Proceedings of the Royal Society of Edinburgh, **144A**, 1113–1126, *2014*

Estimates on the non-real eigenvalues of regular indefinite Sturm–Liouville problems

Jussi Behrndt
Institut für Numerische Mathematik, Technische Universität Graz,
Steyrergasse 30, 8010 Graz, Austria (behrndt@tugraz.at)

Shaozhu Chen
Department of Mathematics, Shandong University (Weihai),
Weihai 264209, People's Republic of China (szchen@sdu.edu.cn)

Friedrich Philipp
Institut für Mathematik, Technische Universität Clausthal,
Erzstraße 1, 38678 Clausthal–Zellerfeld, Germany
(fmphilipp@gmail.com)

Jiangang Qi
Department of Mathematics, Shandong University (Weihai),
Weihai 264209, People's Republic of China (qjg816@163.com)

(MS received 28 July 2013; accepted 4 October 2013)

Regular Sturm–Liouville problems with indefinite weight functions may possess finitely many non-real eigenvalues. In this paper we prove explicit bounds on the real and imaginary parts of these eigenvalues in terms of the coefficients of the differential expression.

1. Introduction

In this paper we consider regular *indefinite* Sturm–Liouville eigenvalue problems of the form

$$\tau(f) = \lambda f \quad \text{with } \tau = \frac{1}{w}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x} + q\right) \tag{1.1}$$

on bounded intervals $(a, b) \subset \mathbb{R}$ with real coefficients $p^{-1}, q, w \in L^1(a, b)$ such that $p > 0$ and $w \neq 0$ almost everywhere (a.e.) on (a, b). Problem (1.1) is supplemented with suitable boundary conditions at the endpoints a and b. The peculiarity here is that the weight function w is not assumed to be positive, and for this reason the eigenvalue problem and the Sturm–Liouville differential expression τ in (1.1) are called *indefinite*.

The history of indefinite Sturm–Liouville eigenvalue problems goes back to the early 20th century, when Haupt [10] and Richardson [16] generalized oscillation results to the indefinite case, and noted that problems of the form (1.1) may have non-real eigenvalues. For more historical details and other classical references we refer the reader to the interesting survey paper [13] by Mingarelli. From a modern and more abstract point of view, the spectral theory of Sturm–Liouville operators with indefinite weights is intimately connected with spectral and perturbation theory of operators that are self-adjoint with respect to the indefinite inner product

$$[f, g] := \int_a^b f(x)\overline{g(x)}w(x)\,\mathrm{d}x, \tag{1.2}$$

where f, g are functions in the weighted L^2-space $L^2_{|w|}(a, b)$. The qualitative spectral properties of the self-adjoint differential operators associated to τ in the Krein space $(L^2_{|w|}(a, b), [\cdot,\cdot])$ are well understood. We emphasize the contribution of Ćurgus and Langer [9], who laid the foundations of this operator-theoretic approach. In particular, the spectrum of any self-adjoint realization consists only of normal eigenvalues. There are at most finitely many non-real eigenvalues, which appear in pairs symmetric with respect to the real axis, and the real eigenvalues accumulate to $+\infty$ and $-\infty$. We refer the reader to the monograph [18] for an overview, and to [1, 2, 4–8, 11, 14] for some other aspects in indefinite Sturm–Liouville theory.

The main objective of this paper is to prove bounds on the non-real spectrum of indefinite Sturm–Liouville operators in terms of the coefficients in the differential expression. This is a challenging open problem according to Mingarelli [13] and Kong *et al*. [12, remark 4.4] (see also [18, remark 11.4.1]). Only very recently first results in this direction have been obtained by the authors of this paper, jointly with Trunk in [3] for a particular singular problem, and in [15, 17] with Xie for regular problems with separated boundary conditions and special weight functions w.

In this paper we investigate the general regular case with arbitrary self-adjoint boundary conditions and a large class of weight functions, thereby extending and completing the results in [15, 17]. The only restriction on the weight w is that we assume the existence of an absolutely continuous function g with $g'^2 p \in L^1(a, b)$ such that $\mathrm{sgn}(g) = \mathrm{sgn}(w)$ a.e. In theorems 3.2 and 3.6 we then obtain bounds for the real and imaginary parts of the non-real eigenvalues of the indefinite Sturm–Liouville eigenvalue problem (1.1), which depend on p, q, g (and thus implicitly on w) and the self-adjoint boundary condition. The techniques in the proofs of our main results are inspired by the methods in [15, 17]. For the case of a weight function with finitely many sign changes, we construct an admissible function g and find bounds that do not depend on g in corollaries 3.3 and 3.7. A particular weight function with infinitely many turning points is treated in example 3.4. Furthermore, for a certain set of real eigenvalues where the eigenfunctions have special sign properties (sometimes called real ghost states) we obtain similar bounds as in theorem 3.2 in theorem 4.3.

The paper has the following structure. After introducing the relevant notions in § 2, we prove the *a priori* bounds on the non-real eigenvalues of indefinite regular Sturm–Liouville operators in § 3. We give estimates on the real exceptional eigenvalues in § 4. A key ingredient in the proofs of the results in §§ 3 and 4 are certain estimates on the norms of the corresponding eigenfunctions and their derivatives in lemmas 3.1, 3.5 and 4.2. In order to improve the flow of the paper we cover the proofs of these lemmas separately in § 5.

2. Preliminaries

Let τ be the indefinite Sturm–Liouville expression from (1.1) with real-valued coefficients $p^{-1}, q, w \in L^1(a,b)$ such that $p > 0$ and $w \neq 0$ a.e. on (a,b). Assume that both sets

$$\{x \in (a,b)\colon w(x) > 0\} \quad \text{and} \quad \{x \in (a,b)\colon w(x) < 0\}$$

have positive Lebesgue measure. Let $L^2_{|w|}(a,b)$ be the linear space (of equivalence classes) of measurable functions $f\colon (a,b) \to \mathbb{C}$ such that $f^2 w \in L^1(a,b)$, and equip this space with the indefinite inner product $[\cdot,\cdot]$ in (1.2).

The differential expression τ is then formally symmetric with respect to $[\cdot,\cdot]$, and hence gives rise to self-adjoint realizations in the Krein space $(L^2_{|w|}(a,b), [\cdot,\cdot])$, that is, τ induces a family of differential operators that are self-adjoint with respect to the Krein space inner product $[\cdot,\cdot]$. In the remainder of this paper, *self-adjoint* refers to self-adjointness with respect to this inner product.

We briefly recollect how the self-adjoint realizations of τ can be parametrized (see [18, § 4.2]). For this, denote by $\mathcal{D}_{\max}$ the maximal domain that consists of all $f \in L^2_{|w|}(a,b)$ such that f, pf' are absolutely continuous and $\tau(f) \in L^2_{|w|}(a,b)$. Any self-adjoint differential operator associated with τ in $(L^2_{|w|}(a,b), [\cdot,\cdot])$ is then of the form

$$A(\mathcal{D})f = \tau(f), \qquad \operatorname{dom} A(\mathcal{D}) = \mathcal{D}, \tag{2.1}$$

where

$$\mathcal{D} = \mathcal{D}_{\mathrm{sep}}(l,r) := \{f \in \mathcal{D}_{\max}\colon (pf')(a) = lf(a),\ (pf')(b) = rf(b)\}$$

with $l, r \in \mathbb{R} \cup \{\infty\}$ or

$$\mathcal{D} = \mathcal{D}_{\mathrm{coup}}(\varphi, R) := \left\{ f \in \mathcal{D}_{\max}\colon \begin{pmatrix} f(b) \\ (pf')(b) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\varphi} R \begin{pmatrix} f(a) \\ (pf')(a) \end{pmatrix} \right\}$$

with $\varphi \in [0, 2\pi)$ and $R \in \mathbb{R}^{2\times 2}$ such that $\det R = 1$. We note that $l = \infty$ or $r = \infty$ in $\mathcal{D}_{\mathrm{sep}}(l,r)$ stand for the Dirichlet boundary conditions at a or b, respectively. For brevity we refer to the above domains as *self-adjoint domains*. To any self-adjoint domain $\mathcal{D}$ we assign a constant $c(\mathcal{D}) \geqslant 0$ as follows:

$$c(\mathcal{D}) := \begin{cases} |l| + |r| & \text{if } \mathcal{D} = \mathcal{D}_{\mathrm{sep}}(l,r) \text{ with } l, r \in \mathbb{R}, \\ |r| & \text{if } \mathcal{D} = \mathcal{D}_{\mathrm{sep}}(\infty, r) \text{ with } r \in \mathbb{R}, \\ |l| & \text{if } \mathcal{D} = \mathcal{D}_{\mathrm{sep}}(l, \infty) \text{ with } l \in \mathbb{R}, \\ 0 & \text{if } \mathcal{D} = \mathcal{D}_{\mathrm{sep}}(\infty, \infty), \\ \dfrac{|r_{11}| + |r_{22}| + 2}{|r_{12}|} & \text{if } \mathcal{D} = \mathcal{D}_{\mathrm{coup}}(\varphi, R) \text{ and } r_{12} \neq 0, \\ |r_{11} r_{21}| & \text{if } \mathcal{D} = \mathcal{D}_{\mathrm{coup}}(\varphi, R) \text{ and } r_{12} = 0, \end{cases} \tag{2.2}$$

where the r_{ij} are the entries of the matrix $R = (r_{ij})_{i,j=1}^2 \in \mathbb{R}^{2\times 2}$ in the case of coupled boundary conditions, i.e. $\mathcal{D} = \mathcal{D}_{\mathrm{coup}}(\varphi, R)$.

Lemma 2.1. *Let $\mathcal{D}$ be a self-adjoint domain and let $\phi \in \mathcal{D}$. We then have that*

$$\operatorname{Im}((p\phi')(b)\overline{\phi(b)} - (p\phi')(a)\overline{\phi(a)}) = 0 \tag{2.3}$$

and, in addition,

$$|(p\phi')(b)\overline{\phi(b)} - (p\phi')(a)\overline{\phi(a)}| \leqslant c(\mathcal{D}) \max\{|\phi(a)|^2, |\phi(b)|^2\}. \tag{2.4}$$

Proof. The identity (2.3) follows from the self-adjointness of $A(\mathcal{D})$. We only show that (2.4) holds in the case $\mathcal{D} = \mathcal{D}_{\mathrm{coup}}(\varphi, R)$. The other cases are evident. Let $\phi \in \mathcal{D}_{\mathrm{coup}}(\varphi, R)$. We then have that

$$\begin{pmatrix} \phi(b) \\ (p\phi')(b) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\varphi} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} \phi(a) \\ (p\phi')(a) \end{pmatrix}, \tag{2.5}$$

and hence, as $\det R = r_{11}r_{22} - r_{12}r_{21} = 1$, also that

$$\begin{pmatrix} r_{22} & -r_{12} \\ -r_{21} & r_{11} \end{pmatrix} \begin{pmatrix} \phi(b) \\ (p\phi')(b) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\varphi} \begin{pmatrix} \phi(a) \\ (p\phi')(a) \end{pmatrix}. \tag{2.6}$$

From (2.5) we obtain

$$r_{12}(p\phi')(a) = \mathrm{e}^{-\mathrm{i}\varphi}\phi(b) - r_{11}\phi(a),$$

and (2.6) yields that

$$r_{12}(p\phi')(b) = r_{22}\phi(b) - \mathrm{e}^{\mathrm{i}\varphi}\phi(a).$$

Hence, if $r_{12} \neq 0$, then

$$(p\phi')(b)\overline{\phi(b)} - (p\phi')(a)\overline{\phi(a)} = \frac{r_{22}|\phi(b)|^2 + r_{11}|\phi(a)|^2 - 2\operatorname{Re}(\mathrm{e}^{\mathrm{i}\varphi}\phi(a)\overline{\phi(b)})}{r_{12}}.$$

This directly implies (2.4). If $r_{12} = 0$, then $r_{11}r_{22} = 1$. Moreover, by (2.6) we have that $r_{22}\phi(b) = \mathrm{e}^{\mathrm{i}\varphi}\phi(a)$, and from (2.5) we obtain

$$\mathrm{e}^{-\mathrm{i}\varphi}(p\phi')(b) = r_{21}\phi(a) + r_{22}(p\phi')(a).$$

This yields that

$$\begin{aligned} (p\phi')(b)\overline{\phi(b)} - (p\phi')(a)\overline{\phi(a)} &= (r_{22}^{-1}(p\phi')(b)\mathrm{e}^{-\mathrm{i}\varphi} - (p\phi')(a))\overline{\phi(a)} \\ &= (r_{22}^{-1}(r_{21}\phi(a) + r_{22}(p\phi')(a)) - (p\phi')(a))\overline{\phi(a)} \\ &= r_{11}r_{21}|\phi(a)|^2, \end{aligned}$$

and (2.4) follows. □

For the estimates on the non-real and exceptional eigenvalues in the next sections we need a set of norms. If $r\colon (a,b) \to [0,\infty)$ is a measurable function, we denote by μ_r the measure on (a,b) with $\mathrm{d}\mu_r = r\,\mathrm{d}t$ and define the weighted L^2-spaces as $L^2_r(a,b) := L^2((a,b),\mu_r)$; this is in accordance with $L^2_{|w|}(a,b)$ defined above. The norm of $L^2_r(a,b)$ is denoted by $\|\cdot\|_{r,2}$. As usual, the L^1-norm and L^∞-norm are denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively.

We close this section with a simple observation, which is exploited in many of the proofs below. Let ϕ be a solution of (1.1), i.e. $\phi, p\phi' \in \mathrm{AC}[a,b]$ and

$$-(p\phi')' + q\phi = \lambda w\phi. \tag{2.7}$$

Multiplying (2.7) with $\bar{\phi}$ and using $(p\phi'\bar{\phi})' = (p\phi')'\bar{\phi} + p|\phi'|^2$ we obtain

$$\lambda w|\phi|^2 = -(p\phi')'\bar{\phi} + q|\phi|^2 = -(p\phi'\bar{\phi})' + p|\phi'|^2 + q|\phi|^2. \tag{2.8}$$

Integration over $[x,b] \subset [a,b]$ gives

$$\lambda \int_x^b w|\phi|^2 = (p\phi')(x)\overline{\phi(x)} - (p\phi')(b)\overline{\phi(b)} + \int_x^b (p|\phi'|^2 + q|\phi|^2), \tag{2.9}$$

and for the real and imaginary part we conclude that

$$(\operatorname{Re}\lambda) \int_x^b w|\phi|^2 = \operatorname{Re}((p\phi')(x)\overline{\phi(x)} - (p\phi')(b)\overline{\phi(b)}) + \int_x^b (p|\phi'|^2 + q|\phi|^2) \tag{2.10}$$

and

$$(\operatorname{Im}\lambda) \int_x^b w|\phi|^2 = \operatorname{Im}((p\phi')(x)\overline{\phi(x)} - (p\phi')(b)\overline{\phi(b)}). \tag{2.11}$$

3. Bounds on non-real eigenvalues

In this section we provide *a priori* bounds on the non-real eigenvalues of the self-adjoint realizations of the regular indefinite Sturm–Liouville expression τ (see theorems 3.2 and 3.6). The following constants are incorporated into these bounds:

$$\alpha := c(\mathcal{D}) + \|q_-\|_1, \qquad \beta := \sqrt{\alpha(1/\|p^{-1}\|_1 + \alpha)} + \alpha, \qquad \gamma := \sqrt{2\beta + 1/\|p^{-1}\|_1}. \tag{3.1}$$

Here (and in the following), $q_-(x) := \min\{0, q(x)\}$, $x \in (a,b)$. Note that α, β and γ only depend on the chosen self-adjoint boundary conditions and the norms $\|q_-\|_1$, $\|p^{-1}\|_1$. In particular, the constants α, β and γ do not depend on the weight function w.

The following lemma is the first of three similar statements that play a key role in the proofs of the eigenvalue estimates in this paper. Its proof can be found in § 5.

LEMMA 3.1. *Let $\mathcal{D}$ be a self-adjoint domain. Then, for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and all solutions $\phi \in \mathcal{D}$ of (2.7), the following estimates hold:*

$$\|\phi'\|_{p,2} \leqslant \beta\|\phi\|_{1/p,2} \quad \textit{and} \quad \|\phi\|_\infty \leqslant \gamma\|\phi\|_{1/p,2}.$$

The next theorem is the first main result of this section. It provides estimates on the real and imaginary parts of the non-real eigenvalues of the operators $A(\mathcal{D})$ in (2.1) for any self-adjoint domain $\mathcal{D}$. For the special case of separated boundary conditions, a different type of estimate can be found in [17, theorem 1.2] and [15, theorem 1.3].

THEOREM 3.2. *Let $\mathcal{D}$ be a self-adjoint domain, let α, β, γ be as in (3.1), and assume that there exists a real-valued function $g \in \mathrm{AC}[a,b]$ with $g(a) = g(b) = 0$*

and $g' \in L^2_p(a,b)$ such that $gw > 0$ a.e. on (a,b). Then, with $\varepsilon > 0$ chosen such that

$$\mu_{1/p}(\{x \in [a,b] \colon p(x)g(x)w(x) < \varepsilon\}) \leqslant \frac{1}{2\gamma^2}, \tag{3.2}$$

the following holds for all eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of the operator $A(\mathcal{D})$:

$$|\operatorname{Im} \lambda| \leqslant \frac{2}{\varepsilon}\beta\gamma\|g'\|_{p,2} \tag{3.3}$$

and

$$|\operatorname{Re} \lambda| \leqslant \frac{2}{\varepsilon}(\beta\gamma\|g'\|_{p,2} + (\beta^2 + \gamma^2\|q\|_1)\|g\|_\infty). \tag{3.4}$$

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be a non-real eigenvalue of $A(\mathcal{D})$ and let $\phi \in \mathcal{D}$ be a corresponding eigenfunction. It is no restriction to assume that

$$\|\phi\|_{1/p,2} = 1. \tag{3.5}$$

Since $g(a) = g(b) = 0$, integration by parts yields that

$$\begin{aligned} \int_a^b g'(x) \int_x^b w(t)|\phi(t)|^2 \,\mathrm{d}t \,\mathrm{d}x &= -\int_a^b g(x) \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b w(t)|\phi(t)|^2 \,\mathrm{d}t \,\mathrm{d}x \\ &= \int_a^b g(x)w(x)|\phi(x)|^2 \,\mathrm{d}x. \end{aligned} \tag{3.6}$$

Let $\Omega := \{x \in [a,b] \colon p(x)g(x)w(x) < \varepsilon\}$ and let $\Omega^c = [a,b] \setminus \Omega$. By assumption we have that $\mu_{p^{-1}}(\Omega) \leqslant 1/2\gamma^2$, and hence we find that

$$\begin{aligned} \int_a^b gw|\phi|^2 &= \int_a^b (pgw)(|\phi|^2 p^{-1}) \geqslant \varepsilon \int_{\Omega^c} |\phi|^2 p^{-1} \\ &= \varepsilon\left(1 - \int_\Omega |\phi|^2 p^{-1}\right) \\ &\geqslant \varepsilon(1 - \|\phi\|_\infty^2 \mu_{p^{-1}}(\Omega)) \geqslant \varepsilon(1 - \gamma^2 \mu_{p^{-1}}(\Omega)) \\ &\geqslant \frac{\varepsilon}{2}, \end{aligned} \tag{3.7}$$

where we have used the estimate $\|\phi\|_\infty^2 \leqslant \gamma^2$ from lemma 3.1 together with (3.5). Combining (3.6) and (3.7) we obtain the estimate

$$\frac{\varepsilon}{2} \leqslant \int_a^b g'(x) \int_x^b w(t)|\phi(t)|^2 \,\mathrm{d}t \,\mathrm{d}x. \tag{3.8}$$

From this, (2.11) and $g(a) = g(b) = 0$, we obtain

$$\begin{aligned} |\operatorname{Im} \lambda| \frac{\varepsilon}{2} &\leqslant \left| \int_a^b g'(x)(\operatorname{Im} \lambda) \int_x^b w(t)|\phi(t)|^2 \,\mathrm{d}t \,\mathrm{d}x \right| \\ &= \left| \int_a^b g'(x) \operatorname{Im}((p\phi')(x)\overline{\phi(x)} - (p\phi')(b)\overline{\phi(b)}) \,\mathrm{d}x \right| \\ &= \left| \int_a^b g'(x) \operatorname{Im}((p\phi')(x)\overline{\phi(x)}) \,\mathrm{d}x \right|, \end{aligned}$$

which can be further estimated as being less than or equal to

$$\int_a^b |g'p\phi'\phi| \leqslant \|\phi\|_\infty \int_a^b |g'|p^{1/2}|\phi'|p^{1/2} \leqslant \|\phi\|_\infty \|g'\|_{p,2}\|\phi'\|_{p,2}.$$

Thus, the assertion on the imaginary part of the eigenvalue λ follows from the estimates $\|\phi'\|_{p,2} \leqslant \beta$ and $\|\phi\|_\infty \leqslant \gamma$ (see lemma 3.1 and (3.5)).

It remains to estimate the real part of λ. For this we define the function $G(x) := \int_x^b p|\phi'|^2 + q|\phi|^2$. From (3.8) and (2.10), we have that

$$\begin{aligned} |\operatorname{Re}\lambda|\frac{\varepsilon}{2} &\leqslant \left| \int_a^b g'(x)(\operatorname{Re}((p\phi')(x)\overline{\phi(x)} - (p\phi')(b)\overline{\phi(b)}) + G(x))\,\mathrm{d}x \right| \\ &= \left| \int_a^b g'(x)\operatorname{Re}((p\phi')(x)\overline{\phi(x)})\,\mathrm{d}x + \int_a^b g'(x)G(x)\,\mathrm{d}x \right|. \end{aligned} \tag{3.9}$$

Integration by parts gives

$$\int_a^b g'(x)G(x)\,\mathrm{d}x = -\int_a^b g(x)G'(x)\,\mathrm{d}x = \int_a^b g(x)(p(x)|\phi'(x)|^2 + q(x)|\phi(x)|^2)\,\mathrm{d}x,$$

and hence from (3.9) we have that

$$\begin{aligned} |\operatorname{Re}\lambda|\frac{\varepsilon}{2} &\leqslant \int_a^b |g'p\phi'\phi| + \int_a^b |g(p|\phi'|^2 + q|\phi|^2)| \\ &\leqslant \|\phi\|_\infty \int_a^b |g'|p^{1/2}|\phi'|p^{1/2} + \|g\|_\infty \int_a^b |p|\phi'|^2 + q|\phi|^2| \\ &\leqslant \|\phi\|_\infty\|g'\|_{p,2}\|\phi'\|_{p,2} + \|g\|_\infty(\|\phi'\|_{p,2}^2 + \|q\|_1\|\phi\|_\infty^2). \end{aligned} \tag{3.10}$$

The inequality (3.10) together with the estimates $\|\phi'\|_{p,2} \leqslant \beta$ and $\|\phi\|_\infty \leqslant \gamma$ (see lemma 3.1 and (3.5)) implies that

$$|\operatorname{Re}\lambda|\frac{\varepsilon}{2} \leqslant \beta\gamma\|g'\|_{p,2} + \|g\|_\infty(\beta^2 + \gamma^2\|q\|_1),$$

which is (3.4). The theorem is proved. □

As the condition in theorem 3.2 concerning the existence of the absolutely continuous function g is somewhat implicit, we show in the next corollary how the theorem becomes more explicit in the case of an indefinite weight function with a finite number of turning points, that is, the interval (a, b) can be segmented into a finite number of intervals, on each of which $\operatorname{sgn}(w)$ is constant. For the special case of one turning point and separated boundary conditions, a different estimate was shown in [17, theorem 1.3] (see also [15, theorem 1.2]).

Corollary 3.3. *Assume that $p = 1$ and that w has n turning points in (a, b). Moreover, let $\mathcal{D}$ be a self-adjoint domain, and let α, β, γ be as in (3.1). Then, with $\varepsilon > 0$ chosen such that*

$$\mu_1(\{x \in (a, b)\colon |w(x)| < \varepsilon\}) \leqslant \frac{1}{4\gamma^2},$$

the following holds for all eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of the operator $A(\mathcal{D})$:

$$|\mathrm{Im}\,\lambda| \leqslant \frac{8}{\varepsilon}\beta\gamma^2(n+1) \quad \textit{and} \quad |\mathrm{Re}\,\lambda| \leqslant \frac{2}{\varepsilon}(4\beta\gamma^2(n+1) + (\beta^2 + \gamma^2\|q\|_1)).$$

Proof. Let $x_1 < \cdots < x_n$ be the turning points of w in (a,b), set $x_0 := a$, $x_{n+1} := b$, and define the constant

$$\nu := \frac{1}{8(n+1)\gamma^2}.$$

Let $k \in \{0,\dots,n\}$. If $x_{k+1} - x_k \geqslant 2\nu$, for $x \in [x_k, x_{k+1}]$ we set

$$g(x) := \mathrm{sgn}(w \mid (x_k, x_{k+1})) \begin{cases} \dfrac{x - x_k}{\nu} & \text{for } x \in [x_k, x_k + \nu], \\ 1 & \text{for } x \in [x_k + \nu, x_{k+1} - \nu], \\ \dfrac{x_{k+1} - x}{\nu} & \text{for } x \in [x_{k+1} - \nu, x_{k+1}]. \end{cases}$$

If $x_{k+1} - x_k < 2\nu$, then we define

$$g(x) := \mathrm{sgn}(w \mid (x_k, x_{k+1})) \frac{(x - x_k)(x_{k+1} - x)}{2\nu^2}, \quad x \in [x_k, x_{k+1}].$$

Obviously, we have $g \in \mathrm{AC}[a,b]$, $g' \in L^2(a,b)$, $g(a) = g(b) = 0$ and $gw > 0$ a.e. Moreover,

$$\int_{x_k}^{x_{k+1}} |g'(x)|^2 \,\mathrm{d}x \leqslant \begin{cases} \dfrac{2}{\nu} & \text{if } x_{k+1} - x_k \geqslant 2\nu, \\ \dfrac{x_{k+1} - x_k}{\nu^2} & \text{if } x_{k+1} - x_k < 2\nu \end{cases}$$
$$\leqslant \frac{2}{\nu}.$$

In addition, it is easy to see that $|g(x)| \leqslant 1$ for every $x \in (a,b)$. Hence, we obtain

$$\|g\|_\infty \leqslant 1 \quad \text{and} \quad \|g'\|_2 \leqslant \sqrt{\frac{2}{\nu}(n+1)} = 4\gamma(n+1).$$

Now, define $\mathcal{S} := \{x \in (a,b) \colon |g(x)| \neq 1\}$ and $\Omega := \{x \in (a,b) \colon g(x)w(x) < \varepsilon\}$. We then have $\mu_1(\mathcal{S}) \leqslant 2\nu(n+1)$, and hence

$$\mu_1(\Omega) \leqslant \mu_1(\{x \in (a,b) \setminus \mathcal{S} \colon |w(x)| < \varepsilon\}) + 2\nu(n+1) \leqslant \frac{1}{2\gamma^2}.$$

The claim now follows from theorem 3.2. □

As the following example illustrates, theorem 3.2 also applies to weight functions with an infinite number of turning points.

Example 3.4. Let $p = 1$ and let $w(x) = \sin(1/x)$ for $x \in [0, 1/\pi]$. Then $w \in L^1(0, 1/\pi)$, but $w \notin \mathrm{AC}[0, 1/\pi]$ (and hence the results in [15] do not apply here). In order to estimate the non-real eigenvalues of (1.1) with some $q \in L^1(0, 1/\pi)$

and some self-adjoint boundary conditions, we set $g(x) := x^4 \sin(1/x)$. Then, g is a function as in theorem 3.2 with

$$\|g\|_\infty \leqslant 0.003 \quad \text{and} \quad \|g'\|_2 \leqslant 0.02.$$

Choose $k_0 \in \mathbb{N}$ such that $(k_0+1)\pi > 4\gamma^2$. If, for $k \in \mathbb{N}$, we set

$$I_k := \left[\frac{1}{(k+1)\pi}, \frac{1}{k\pi}\right],$$

we then have

$$\mu_1\biggl(\bigcup_{k=k_0+1}^{\infty} I_k\biggr) < \frac{1}{4\gamma^2},$$

so, for (3.2) to hold, it suffices to find $\varepsilon > 0$ such that

$$\mu_1\biggl(\biggl\{x \in \bigcup_{k=1}^{k_0} I_k \colon x^4 \sin^2(1/x) < \varepsilon\biggr\}\biggr) < \frac{1}{4\gamma^2}. \tag{3.11}$$

For this, we first observe that for $x \in I_k$ we have $x^2|\sin(1/x)| \geqslant p_k(x)$, where

$$p_k(x) := \frac{1}{\pi}(1 - k\pi x)((k+1)\pi x - 1), \quad x \in I_k.$$

It is easily seen that, for $\varepsilon > 0$ small enough, we have

$$\mu_1(\{x \in I_k \colon p_k(x) < \sqrt{\varepsilon}\}) = \frac{1 - \sqrt{1 - 4\sqrt{\varepsilon}k(k+1)\pi}}{k(k+1)\pi} \leqslant \frac{4\sqrt{\varepsilon}k(k+1)\pi}{k(k+1)\pi} = 4\sqrt{\varepsilon}.$$

Hence, with $\sqrt{\varepsilon} := 1/4k_0(k_0+1)\pi$ we can estimate the left-hand side of (3.11) by

$$\sum_{k=1}^{k_0} \mu_1(\{x \in I_k \colon x^2|\sin(1/x)| < \sqrt{\varepsilon}\}) \leqslant 4k_0\sqrt{\varepsilon} = \frac{1}{(k_0+1)\pi} < \frac{1}{4\gamma^2},$$

so that (3.11), and hence (3.2), is satisfied. We now find estimates on the non-real eigenvalues by making use of (3.3) and (3.4) in theorem 3.2.

In the next lemma we prove different estimates from those in lemma 3.1 for $\|\phi'\|_{p,2}$ and $\|\phi\|_\infty$ under the assumption that the weight function w is such that

$$\int_a^b w \neq 0. \tag{3.12}$$

These involve the constant α in (3.1) and the constant δ defined by

$$\delta := 2 + 2\frac{\|w\|_1}{|\int_a^b w|}. \tag{3.13}$$

The proof of lemma 3.5 can be found in § 5.

Lemma 3.5. *Assume that the weight function w satisfies (3.12). Then, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and all solutions $\phi \in \mathcal{D}$ of (2.7), the following estimates hold:*

$$\|\phi'\|_{p,2} \leqslant \alpha\delta\|\phi\|_{1/p,2} \quad \textit{and} \quad \|\phi\|_\infty \leqslant \sqrt{\alpha}\delta\|\phi\|_{1/p,2}.$$

By the same reasoning as in the proof of theorem 3.2, the estimates on $\|\phi'\|_{p,2}$ and $\|\phi\|_\infty$ yield bounds on the non-real eigenvalues of the self-adjoint realizations of the regular indefinite Sturm–Liouville expression τ. We note that the estimates in theorems 3.2 and 3.6 are not directly comparable, but can of course be combined if w satisfies assumption (3.12).

Theorem 3.6. *Assume that the weight function w satisfies (3.12), let $\mathcal{D}$ be a self-adjoint domain, let α and δ be as above and assume that there exists a real-valued function $g \in \mathrm{AC}[a,b]$ with $g(a) = g(b) = 0$ and $g' \in L^2_p(a,b)$ such that $gw > 0$ a.e. on (a,b). Then, with $\varepsilon > 0$ chosen such that*

$$\mu_{1/p}(\{x \in [a,b] \colon p(x)g(x)w(x) < \varepsilon\}) \leqslant \frac{1}{2\alpha\delta^2},$$

the following holds for all eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of the operator $A(\mathcal{D})$:

$$|\mathrm{Im}\,\lambda| \leqslant \frac{2}{\varepsilon}\alpha^{3/2}\delta^2\|g'\|_{p,2} \tag{3.14}$$

and

$$|\mathrm{Re}\,\lambda| \leqslant \frac{2}{\varepsilon}\alpha\delta^2(\sqrt{\alpha}\|g'\|_{p,2} + (\alpha + \|q\|_1)\|g\|_\infty). \tag{3.15}$$

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be an eigenvalue corresponding to some eigenfunction $\phi \in \mathcal{D}$, and assume that ϕ satisfies (3.5). The same reasoning as in (3.6) and (3.7) leads to (3.8), and hence to the estimates

$$|\mathrm{Im}\,\lambda| \leqslant \frac{2}{\varepsilon}\|\phi\|_\infty\|g'\|_{p,2}\|\phi'\|_{p,2}$$

and

$$|\mathrm{Re}\,\lambda| \leqslant \frac{2}{\varepsilon}(\|\phi\|_\infty\|g'\|_{p,2}\|\phi'\|_{p,2} + \|g\|_\infty(\|\phi'\|^2_{p,2} + \|q\|_1\|\phi\|^2_\infty)).$$

The assertions now follow from $\|\phi'\|_{p,2} \leqslant \alpha\delta$ and $\|\phi\|_\infty \leqslant \sqrt{\alpha}\delta$ (see lemma 3.5 and (3.5)). □

The next corollary is a variant of corollary 3.3 and can be proved in the same way.

Corollary 3.7. *Assume that $p = 1$, that w satisfies (3.12) and has n turning points in (a,b). Moreover, let $\mathcal{D}$ be a self-adjoint domain and let α and δ be as in (3.1) and (3.13). Then, with $\varepsilon > 0$ chosen such that*

$$\mu_1(\{x \in (a,b) \colon |w(x)| < \varepsilon\}) \leqslant \frac{1}{4\alpha\delta^2},$$

the following holds for all eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of the operator $A(\mathcal{D})$:

$$|\mathrm{Im}\,\lambda| \leqslant \frac{8}{\varepsilon}\alpha^2\delta^3(n+1) \quad \text{and} \quad |\mathrm{Re}\,\lambda| \leqslant \frac{2}{\varepsilon}\alpha\delta^2(4\alpha\delta(n+1) + \alpha + \|q\|_1).$$

Remark 3.8. If we regard the existence of the function g in theorems 3.2 and 3.6 as a condition on the weight function w, it turns out that the condition $g(a) = g(b) = 0$

is redundant. To see this, let $\tilde{g}$ be an absolutely continuous function on $[a,b]$ with $\tilde{g}' \in L^2_p(a,b)$ and $\tilde{g}w > 0$, choose a function $h \in \mathrm{AC}[a,b]$ such that

$$h(x) > 0 \quad \text{for all } x \in (a,b), \qquad h(a) = h(b) = 0 \qquad \text{and} \qquad h' \in L^2_p(a,b), \tag{3.16}$$

and set $g := h\tilde{g}$. Then,

$$g \in \mathrm{AC}[a,b], \qquad g(a) = g(b) = 0, \qquad gw > 0 \quad \text{and} \quad g' = h'\tilde{g} + h\tilde{g}' \in L^2_p(a,b).$$

We note that a function h with the above-mentioned properties can be defined as follows. Choose $x_0 \in (a,b)$ such that $\int_a^{x_0} 1/\sqrt{p} = \int_{x_0}^b 1/\sqrt{p}$ and let

$$h(x) := \int_a^x \frac{\operatorname{sgn}(x_0 - t)}{\sqrt{p(t)}}\,\mathrm{d}t, \quad x \in [a,b].$$

We also mention that the condition $g(a) = g(b) = 0$ is not redundant for the eigenvalue estimates in theorems 3.2 and 3.6.

4. Bounds on exceptional real eigenvalues

Let $A(\mathcal{D})$ be a self-adjoint realization of the indefinite Sturm–Liouville expression τ defined on some self-adjoint domain $\mathcal{D}$. It is well known that the resolvent of $A(\mathcal{D})$ is a compact operator and that the real eigenvalues of $A(\mathcal{D})$ accumulate to $+\infty$ and $-\infty$. Moreover, the real eigenvalues have the following *sign properties* (see [9]).

PROPOSITION 4.1. *Let $\mathcal{D}$ be a self-adjoint domain. There then exist at most finitely many real eigenvalues $\lambda \neq 0$ of $A(\mathcal{D})$ with a corresponding eigenfunction ϕ such that*

$$\lambda[\phi,\phi] = \lambda \int_a^b |\phi|^2 w \leqslant 0. \tag{4.1}$$

These eigenvalues are called real exceptional eigenvalues *of $A(\mathcal{D})$.*

We mention that $\lambda = 0$ is said to be an exceptional eigenvalue of $A(\mathcal{D})$ if there exists a function ψ in the root subspace of $A(\mathcal{D})$ corresponding to 0 such that $[A(\mathcal{D})\psi,\psi] < 0$. Furthermore, we note that in [13] the eigenfunctions corresponding to (non-zero) real exceptional eigenvalues satisfying (4.1) are called *real ghost states*.

In what follows we provide estimates on the real exceptional eigenvalues along the lines of theorem 3.2. The following preparatory lemma is the analog of lemma 3.1 and is also proved in § 5.

LEMMA 4.2. *For all $\lambda \in \mathbb{R} \setminus \{0\}$ and all solutions $\phi \in \mathcal{D}$ of (2.7) that satisfy (4.1), we have*

$$\|\phi'\|_{p,2} \leqslant \beta\|\phi\|_{1/p,2} \quad \textit{and} \quad \|\phi\|_\infty \leqslant \gamma\|\phi\|_{1/p,2}.$$

Lemma 4.2 implies the following variant of theorem 3.2; its proof remains the same. We leave it to the reader to formulate a variant of corollary 3.3 for real exceptional eigenvalues.

THEOREM 4.3. *Let $\mathcal{D}$ be a self-adjoint domain, let α, β, γ be as above, and assume that there exists a real-valued function $g \in \mathrm{AC}[a,b]$ such that $gw > 0$ a.e. on (a,b), $g(a) = g(b) = 0$ and $g' \in L^2_p(a,b)$. Then, with $\varepsilon > 0$ chosen such that*

$$\mu_{1/p}(\{x \in [a,b] \colon p(x)g(x)w(x) < \varepsilon\}) \leqslant \frac{1}{2\gamma^2},$$

the following holds for all real exceptional eigenvalues λ of the operator $A(\mathcal{D})$:

$$|\lambda| \leqslant \frac{2}{\varepsilon}(\beta\gamma\|g'\|_{p,2} + (\beta^2 + \gamma^2\|q\|_1)\|g\|_\infty).$$

5. Proofs of lemmas 3.1, 3.5 and 4.2

In this section we provide the remaining proofs of lemmas 3.1, 3.5 and 4.2.

Proof of lemma 3.1. Choosing $x = a$ in (2.11) and taking into account (2.3) and $\operatorname{Im}\lambda \neq 0$, we find that

$$\int_a^b w|\phi|^2 = 0. \tag{5.1}$$

From (2.9) we then obtain

$$\begin{aligned}\|\phi'\|_{p,2}^2 &= (p\phi')(b)\overline{\phi(b)} - (p\phi')(a)\overline{\phi(a)} - \int_a^b q|\phi|^2 \\ &\leqslant c(\mathcal{D})\max\{|\phi(a)|^2, |\phi(b)|^2\} + \int_a^b |q_-||\phi|^2 \\ &\leqslant (c(\mathcal{D}) + \|q_-\|_1)\|\phi\|_\infty^2 \\ &= \alpha\|\phi\|_\infty^2. \end{aligned}\tag{5.2}$$

For $x, y \in [a,b]$, $y < x$, we have that

$$\begin{aligned}|\phi(x)|^2 - |\phi(y)|^2 &= \int_y^x (|\phi|^2)' = \int_y^x (\phi'\bar{\phi} + \phi\bar{\phi}') = 2\int_y^x \operatorname{Re}(\phi'\bar{\phi}) \leqslant 2\int_y^x |\phi'\phi| \\ &\leqslant 2\int_a^b |\phi'\phi| = 2\int_a^b |\phi'|p^{1/2}|\phi|p^{-1/2} \leqslant 2\|\phi'\|_{p,2}\|\phi\|_{1/p,2},\end{aligned}$$

where we have used the Cauchy–Schwarz inequality in the last estimate. Multiplying the above inequality with $p^{-1}(y)$ and integrating over $[a,b]$ with respect to y gives

$$|\phi(x)|^2\|p^{-1}\|_1 - \|\phi\|_{1/p,2}^2 \leqslant 2\|\phi'\|_{p,2}\|\phi\|_{1/p,2}\|p^{-1}\|_1$$

for all $x \in [a,b]$. Hence, it follows that

$$\|\phi\|_\infty^2 \leqslant 2\|\phi'\|_{p,2}\|\phi\|_{1/p,2} + \|p^{-1}\|_1^{-1}\|\phi\|_{1/p,2}^2. \tag{5.3}$$

Therefore, we obtain from (5.2) that

$$\|\phi'\|_{p,2}^2 \leqslant \alpha\|\phi\|_\infty^2 \leqslant 2\alpha\|\phi'\|_{p,2}\|\phi\|_{1/p,2} + \alpha\|p^{-1}\|_1^{-1}\|\phi\|_{1/p,2}^2.$$

This yields that

$$(\|\phi'\|_{p,2} - \alpha\|\phi\|_{1/p,2})^2 \leqslant \alpha\|p^{-1}\|_1^{-1}\|\phi\|_{1/p,2}^2 + \alpha^2\|\phi\|_{1/p,2}^2 = \alpha(\|p^{-1}\|_1^{-1} + \alpha)\|\phi\|_{1/p,2}^2,$$

and hence

$$\|\phi'\|_{p,2} \leqslant \sqrt{\alpha(1/\|p^{-1}\|_1 + \alpha)}\|\phi\|_{1/p,2} + \alpha\|\phi\|_{1/p,2} = \beta\|\phi\|_{1/p,2}, \tag{5.4}$$

so the first estimate in the lemma is proved. The second estimate follows from the first and (5.3). Indeed, with the help of (5.4) we obtain from (5.3) that

$$\|\phi\|_\infty^2 \leqslant 2\beta\|\phi\|_{1/p,2}^2 + \|p^{-1}\|_1^{-1}\|\phi\|_{1/p,2}^2$$

holds, which implies that $\|\phi\|_\infty \leqslant \sqrt{2\beta + 1/\|p^{-1}\|_1}\|\phi\|_{1/p,2} = \gamma\|\phi\|_{1/p,2}$. □

Proof of lemma 3.5. Let $W(x) := \int_a^x w$, $x \in [a,b]$, and observe that integration by parts yields that

$$\int_a^b W(|\phi|^2)' = W(b)|\phi(b)|^2 - \int_a^b w|\phi|^2 = W(b)|\phi(b)|^2,$$

where we have used (5.1) in the last step. This implies that

$$\begin{aligned} W(b)|\phi(x)|^2 &= -W(b)(|\phi(b)|^2 - |\phi(x)|^2) + \int_a^b W(|\phi|^2)' \\ &= -W(b)\int_x^b (|\phi|^2)' + \int_a^b W(|\phi|^2)', \end{aligned} \tag{5.5}$$

and hence, with $\|W\|_\infty \leqslant \|w\|_1$ and $W(b) = \int_a^b w$, we conclude that

$$\begin{aligned} |\phi(x)|^2 &= -\int_x^b (|\phi|^2)' + \frac{1}{W(b)}\int_a^b W(|\phi|^2)' \\ &\leqslant 2\int_a^b |\phi'\phi| + 2\frac{\|W\|_\infty}{|W(b)|}\int_a^b |\phi'\phi| \\ &\leqslant \left(2 + 2\frac{\|w\|_1}{|\int_a^b w|}\right)\int_a^b |\phi'|p^{1/2}|\phi|p^{-1/2} \\ &\leqslant \delta\|\phi'\|_{p,2}\|\phi\|_{1/p,2} \end{aligned} \tag{5.6}$$

holds for all $x \in [a,b]$. This leads to the estimate

$$\|\phi\|_\infty^2 \leqslant \delta\|\phi'\|_{p,2}\|\phi\|_{1/p,2}. \tag{5.7}$$

As in the proof of lemma 3.1, we have that $\|\phi'\|_{p,2}^2 \leqslant \alpha\|\phi\|_\infty^2$ (see (5.2)), which together with (5.7) yields that

$$\|\phi'\|_{p,2} \leqslant \alpha\delta\|\phi\|_{1/p,2}.$$

Plugging this into (5.7) gives

$$\|\phi\|_\infty \leqslant \sqrt{\alpha}\delta\|\phi\|_{1/p,2},$$

which completes the proof of lemma 3.5. □

Proof of lemma 4.2. For a solution $\phi \in \mathcal{D}$ we have

$$\lambda \int_a^b w|\phi|^2 = (p\phi')(a)\overline{\phi(a)} - (p\phi')(b)\overline{\phi(b)} + \int_a^b (p|\phi'|^2 + q|\phi|^2); \tag{5.8}$$

see (2.9) with $x = a$. The assumption (4.1) then implies the estimate

$$\|\phi'\|_{p,2}^2 \leqslant (p\phi')(b)\overline{\phi(b)} - (p\phi')(a)\overline{\phi(a)} - \int_a^b q|\phi|^2,$$

and hence the estimate $\|\phi'\|_{p,2}^2 \leqslant \alpha\|\phi\|_\infty^2$ in (5.2) remains valid. Thus, the rest of the proof of lemma 3.1 also holds under the present assumptions on ϕ and yields the estimates for $\|\phi'\|_{p,2}$ and $\|\phi\|_\infty$. □

References

1 R. Beals. Indefinite Sturm–Liouville problems and half-range completeness. *J. Diff. Eqns* **56** (1985), 391–407.

2 J. Behrndt and C. Trunk. On the negative squares of indefinite Sturm–Liouville operators. *J. Diff. Eqns* **238** (2007), 491–519.

3 J. Behrndt, F. Philipp and C. Trunk. Bounds on the non-real spectrum of differential operators with indefinite weights. *Math. Annalen* **357** (2013), 185–213.

4 P. Binding and M. Möller. Negativity indices for definite and indefinite Sturm–Liouville problems. *Math. Nachr.* **283** (2010), 180–192.

5 P. Binding and H. Volkmer. Oscillation theory for Sturm–Liouville problems with indefinite coefficients. *Proc. R. Soc. Edinb.* A **131** (2001), 989–1002.

6 P. Binding and H. Volkmer. Existence and asymptotics of eigenvalues of indefinite systems of Sturm–Liouville and Dirac type. *J. Diff. Eqns* **172** (2001), 116–133.

7 P. Binding, P. Browne and B. Watson. Spectral asymptotics for Sturm–Liouville equations with indefinite weight. *Trans. Am. Math. Soc.* **354** (2002), 4043–4065.

8 P. Binding, H. Langer and M. Möller. Oscillation results for indefinite Sturm–Liouville problems with an indefinite weight function. *J. Computat. Appl. Math.* **171** (2004), 93–101.

9 B. Ćurgus and H. Langer. A Krein space approach to symmetric ordinary differential operators with an indefinite weight function. *J. Diff. Eqns* **79** (1989), 31–61.

10 O. Haupt. Über eine Methode zum Beweise von Oszillationstheoremen. *Math. Annalen* **76** (1915), 67–104.

11 I. M. Karabash, A. S. Kostenko and M. M. Malamud. The similarity problem for J-nonnegative Sturm–Liouville operators. *J. Diff. Eqns* **246** (2009), 964–997.

12 Q. Kong, M. Möller, H. Wu and A. Zettl. Indefinite Sturm–Liouville problems. *Proc. R. Soc. Edinb.* A **133** (2003), 639–652.

13 A. B. Mingarelli. A survey of the regular weighted Sturm–Liouville problem: the non-definite case. In *Proc. int. workshop on applied differential equations, Beijing, China*, pp. 109–137 (Singapore: World Scientific, 1986).

14 F. Philipp. Indefinite Sturm–Liouville operators with periodic coefficients. *Operat. Matrices* **7** (2013), 777–811.

15 J. Qi and S. Chen. A priori bounds and existence of non-real eigenvalues of indefinite Sturm–Liouville problems. *J. Spectral Theory* **4** (2014), 53–63.

16 R. G. D. Richardson. Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order. *Am. J. Math.* **40** (1918), 283–316.

17 B. Xie and J. Qi. Non-real eigenvalues of indefinite Sturm–Liouville problems. *J. Diff. Eqns* **255** (2013), 2291–2301.

18 A. Zettl. *Sturm–Liouville theory* (Providence, RI: American Mathematical Society, 2005).

(*Issued 5 December 2014*)