the continuations $\tilde{g}_\pm$ of $g_\pm$ by zero onto $\mathbb{R}^n$ both are in $H^s(\mathbb{R}^n)$ (see [29, theorem 1.4.4.4

and corollary 1.4.4.5] and note that the proofs of these statements in [29] also cover the case of an unbounded domain with a compact smooth boundary). Therefore, $Rf = rf = \tilde{g}_+ + \tilde{g}_- \in H^s(\mathbb{R}^n)$ and, hence, $H^s(\mathbb{R}^n)$ is invariant for $R$. Furthermore, $R$ is an isomorphism in $L^2(\mathbb{R}^n)$ and the estimate $R^2 \geqslant \operatorname{ess\,inf} r^2 > 0$ holds, i.e. $R$ possesses all the properties of the operator $W$ in condition (i) of theorem 4.4. □

Remark 5.2. If, for example, the function $r$ is equal to a (negative) constant outside some bounded subset of $\mathbb{R}^n$ and $r_\pm$ belong to the Hölder spaces $C^{0,\alpha}(\bar{\Omega}_\pm)$ for some $\alpha > 0$, then it follows from [29, theorem 1.4.1.1] and a similar argument as in the proof of lemma 5.1 that $H^s(\Omega_\pm)$, $s \in (0, \alpha)$, are invariant subspaces of $R_\pm$.

For completeness, we state the following immediate consequence of theorem 4.4 and lemma 5.1.

Corollary 5.3. *Assume that* $\min \sigma_{\mathrm{ess}}(A) \leqslant 0$ *and* $R_\pm(H^s(\Omega_\pm)) \subset H^s(\Omega_\pm)$ *holds for some* $s \in (0, \frac{1}{2})$. *Then the non-real spectrum of* $T$ *is bounded.*

In the following theorem, we obtain more precise statements on the qualitative spectral properties of $T$. The proof is based on a multidimensional variant of Glazmann's decomposition method from the theory of ordinary differential operators (see, for example, [3, 10, 15, 19, 38]).

Theorem 5.4. *Assume that* $\min \sigma_{\mathrm{ess}}(A) \leqslant 0$. *If* $\rho(T) \neq \emptyset$, *then the essential spectrum of* $T$ *is real, bounded from above, and* $\sigma_{\mathrm{ess}}(T) \cap [0, \infty) \neq \emptyset$ *holds. Moreover, the non-real spectrum of* $T$ *consists of normal eigenvalues that are symmetric with respect to the real line and which may accumulate to points in* $\sigma_{\mathrm{ess}}(T)$.

*If, in particular,* $R_\pm(H^s(\Omega_\pm)) \subset H^s(\Omega_\pm)$ *holds for some* $s \in (0, \frac{1}{2})$, *then the assumption* $\rho(T) \neq \emptyset$ *is satisfied, the above assertions hold and the non-real spectrum of* $T$ *is bounded.*

*Proof.* Besides the operators $A$ and $T$, we will make use of the self-adjoint elliptic differential operators

$$A_\pm f_\pm := \ell(f_\pm), \qquad \operatorname{dom} A_\pm = H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm), \tag{5.1}$$

in $L^2(\Omega_\pm)$ and the weighted differential operators

$$B_\pm f_\pm := \mathcal{L}_\pm(f_\pm) = \frac{1}{r_\pm}\ell(f_\pm), \qquad \operatorname{dom} B_\pm = H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm), \tag{5.2}$$

which are self-adjoint in the weighted $L^2$-space $L^2(\Omega_\pm, \pm r_\pm)$, where the (positive definite) scalar products $\langle \cdot, \cdot \rangle_\pm$ are defined as

$$\langle f_\pm, g_\pm \rangle_\pm := \int_{\Omega_\pm} f_\pm(x)\overline{g_\pm(x)}(\pm r_\pm(x))\,\mathrm{d}x, \quad f_\pm, g_\pm \in L^2(\Omega_\pm). \tag{5.3}$$

Observe that the orthogonal sums $A_+ \oplus A_-$ and $B_+ \oplus B_-$ are self-adjoint operators in $L^2(\Omega)$ and $L^2(\Omega, r)$, respectively. Furthermore, since the boundary $\mathcal{C}$ is compact and smooth, it can be shown that the resolvent differences

$$(A - \lambda)^{-1} - ((A_+ \oplus A_-) - \lambda)^{-1}, \qquad \lambda \in \rho(A) \cap \rho(A_+ \oplus A_-), \tag{5.4}$$

and

$$(T-\lambda)^{-1} - ((B_+ \oplus B_-) - \lambda)^{-1}, \qquad \lambda \in \rho(T) \cap \rho(B_+ \oplus B_-), \tag{5.5}$$

are compact operators in $L^2(\mathbb{R}^n)$ (see [12] and theorem 5.7).

Recall that the spectra of $A_\pm$ are bounded from below and, moreover, as $\Omega_+$ is assumed to be bounded, the spectrum of $A_+$ consists of normal eigenvalues (see theorems 3.1 and 4.1). Furthermore, the differential operators in (5.1) and (5.2) are connected via

$$B_\pm = R_\pm^{-1} A_\pm, \tag{5.6}$$

where $R_\pm$ are the multiplication operators with the functions $r_\pm$ in the spaces $L^2(\Omega_\pm)$. For $\lambda \in \rho(A_+) \cap \rho(B_+)$, the resolvents of $A_+$ and $B_+$ are connected via

$$(B_+ - \lambda)^{-1} = (A_+ - \lambda)^{-1} R_+ - \lambda (A_+ - \lambda)^{-1} (I - R_+)(B_+ - \lambda)^{-1}$$

and since $(A_+ - \lambda)^{-1}$ is compact, the same holds for the resolvent of $B_+$ (see the proof of theorem 3.2). Thus, the spectrum of $B_+$ is also bounded from below and consists of normal eigenvalues which accumulate to $+\infty$.

Next the spectrum of $B_-$ will be described in terms of the spectrum of $A_-$. Since the resolvent difference in (5.4) is compact and $\sigma_{\mathrm{ess}}(A_+) = \emptyset$, we conclude that

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A_+ \oplus A_-) = \sigma_{\mathrm{ess}}(A_-).$$

Furthermore, the assumption $\min \sigma_{\mathrm{ess}}(A) \leqslant 0$ implies that $A_-$ is bounded from below with negative lower bound

$$\nu := \min \sigma(A_-) \leqslant \min \sigma_{\mathrm{ess}}(A_-) \leqslant 0. \tag{5.7}$$

Denote the usual scalar product in $L^2(\Omega_-)$ by $(\cdot,\cdot)_-$ and let $\gamma$ be the supremum of the weight function $r_-$ on $\Omega_-$. Then we have $\gamma < 0$ and $(-r_-(x))^{-1} \leqslant (-\gamma)^{-1}$ for all $x \in \Omega_-$. Moreover, from the estimate

$$(f_-, f_-)_- = \int_{\Omega_-} \frac{1}{-r_-(x)} |f_-(x)|^2 (-r_-(x)) \, \mathrm{d}x \leqslant \frac{1}{-\gamma} \langle f_-, f_- \rangle_-, \quad f \in L^2(\Omega_-),$$

we obtain together with (5.3) and (5.6) that

$$\langle B_- f_-, f_- \rangle_- = (-A_- f_-, f_-)_- \leqslant -\nu (f_-, f_-)_- \leqslant \frac{\nu}{\gamma} \langle f_-, f_- \rangle_-$$

holds for all $f_- \in \operatorname{dom} B_-$, i.e. the spectrum $\sigma(B_-)$ and the essential spectrum $\sigma_{\mathrm{ess}}(B_-)$ are bounded from above by the positive constant $\nu/\gamma$. Observe that $\max \sigma_{\mathrm{ess}}(B_-) \geqslant 0$ holds, since otherwise $\min \sigma_{\mathrm{ess}}(-B_-)$ is positive and $A_- = (-R_-)(-B_-)$ implies that also $\min \sigma_{\mathrm{ess}}(A_-)$ is positive which contradicts (5.7).

Summing up, we have shown that the essential spectrum of $B_+ \oplus B_-$ is real, bounded from above and $\sigma_{\mathrm{ess}}(B_+ \oplus B_-) \cap [0, \infty) \neq \emptyset$. Since the resolvent difference (5.5) is compact, we obtain $\sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(B_+ \oplus B_-) = \sigma_{\mathrm{ess}}(B_-)$ which, together with corollary 5.3, yields the statements. $\square$

In the following, we will show that the non-real eigenvalues of $T$ and the corresponding eigenspaces can be characterized with the help of Dirichlet-to-Neumann maps associated to the restrictions of the elliptic differential expression $\mathcal{L}$ on $\Omega_\pm$.

For this, recall first that the mapping $C^\infty(\overline{\Omega}_\pm) \ni f_\pm \mapsto \{f_\pm|_{\mathcal{C}}, \partial f_\pm/\partial\nu_\pm|_{\mathcal{C}}\}$ extends to a continuous surjective mapping

$$H^2(\Omega_\pm) \ni f_\pm \mapsto \left\{ f_\pm|_{\mathcal{C}}, \frac{\partial f_\pm}{\partial \nu_\pm}\bigg|_{\mathcal{C}} \right\} \in H^{3/2}(\mathcal{C}) \times H^{1/2}(\mathcal{C}), \tag{5.8}$$

where

$$\frac{\partial f_\pm}{\partial \nu_\pm}\bigg|_{\mathcal{C}} := \sum_{j,k=1}^{n} a_{jk} \mathfrak{n}_{\pm,j} \frac{\partial f_\pm}{\partial x_k}\bigg|_{\mathcal{C}}$$

and $\mathfrak{n}_\pm(x) = (\mathfrak{n}_{\pm,1}(x), \ldots, \mathfrak{n}_{\pm,n}(x))$ is the unit vector at the point $x \in \mathcal{C}$ pointing out of $\Omega_\pm$. The next simple lemma is based on a standard decomposition argument. We provide a complete proof for the reader's convenience.

Lemma 5.5. *For every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\varphi \in H^{3/2}(\mathcal{C})$, there exist unique functions $f_{\pm,\lambda}(\varphi) \in H^2(\Omega_\pm)$ such that*

$$\mathcal{L}_\pm f_{\pm,\lambda}(\varphi) = \lambda f_{\pm,\lambda}(\varphi) \quad \text{and} \quad f_{\pm,\lambda}(\varphi)|_{\mathcal{C}} = \varphi.$$

*Proof.* It is sufficient to show that, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the linear subspace

$$\mathcal{S} := \{h_+ \oplus h_- \in H^2(\Omega_+) \oplus H^2(\Omega_-) \colon h_+|_{\mathcal{C}} = h_-|_{\mathcal{C}}\} \tag{5.9}$$

admits the direct sum decomposition

$$\mathcal{S} = \{g_+ \oplus g_- \in \mathcal{S} \colon g_\pm|_{\mathcal{C}} = 0\} \dot{+} \{h_{+,\lambda} \oplus h_{-,\lambda} \in \mathcal{S} \colon \mathcal{L}_\pm h_{\pm,\lambda} = \lambda h_{\pm,\lambda}\}. \tag{5.10}$$

In fact, it follows from (5.8) that the trace map $h \mapsto h|_{\mathcal{C}}$ defined on $\mathcal{S}$ in (5.9) maps onto $H^{3/2}(\mathcal{C})$, and since the first term on the right-hand side of (5.10) is its kernel, it follows that the trace map maps the second term on the right-hand side of (5.10) bijectively onto $H^{3/2}(\mathcal{C})$.

In order to prove the decomposition (5.10), note first that the inclusion $\supset$ in (5.10) holds. Hence, it remains to verify the inclusion $\subset$. For this, let $h_+ \oplus h_- \in \mathcal{S}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be fixed. Since the boundary $\mathcal{C}$ of $\Omega_\pm$ is assumed to be compact and smooth, it follows that the differential operators $B_\pm$ in (5.2) are defined on

$$\begin{aligned} \operatorname{dom} B_\pm &= H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm) \\ &= \{f \in H^2(\Omega_\pm) \colon f_\pm|_{\mathcal{C}} = 0\}. \end{aligned} \tag{5.11}$$

Hence, the first set on the right-hand side of (5.10) coincides with $\operatorname{dom}(B_+ \oplus B_-)$. Since the spectrum of $B_+ \oplus B_-$ is contained in $\mathbb{R}$ (see the proof of theorem 5.4) $B_+ \oplus B_- - \lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is a bijection from its domain onto $L^2(\mathbb{R}^n)$. Thus, there exists $g_+ \oplus g_- \in \operatorname{dom}(B_+ \oplus B_-)$ such that

$$(\mathcal{L}_+ - \lambda)h_+ \oplus (\mathcal{L}_- - \lambda)h_- = (B_+ - \lambda)g_+ \oplus (B_- - \lambda)g_-.$$

Therefore, $\mathcal{L}_\pm(h_\pm - g_\pm) = \lambda(h_\pm - g_\pm)$ and, hence,

$$h_+ \oplus h_- = g_+ \oplus g_- + ((h_+ - g_+) \oplus (h_- - g_-))$$

shows that the inclusion $\subset$ in (5.10) is also valid. The sum in (5.10) is direct, since $\sigma(B_+ \oplus B_-) \subset \mathbb{R}$; indeed, each element in the intersection of the sets on the right-hand side of (5.10) would be an eigenfunction of $B_+ \oplus B_-$ corresponding to $\lambda \in \mathbb{C} \setminus \mathbb{R}$. $\square$

For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\varphi \in H^{3/2}(\mathcal{C})$ and $f_{\pm,\lambda}(\varphi) \in H^2(\Omega_\pm)$ as in lemma 5.5, we define

$$M(\lambda)\colon H^{3/2}(\mathcal{C}) \to H^{1/2}(\mathcal{C}), \qquad \varphi \mapsto \frac{\partial f_{+,\lambda}(\varphi)}{\partial \nu_+}\Big|_{\mathcal{C}} + \frac{\partial f_{-,\lambda}(\varphi)}{\partial \nu_-}\Big|_{\mathcal{C}}. \tag{5.12}$$

Roughly speaking, $M$ is the sum of the Dirichlet-to-Neumann maps associated to $\mathcal{L}_\pm$ which map the Dirichlet boundary values of solutions of $\mathcal{L}_\pm f_\pm = \lambda f_\pm$ onto their Neumann boundary values. A similar function in a 'definite' setting also appears in [43]. In the next theorem we show how the non-real eigenvalues of $T$ can be described with the help of the function $M$.

THEOREM 5.6. *Let the operator function $\lambda \mapsto M(\lambda)$ be defined as in (5.12) and assume that $\rho(T) \neq \emptyset$. Then*

$$\sigma(T) \cap (\mathbb{C} \setminus \mathbb{R}) = \{\lambda \in \mathbb{C} \setminus \mathbb{R}\colon \ker M(\lambda) \neq \{0\}\}$$

*and* $\ker(T - \lambda) = \{f \in H^2(\mathbb{R}^n)\colon M(\lambda) f|_{\mathcal{C}} = 0\}$ *for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

*Proof.* Assume first that $\lambda \in \mathbb{C} \setminus \mathbb{R}$ belongs to the spectrum of $T$. Then, by theorems 4.3 and 5.4, the point $\lambda$ is a normal eigenvalue of $T$ and, hence, $\mathcal{L} f = \lambda f$ holds for some non-trivial $f \in \operatorname{dom} T = H^2(\mathbb{R}^n)$. In particular, the restrictions $f_\pm$ of $f$ onto $\Omega_\pm$ belong to $H^2(\Omega_\pm)$ and we have

$$\mathcal{L}_\pm f_\pm = \lambda f_\pm, \qquad f_+|_{\mathcal{C}} = f_-|_{\mathcal{C}}, \qquad \frac{\partial f_+}{\partial \nu_+}\Big|_{\mathcal{C}} = -\frac{\partial f_-}{\partial \nu_-}\Big|_{\mathcal{C}}. \tag{5.13}$$

By (5.8) we have $\varphi := f_\pm|_{\mathcal{C}} \in H^{3/2}(\mathcal{C})$ and, hence, $f_\pm = f_{\pm,\lambda}(\varphi)$ in the notation of lemma 5.5. The third property in (5.13) implies

$$\begin{aligned} M(\lambda)\varphi &= \frac{\partial f_{+,\lambda}(\varphi)}{\partial \nu_+}\Big|_{\mathcal{C}} + \frac{\partial f_{-,\lambda}(\varphi)}{\partial \nu_-}\Big|_{\mathcal{C}} \\ &= \frac{\partial f_+}{\partial \nu_+}\Big|_{\mathcal{C}} + \frac{\partial f_-}{\partial \nu_-}\Big|_{\mathcal{C}} \\ &= 0 \end{aligned}$$

and, hence, $\varphi \in \ker M(\lambda)$. Furthermore, $\varphi$ is non-zero, since otherwise $f_\pm \in H^2(\Omega_\pm)$ would be non-trivial solutions of the Dirichlet problems $\mathcal{L}_\pm f_\pm = \lambda f_\pm$, $f_\pm|_{\mathcal{C}} = 0$, which do not exist due to $\lambda \notin \mathbb{R}$. In other words, since the self-adjoint operators $B_\pm$ in (5.2) do not have non-real eigenvalues, we conclude $\varphi \neq 0$.

For the converse, let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\varphi \in \ker M(\lambda)$ with $\varphi \neq 0$. By lemma 5.5, there exist unique functions $f_{\pm,\lambda}(\varphi) \in H^2(\Omega_\pm)$ such that $\mathcal{L}_\pm f_{\pm,\lambda}(\varphi) = \lambda f_{\pm,\lambda}(\varphi)$ and $f_{\pm,\lambda}(\varphi)|_{\mathcal{C}} = \varphi$ hold. Since $M(\lambda)\varphi = 0$, we have

$$\frac{\partial f_{+,\lambda}(\varphi)}{\partial \nu_+}\Big|_{\mathcal{C}} = -\frac{\partial f_{-,\lambda}(\varphi)}{\partial \nu_-}\Big|_{\mathcal{C}}. \tag{5.14}$$

Define the function $f = f_+ \oplus f_- \in L^2(\mathbb{R}^n)$ by $f_\pm := f_{\pm,\lambda}(\varphi)$ and let $g \in \operatorname{dom} T$. Then $f \neq 0$ and

$$\begin{aligned} [\mathcal{L} f, g] - [f, Tg] &= (\ell f, g) - (f, \ell g) \\ &= (\ell_+ f_+, g_+)_+ - (f_+, \ell_+ g_+)_+ + (\ell_- f_-, g_-)_- - (f_-, \ell_- g_-)_-, \end{aligned} \tag{5.15}$$

where $[\cdot,\cdot]$ is the indefinite inner product in (2.4), $(\cdot,\cdot)$ is the usual scalar product in $L^2(\mathbb{R}^n)$ and $(\cdot,\cdot)_\pm$ denote the scalar products in $L^2(\Omega_\pm)$. Since the function $g \in \operatorname{dom} T$ satisfies

$$g_+|_\mathcal{C} = g_-|_\mathcal{C} \quad \text{and} \quad \frac{\partial g_+}{\partial \nu_+}\bigg|_\mathcal{C} = -\frac{\partial g_-}{\partial \nu_-}\bigg|_\mathcal{C}$$

it follows from Green's identity, $f_+|_\mathcal{C} = f_-|_\mathcal{C}$ and (5.14) that (5.15) is equal to

$$\left(f_+|_\mathcal{C}, \frac{\partial g_+}{\partial \nu_+}\bigg|_\mathcal{C}\right) - \left(\frac{\partial f_+}{\partial \nu_+}\bigg|_\mathcal{C}, g_+|_\mathcal{C}\right) + \left(f_-|_\mathcal{C}, \frac{\partial g_-}{\partial \nu_-}\bigg|_\mathcal{C}\right) - \left(\frac{\partial f_-}{\partial \nu_-}\bigg|_\mathcal{C}, g_-|_\mathcal{C}\right) = 0.$$

This is true for any $g \in \operatorname{dom} T$, and since $T$ is self-adjoint with respect to $[\cdot,\cdot]$, we conclude from $[\mathcal{L}f, g] = [f, Tg]$ that $f \in \operatorname{dom} T$ and $Tf = \mathcal{L}f$. Moreover, from $\mathcal{L}_\pm f_\pm = \lambda f_\pm$, we obtain $f \in \ker(T-\lambda)$, i.e. $\lambda$ is an eigenvalue of $T$ with corresponding eigenfunction $f$. □

The next theorem provides a variant of Krein's formula which shows how the resolvent of the indefinite elliptic operator $T$ differs from the resolvent of the orthogonal sum of the weighted differential operators (see (5.2) and (5.11))

$$B_\pm f_\pm = \mathcal{L}_\pm(f_\pm) = \frac{1}{r_\pm}\ell_\pm(f_\pm), \qquad \operatorname{dom} B_\pm = H^2(\Omega_\pm) \cap H^1_0(\Omega_\pm).$$

The operators $T$ and $B_+ \oplus B_-$ are viewed as operators in $L^2(\mathbb{R}^n)$. We note first that the statements in lemma 5.5 and theorem 5.6 remain true if the set $\mathbb{C}\setminus\mathbb{R}$ is replaced by the resolvent set of the operator $B_+ \oplus B_-$. This set contains $\mathbb{C}\setminus\mathbb{R}$ and may also contain subsets of the real line. For $\lambda \in \rho(B_+ \oplus B_-)$, define the mapping $\gamma(\lambda)\colon L^2(\mathcal{C}) \to L^2(\mathbb{R}^n)$ by

$$\gamma(\lambda)\varphi := f_{+,\lambda}(\varphi) \oplus f_{-,\lambda}(\varphi), \qquad \operatorname{dom}\gamma(\lambda) = H^{3/2}(\mathcal{C}),$$

where $f_{\pm,\lambda}(\varphi)$ are the unique solutions of $\mathcal{L}_\pm u_\pm = \lambda u_\pm$, $u_\pm|_\mathcal{C} = \varphi$ (see lemma 5.5).

Theorem 5.7 is an indefinite variant of [4, part (ii) of theorem 4.4] and can be proved in almost the same way. We therefore only sketch some ideas from the proof and refer the interested reader to [4, § 4] for the details (see also [8]). Recall that the multiplication operator $R$ is an isomorphism in $L^2(\mathbb{R}^n)$.

Theorem 5.7. *For all $\lambda \in \rho(T) \cap \rho(B_+ \oplus B_-)$, the difference of the resolvents of $T$ and $B_+ \oplus B_-$ is a compact operator in $L^2(\mathbb{R}^n)$ given by*

$$(T-\lambda)^{-1} - ((B_+ \oplus B_-) - \lambda)^{-1} = \gamma(\lambda)M(\lambda)^{-1}\gamma(\bar\lambda)^*R. \tag{5.16}$$

*Proof.* A slight modification of [4, proposition 4.3] yields that $\gamma(\lambda)$ is a densely defined bounded operator from $L^2(\mathcal{C})$ into $L^2(\mathbb{R}^n)$ and that the adjoint operator $\gamma(\bar\lambda)^*\colon L^2(\mathbb{R}^n) \to L^2(\mathcal{C})$ has the property

$$\gamma(\bar\lambda)^*R((B_+ \oplus B_-) - \lambda)f = -\frac{\partial f_+}{\partial \nu_+}\bigg|_\mathcal{C} - \frac{\partial f_-}{\partial \nu_-}\bigg|_\mathcal{C},$$

where $f = f_+ \oplus f_- \in \operatorname{dom}(B_+ \oplus B_-)$. In particular, taking $\operatorname{ran}\gamma(\bar\lambda)^* \subset H^{1/2}(\mathcal{C})$ with lemma 5.5, theorem 5.6 and (5.12), we conclude that the right-hand side of

(5.16) is well defined for all $\lambda \in \rho(T) \cap \rho(B_+ \oplus B_-)$. The same reasoning as in the proof of [4, part (ii) of theorem 4.4] shows the relation (5.16) for the difference of the resolvents of $T$ and $B_+ \oplus B_-$ in $L^2(\mathbb{R}^n)$. Moreover, it follows from [36] in the same way as in [4, corollaries 3.6 and 4.6] that, for $\lambda \in \rho(T) \cap \rho(B_+ \oplus B_-)$, the closure of $M(\lambda)^{-1}$ in $L^2(\mathcal{C})$ is a compact operator in $L^2(\mathcal{C})$. Therefore, the right-hand side of (5.16) is a compact operator in $L^2(\mathbb{R}^n)$. □

Remark 5.8. We note that the resolvent difference in (5.16) is not only compact but belongs to certain Schatten–von Neumann ideals that depend on the dimension $n$ (see [11, 12, 30, 37]).

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