Evolution of Aharonov-Berry superoscillations in Dirac δ-potential

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Abstract

The main goal of this note is to study the time evolution of superoscillations under the 1D-Schrödinger equation with attractive or repulsive Dirac δ-potential located at the origin of the real line. Such potentials are of particular interest since they simulate short range interactions and the corresponding quantum system is an explicitely solvable model. Moreover, we give the large time asymptotics of this solution, which turns out to be different for the repulsive and the attractive model. The method that we use to study the time evolution of superoscillations is based on the continuity of the time evolution operator acting in a space of exponentially bounded entire functions.

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1 Introduction

Superoscillatory functions and their evolution in time under different Hamiltonians have deserved a lot of attention in the physical and mathematical literature in the recent past. The main physical motivation is the fact that superoscillatory functions may appear as the outcome of weak measurements, which were introduced by Aharonov and collaborators. In a series of papers Aharonov, Berry and coauthors have shown various important aspects of superoscillations, and without claiming completeness we mention [1, 10, 12, 16, 17, 18]. More recently superoscillations were also investigated from a mathematical point of view, see, e.g., [2, 3, 4, 5, 6, 8, 23]. The list of contributions related to the mathematical aspects of superoscillations is much longer, thus we refer the interested reader to the survey papers [9, 11, 14] and the references therein. A particularly important issue is to understand how superoscillations

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evolve in time under different Hamiltonians. The Hamiltonian of the Dirac δ-potential will be investigated in this note.

The standard example of superoscillatory functions is the following: For every real number $a \in \mathbb{R}$ with $|a| > 1$ consider the sequence of entire functions

$$F_n(z,a) = \sum_{j=0}^{n} C_j(n,a) e^{i(1-2j/n)z}, \qquad z \in \mathbb{C}, \tag{1.1}$$

with coefficients

$$C_j(n,a) = \binom{n}{j} \left(\frac{1+a}{2}\right)^{n-j} \left(\frac{1-a}{2}\right)^{j}, \tag{1.2}$$

where $\binom{n}{j}$ denotes the binomial coefficient. The special property of this sequence is, that although every function F_n is a superposition of waves with frequencies in $[-1,1]$, the whole sequence converges to

$$\lim_{n\to\infty} F_n(z,a) = e^{iaz}, \qquad z \in \mathbb{C}, \tag{1.3}$$

a plane wave with frequency $|a| > 1$. This intuitively explains why such a sequence is called superoscillatory. One can prove that the limit (1.3) converges uniformly on all compact subsets of $\mathbb{C}$ and even in a certain stronger sense, see [24, Theorem 2.1]. However, the convergence (1.3) is not uniform on all of $\mathbb{C}$, see [3, Proposition 4.2].

The following definitions are now inspired by this example and put the notion of superoscillations into a mathematical framework. We start with the definition of a generalized Fourier sequences as a generalization of (1.1).

Definition 1.1. *A* generalized Fourier sequence *is a sequence of the form*

$$F_n(z) = \sum_{j=0}^{n} C_j(n) e^{ik_j(n)z}, \qquad z \in \mathbb{C}, \tag{1.4}$$

where $C_j(n) \in \mathbb{C}$ and $k_j(n) \in \mathbb{R}$.

The key feature of the above example (1.1) is the convergence (1.3) to a plain wave with a higher frequency. This superoscillatory property is formalized in the next definition, where it is convenient to use a suitable space of exponentially bounded entire functions. More precisely, for a fixed $\gamma \in (0,\infty]$ we shall denote the collection of entire functions f that satisfy the estimate $|f(z)| \leq Ae^{B|z|}$, $z \in \mathbb{C}$, for some $A \geq 0$ and $B \in [0,\gamma)$, by $A_{1,\gamma}(\mathbb{C})$. For more details on this space and the corresponding notion of convergence, we refer the reader to Section 2.

Definition 1.2. *Let $\gamma \in (0,\infty]$. A generalized Fourier sequence $(F_n) \subset A_{1,\gamma}(\mathbb{C})$ of the form* (1.4) *is said to be* superoscillating *in $A_{1,\gamma}(\mathbb{C})$ if:*

(i) *There exists $k' \in [0,\gamma)$ such that $|k_j(n)| \leq k'$ for all $n \in \mathbb{N}_0$ and $j = 0,\dots,n$.*

(ii) *There exists $a \in \mathbb{R}$ with $k' < |a| < \gamma$ such that F_n converges to $z \mapsto e^{iaz}$ in $A_{1,\gamma}(\mathbb{C})$, that is, for some $B \in [0,\gamma)$ one has*

$$\lim_{n\to\infty} \left\| (F_n - e^{ia\,\cdot})e^{-B|\cdot|} \right\|_\infty = 0. \tag{1.5}$$

The main objective in this note is to study the time evolution of the solution of the Schrödinger equation with Dirac δ-potential,

$$i\frac{\partial}{\partial t}\Psi(t,x) = \left(-\frac{\partial^2}{\partial x^2} + 2c\delta_0(x)\right)\Psi(t,x), \qquad t>0,\ x\in\mathbb{R}, \tag{1.6}$$

and superoscillating inital data. Here $c \in \mathbb{R}\setminus\{0\}$ is, up to the factor 2, the potential strength and can be either attractive ($c<0$) or repulsive ($c>0$), but is considered to be time independent. In the context of solvable models in quantum mechanics the spectral theory of Schrödinger operators with δ-potentials has deserved a lot of attention in the last decades; for a first glance we refer the reader to the monograph [13] and the contributions [15, 20, 25]. The formal equation (1.6) is made mathematically rigorous by treating the distribution δ_0 via boundary conditions at $x=0$:

$$i\frac{\partial}{\partial t}\Psi(t,x) = -\frac{\partial^2}{\partial x^2}\Psi(t,x), \qquad t>0,\ x\in\mathbb{R}\setminus\{0\}, \tag{1.7}$$

$$\Psi(t,0^+) = \Psi(t,0^-), \qquad t>0, \tag{1.8}$$

$$\frac{\partial}{\partial x}\Psi(t,0^+) - \frac{\partial}{\partial x}\Psi(t,0^-) = 2c\Psi(t,0), \qquad t>0. \tag{1.9}$$

The continuity condition (1.8) is understood in the sense that $\Psi(t,0^\pm) := \lim_{\varepsilon\searrow 0}\Psi(t,\pm\varepsilon)$ both exist, are finite and coincide. Hence $\Psi(t,\cdot)$ can be continuously extended to the whole real line by $\Psi(t,0) := \Psi(t,0^\pm)$, which appers on the right-hand side of the transmission condition (1.9).

One can deduce (on a formal level) from [19, 22, 26, 27] that the solution of the corresponding Cauchy problem admits the representation

$$\Psi(t,x) = \int_{\mathbb{R}} G(x,y,t)\Psi(0,y)dy, \qquad t>0,\ x\in\mathbb{R}, \tag{1.10}$$

where the Green function has the explicit form

$$G(x,y,t) = -\frac{c}{2}e^{-\frac{(|x|+|y|)^2}{4it}}\Lambda\left(c\sqrt{it} + \frac{|x|+|y|}{2\sqrt{it}}\right) + \frac{1}{2\sqrt{i\pi t}}e^{-\frac{(x-y)^2}{4it}}; \tag{1.11}$$

here the entire function Λ is defined (using the error function erf) as

$$\Lambda(z) = e^{z^2}(1-\operatorname{erf}(z)), \qquad \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}d\xi, \qquad z\in\mathbb{C}. \tag{1.12}$$

Although we will not use results from the theory of Mittag-Leffler functions, we mention for completeness that Λ in (1.12) is connected to the special Mittag-Leffler function $E_{\frac{1}{2},1}$ by $\Lambda(z) = E_{\frac{1}{2},1}(-z)$, see [28].

For our purposes it is important to provide an explicit representation of the solution of the Schrödinger equation (1.6), or, more precisely, of the problem (1.7) – (1.9), with a plane wave e^{iax} as initial condition. This is the content of our first result. For the convenience of the reader we give a self-contained direct proof in Section 3 instead of using the Green function above.

Theorem 1.3. *Let $c \in \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{R}$. Then the solution Ψ of* (1.7) – (1.9) *with initial condition $\Psi(0,x) = e^{iax}$, has the form*

$$\Psi(t,x) = \Psi_{\text{free}}(t,x) + \Psi_\delta^{(0)}(t,x) + \Psi_\delta^{(+)}(t,x) + \Psi_\delta^{(-)}(t,x), \qquad t > 0,\ x \in \mathbb{R}, \tag{1.13}$$

where the four terms of the wave function are

$$\begin{aligned} \Psi_{\text{free}}(t,x) &= e^{iax - ia^2 t}, \\ \Psi_\delta^{(0)}(t,x) &= \frac{c^2}{c^2 + a^2} e^{-\frac{x^2}{4it}} \Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right), \\ \Psi_\delta^{(\pm)}(t,x) &= -\frac{c}{2(c \mp ia)} e^{-\frac{x^2}{4it}} \Lambda\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right). \end{aligned} \tag{1.14}$$

The long-time behaviour of this plane wave disturbed by a δ-potential (1.13) is investigated in the second result of this note. Again we refer the reader to Section 3 for a detailed proof.

Theorem 1.4. *Let $a, c \in \mathbb{R} \setminus \{0\}$. For every fixed $x \in \mathbb{R}$ and $c > 0$ the solution Ψ in* (1.13) *has the asymptotic behaviour*

$$\Psi(t,x) = e^{-ia^2 t}\left(e^{iax} - \frac{c}{c - i|a|} e^{i|ax|}\right) + \mathcal{O}\left(\frac{1}{t}\right), \qquad t \to \infty, \tag{1.15}$$

and for every fixed $x \in \mathbb{R}$ and $c < 0$ the solution Ψ in (1.13) *has the asymptotic behaviour*

$$\Psi(t,x) = e^{-ia^2 t}\left(e^{iax} - \frac{c}{c - i|a|} e^{i|ax|}\right) + \frac{2c^2}{c^2 + a^2} e^{c|x| + ic^2 t} + \mathcal{O}\left(\frac{1}{t}\right), \qquad t \to \infty. \tag{1.16}$$

In the repulsive case $c > 0$ we have a δ-potential barrier and scattering solutions with positive energy. Equation (1.15) shows, that for large times the wave keeps oscillating as $e^{-ia^2 t}$, as the free wave Ψ_{free} does, but with a different complex prefactor, which means a different amplitude as well as a phase shift. In the attractive case $c < 0$ we have a δ-potential well with a bound state whose negative energy is proportional to $-c^2$. The additional term

$$\frac{2c^2}{c^2 + a^2} e^{c|x| + ic^2 t}, \tag{1.17}$$

that appears in the asymptotic solution (1.16), is the damped wave that interacts with the δ-potential well. In fact, the exponential damping $e^{c|x|}$ in space, as well as the oscillations $e^{ic^2 t}$ in time, depend on c. The prefactor $\frac{2c^2}{c^2+a^2}$ shows that (1.17) becomes negligible (also in a neighborhood of the origin) when $|a|$ is much larger than $|c|$.

We also remark that in Theorem 1.4 only the solution Ψ of (1.6) with initial condition e^{iax} is considered. Of course, by the linearity of the Schrödinger equation, one can also allow any superposition of plane waves as initial condition and gets the respective superposition of waves from (1.15) or (1.16) as asymptotics of the corresponding solution.

The next theorem can be viewed as the main result of this note. Now we discuss the time evolution of a superoscillatory sequence (F_n) of the form (1.4) as initial data in the Cauchy problem (1.7) – (1.9).

Theorem 1.5. *Let $c \in \mathbb{R} \setminus \{0\}$ and let (F_n) be a superoscillating sequence in $A_{1,|c|}(\mathbb{C})$, that is, F_n is of the form (1.4) and converges to $z \mapsto e^{iaz}$ in $A_{1,|c|}(\mathbb{C})$, see Definition 1.2. Then the solutions Ψ_n of (1.7) – (1.9) with initial condition $\Psi_n(0, x) = F_n(x)$ converge uniformly on every compact subset of $(0, \infty) \times \mathbb{R}$ to the solution Ψ in (1.13) with initial condition $\Psi(0, x) = e^{iax}$.*

Note that by Definition 1.2, the plain wave e^{iax} in Theorem 1.5 is only allowed to have a frequency $|a| < |c|$. This condition seems to be necessary for mathematical reasons only (in order to apply the strategy of convolution operators).

The proof of Theorem 1.5 is given at the end of Section 3. The key idea of the proof is to use the explicit representation of the solutions Ψ_n and Ψ and to study the time evolution operator $F_n \mapsto \Psi_n$ in the space $A_{1,|c|}(\mathbb{C})$; cf. Lemma 3.2 and Lemma 3.3. A similar technique was also used in [4, 5, 6, 9, 11, 14, 21].

Theorems 1.4 and 1.5 can be combined to the concept of the existence of a shift (for the limit function) and longevity of superoscillations in time. Namely, if we take a sequence (F_n) of the form (1.4) and define $\Phi_\lambda(t, x)$ as the solution of (1.6) with initial data $e^{i\lambda x}$ then, due to the linearity of the Schrödinger equation, the solution $\Psi_n(t, x)$ can be written in the form

$$\Psi_n(t, x) = \sum_{j=0}^{n} C_j(n) \Phi_{k_j(n)}(t, x). \tag{1.18}$$

As pointed out by Aharonov, we can view the right-hand side of (1.18) as an approximation of the function $\lambda \mapsto \Phi_\lambda(t, x)$ for fixed $t > 0$ and $x \in \mathbb{R}$. Indeed, if we compute $\Phi_\lambda(t, x)$ in the points $\lambda = k_j(n) \in [-k', k']$ then the limit $\Psi_n \to \Psi$ in Theorem 1.5 determines $\Phi_a(t, x)$ in points $a \in (-|c|, |c|)$, also outside of $[-k', k']$.

2 Exponentially bounded entire functions

In this preparatory section we introduce a space of exponentially bounded entire functions and a corresponding notion of convergence.

Definition 2.1. *For $\gamma \in (0, \infty]$ we define the space*

$$A_{1,\gamma}(\mathbb{C}) := \left\{ f \in \mathcal{H}(\mathbb{C}) : \exists A \geq 0, B \in [0, \gamma) \text{ such that } |f(z)| \leq Ae^{B|z|} \text{ for all } z \in \mathbb{C} \right\}, \tag{2.1}$$

where $\mathcal{H}(\mathbb{C})$ denotes the space of entire functions. We say that a sequence $(f_n) \subset A_{1,\gamma}(\mathbb{C})$ converges to $f \in A_{1,\gamma}(\mathbb{C})$ in $A_{1,\gamma}(\mathbb{C})$ if there exists some $B \in [0, \gamma)$ such that

$$\lim_{n \to \infty} \left\| (f_n - f) e^{-B|\,\cdot\,|} \right\|_\infty = 0. \tag{2.2}$$

This type of convergence will be denoted by $f_n \xrightarrow{A_{1,\gamma}} f$.

Observe that convergence in $A_{1,\gamma}(\mathbb{C})$ implies the uniform convergence on all compact subsets of $\mathbb{C}$. Note also that the space $A_{1,\infty}(\mathbb{C})$ coincides with the space

$$A_1(\mathbb{C}) := \left\{ f \in \mathcal{H}(\mathbb{C}) : \exists A, B \geq 0 \text{ such that } |f(z)| \leq Ae^{B|z|} \text{ for all } z \in \mathbb{C} \right\}, \tag{2.3}$$

which appears often in the treatement of superoscillating functions. Moreover, the $A_{1,\infty}(\mathbb{C})$-convergence from Definition 2.1 coincides with the usual notion of convergence in the space

$A_1(\mathbb{C})$ and also Lemma 2.2 in the special case $\gamma = \infty$ reduces to [14, Lemma 2.2]. For more details on the space $A_1(\mathbb{C})$ we refer the reader to [6, Chapter 4] or [14]. However, for the present purposes the space $A_1(\mathbb{C})$ is not suitable and has to be replaced by $A_{1,\gamma}(\mathbb{C})$ for some $\gamma \in (0, \infty)$; cf. Remark 3.4.

The next lemma provides a characterization of the space $A_{1,\gamma}(\mathbb{C})$ in terms of bounds for the coefficients in the power series expansion of an entire function. In the following, for $f \in \mathcal{H}(\mathbb{C})$ we shall use the notation

$$f(z) = \sum_{k=0}^{\infty} f^{(k)} z^k, \qquad z \in \mathbb{C}, \tag{2.4}$$

where the coefficients $f^{(k)}$ can be expressed by Cauchy's integral formula in the form

$$f^{(k)} = \frac{1}{k!} \frac{d^k}{dz^k} f(z)\Big|_{z=0} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}} dz, \qquad r > 0. \tag{2.5}$$

Lemma 2.2. *Let $\gamma \in (0, \infty]$ and consider the space $A_{1,\gamma}(\mathbb{C})$ from Definition 2.1. Then*

$$A_{1,\gamma}(\mathbb{C}) = \left\{ f \in \mathcal{H}(\mathbb{C}) : \exists A \geq 0, B \in [0, \gamma) \text{ such that } |f^{(k)}| \leq A \frac{B^k}{k!} \text{ for all } k \in \mathbb{N}_0 \right\} \tag{2.6}$$

and for any sequence $(f_n) \subset A_{1,\gamma}(\mathbb{C})$ and $f \in A_{1,\gamma}(\mathbb{C})$ we have $f_n \overset{A_{1,\gamma}}{\longrightarrow} f$ if and only if there exists a sequence $(A_n) \geq 0$ and $B \in [0, \gamma)$ such that

(i) $\left|f_n^{(k)} - f^{(k)}\right| \leq A_n \frac{B^k}{k!}$ *for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$;*

(ii) $\lim_{n\to\infty} A_n = 0$.

Proof. Let us first prove the identity (2.6). For the inclusion " $\supseteq$ " assume that the coefficients $f^{(k)}$ satisfy

$$|f^{(k)}| \leq A \frac{B^k}{k!}, \qquad k \in \mathbb{N}_0, \tag{2.7}$$

for some $A \geq 0$ and $B \in [0, \gamma)$. Then we can estimate f pointwise by

$$|f(z)| \leq \sum_{k=0}^{\infty} |f^{(k)}||z|^k \leq A \sum_{k=0}^{\infty} \frac{B^k}{k!} |z|^k = A e^{B|z|}, \qquad z \in \mathbb{C}, \tag{2.8}$$

which implies $f \in A_{1,\gamma}(\mathbb{C})$. For the inverse inclusion " $\subseteq$ " in (2.6) let $f \in A_{1,\gamma}(\mathbb{C})$, that is, $f \in \mathcal{H}(\mathbb{C})$ and satisfies

$$|f(z)| \leq A e^{B|z|}, \qquad z \in \mathbb{C}, \tag{2.9}$$

for some $A \geq 0$ and $B \in [0, \gamma)$. It is no restriction to assume that $B > 0$, which will be done in the following. Using (2.5) and (2.9) we obtain the estimate

$$|f^{(k)}| = \frac{1}{2\pi} \left| \int_{|z|=r} \frac{f(z)}{z^{k+1}} dz \right| \leq \frac{1}{2\pi} \sup_{|z|=r} \left| \frac{f(z)}{z^{k+1}} \right| 2\pi r \leq \frac{A e^{Br}}{r^k}, \tag{2.10}$$

for any $r > 0$. It is easy to see that the right-hand side of (2.10), regarded as a function of r, has its minimum at $r = \frac{k}{B}$, which leads to the estimate

$$|f^{(k)}| \leq A \left(\frac{eB}{k}\right)^k, \qquad k \in \mathbb{N}_0. \tag{2.11}$$

Now choose some $\widetilde{B} \in (B, \gamma)$. Then by the asymptotic behaviour $k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$ of the factorial we get

$$\left(\frac{eB}{k}\right)^k \frac{k!}{\widetilde{B}^k} \sim \sqrt{2\pi k}\left(\frac{B}{\widetilde{B}}\right)^k,$$

and since $\widetilde{B} > B$ we conclude that the right-hand side tends to zero as $k \to \infty$. Hence there exists a constant $\widetilde{C} > 0$, only depending on B and $\widetilde{B}$, such that

$$\left(\frac{eB}{k}\right)^k \frac{k!}{\widetilde{B}^k} \leq \widetilde{C}, \qquad k \in \mathbb{N}_0.$$

Using this in (2.11), finally gives the estimate

$$|f^{(k)}| \leq A\widetilde{C}\frac{\widetilde{B}^k}{k!}, \qquad k \in \mathbb{N}_0, \tag{2.12}$$

and we have shown that f is contained in the right-hand side of (2.6).

In order to prove the equivalence of the $A_{1,\gamma}(\mathbb{C})$-convergence with the conditions (i) and (ii), assume first that $f_n \xrightarrow{A_{1,\gamma}} f$ for some sequence $(f_n) \subset A_{1,\gamma}(\mathbb{C})$ and $f \in A_{1,\gamma}(\mathbb{C})$. Then by Definition 2.1 there exists $B \in [0, \gamma)$ such that

$$\lim_{n\to\infty} \left\| (f_n - f)e^{-B|\,\cdot\,|} \right\|_\infty = 0. \tag{2.13}$$

If we choose $A_n := \|(f_n - f)e^{-B|\,\cdot\,|}\|_\infty$ then $|f_n(z) - f(z)| \leq A_n e^{B|z|}$, $z \in \mathbb{C}$, and in the same way as in the above argument (showing that (2.9) implies (2.12)) one concludes

$$\left|f_n^{(k)} - f^{(k)}\right| \leq A_n\widetilde{C}\frac{\widetilde{B}^k}{k!}, \qquad k \in \mathbb{N}_0,\ n \in \mathbb{N}.$$

Hence (i) is satisfied, and from (2.13) we conclude that (ii) holds as well.

Conversely, assume that (i) and (ii) hold true for some sequence $(f_n) \subset A_{1,\gamma}(\mathbb{C})$ and $f \in A_{1,\gamma}(\mathbb{C})$, and some sequence $(A_n) \geq 0$ and $B \in [0, \gamma)$. Then we obtain in the same way as above (showing that (2.7) leads to (2.8)) that

$$|f_n(z) - f(z)| \leq A_n e^{B|z|}, \qquad z \in \mathbb{C}.$$

As $\lim\limits_{n\to\infty} A_n = 0$ we conclude $\lim\limits_{n\to\infty} \|(f_n - f)e^{-B|\,\cdot\,|}\|_\infty = 0$, which shows $f_n \xrightarrow{A_{1,\gamma}} f$. $\square$

3 Proofs of Theorem 1.3, Theorem 1.4 and Theorem 1.5

In this section we prove the main results of this note. First, Lemma 3.1 provides some basic properties of the function Λ from (1.12), which will be used throughout this section.

Lemma 3.1. *For the function Λ in (1.12) the following statements hold:*

(i) $\Lambda(-z) = 2e^{z^2} - \Lambda(z)$;

(ii) $\frac{d}{dz}\Lambda(z) = 2z\Lambda(z) - \frac{2}{\sqrt{\pi}}$;

(iii) *for* $|z| \to \infty$ *one has*

$$\Lambda(z) = \begin{cases} \frac{1}{\sqrt{\pi}z} + \mathcal{O}\left(\frac{1}{|z|^2}\right), & \text{if } \operatorname{Re}(z) \geq 0, \\ 2e^{z^2} + \frac{1}{\sqrt{\pi}z} + \mathcal{O}\left(\frac{1}{|z|^2}\right), & \text{if } \operatorname{Re}(z) \leq 0; \end{cases} \tag{3.1}$$

(iv) Λ *admits the power series representation*

$$\Lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\frac{n}{2}+1)} z^n. \tag{3.2}$$

Proof. (i) Since $\operatorname{erf}(-z) = -\operatorname{erf}(z)$, we conclude from the definition of Λ in (1.12), that

$$\Lambda(-z) = e^{z^2}(1+\operatorname{erf}(z)) = 2e^{z^2} - e^{z^2}(1-\operatorname{erf}(z)) = 2e^{z^2} - \Lambda(z).$$

(ii) Since $\frac{d}{dz}\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}e^{-z^2}$, we conclude

$$\frac{d}{dz}\Lambda(z) = 2ze^{z^2}(1-\operatorname{erf}(z)) - e^{z^2}\frac{2}{\sqrt{\pi}}e^{-z^2} = 2z\Lambda(z) - \frac{2}{\sqrt{\pi}}.$$

(iii) Inserting the integral representation of the error function into Λ in (1.12), gives

$$\Lambda(z) = \frac{2}{\sqrt{\pi}}e^{z^2}\left(\int_0^\infty e^{-\xi^2}d\xi - \int_0^z e^{-\xi^2}d\xi\right). \tag{3.3}$$

Now we use that the complex integral over the entire function $e^{-\xi^2}$ is path independent and that $\lim_{x\to\infty}\int_x^{x+z} e^{-\xi^2}d\xi = 0$. Hence the two integrals on the right-hand side of (3.3) can be replaced by a path integral from z to ∞ parallel to the real axis, which implies

$$\Lambda(z) = \frac{2}{\sqrt{\pi}}e^{z^2}\int_0^\infty e^{-(z+s)^2}ds = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2-2zs}ds. \tag{3.4}$$

Using partial integration in (3.4) leads to

$$\Lambda(z) = -\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{z+s}\frac{d}{ds}e^{-s^2-2zs}ds = \frac{1}{\sqrt{\pi}}\left(\frac{1}{z} - \int_0^\infty \frac{e^{-s^2-2zs}}{(z+s)^2}ds\right).$$

In the case $\operatorname{Re}(z) \geq 0$ we can use $e^{-2\operatorname{Re}(z)s} \leq 1$ as well as $|z+s|^2 \geq |z|^2$ to estimate the integral on the right-hand side by

$$\left|\int_0^\infty \frac{e^{-s^2-2zs}}{(z+s)^2}ds\right| \leq \int_0^\infty \frac{e^{-2\operatorname{Re}(z)s-s^2}}{|z+s|^2}ds \leq \int_0^\infty \frac{e^{-s^2}}{|z|^2}ds = \frac{\sqrt{\pi}}{2|z|^2},$$

which shows the asymptotic behaviour for $\mathrm{Re}(z) \geq 0$. The case $\mathrm{Re}(z) \leq 0$ follows from (i).

(iv) In order to prove (3.2), we rewrite (3.4) in the form

$$\begin{aligned}\Lambda(z) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2-2sz} ds \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-2z)^n}{n!} \int_0^\infty e^{-s^2} s^n ds \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-2)^n \Gamma(\frac{n+1}{2})}{n!} z^n,\end{aligned}$$

where dominated convergence theorem was used in the second equality. Then the Legendre duplication formula

$$\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n}{2}+1\right) = \frac{\sqrt{\pi}}{2^n} \Gamma(n+1) = \frac{\sqrt{\pi}}{2^n} n!$$

implies the power series representation (3.2). □

Now the explicit solution Ψ of (1.6) with initial condition $\Psi(0,x) = e^{iax}$, stated in Theorem 1.3, will be proven. For the convenience of the reader we prefer not to use the Green function (1.11) and the corresponding integral representation (1.10) of the solution (sometimes it is not exactly stated for which classes of initial conditions the integral exists), but instead we give a self-contained direct proof.

Proof of Theorem 1.3. To compute the derivatives of Ψ_{free}, $\Psi^{(0)}$ and $\Psi^{(\pm)}$ for $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$ we use Lemma 3.1 (ii). A straightforward calculation leads to

$$\begin{aligned}\frac{\partial}{\partial t} \Psi_{\mathrm{free}}(t,x) &= -ia^2 e^{iax - ia^2 t}, \\ \frac{\partial}{\partial x} \Psi_{\mathrm{free}}(t,x) &= iae^{iax - ia^2 t}, \\ \frac{\partial^2}{\partial x^2} \Psi_{\mathrm{free}}(t,x) &= -a^2 e^{iax - ia^2 t},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t} \Psi_\delta^{(0)}(t,x) &= \frac{ic^2}{c^2+a^2} e^{-\frac{x^2}{4it}} \left[c^2 \Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right) + \frac{1}{it\sqrt{\pi}} \left(\frac{|x|}{2\sqrt{it}} - c\sqrt{it}\right)\right], \\ \frac{\partial}{\partial x} \Psi_\delta^{(0)}(t,x) &= \frac{\mathrm{sgn}(x) c^2}{c^2+a^2} e^{-\frac{x^2}{4it}} \left[c \Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right) - \frac{1}{\sqrt{i\pi t}}\right], \\ \frac{\partial^2}{\partial x^2} \Psi_\delta^{(0)}(t,x) &= \frac{c^2}{c^2+a^2} e^{-\frac{x^2}{4it}} \left[c^2 \Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right) + \frac{1}{it\sqrt{\pi}} \left(\frac{|x|}{2\sqrt{it}} - c\sqrt{it}\right)\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t} \Psi_\delta^{(\pm)}(t,x) &= \frac{ic}{2(c \mp ia)} e^{-\frac{x^2}{4it}} \left[a^2 \Lambda\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right) - \frac{1}{it\sqrt{\pi}} \left(\frac{|x|}{2\sqrt{it}} \mp ia\sqrt{it}\right)\right], \\ \frac{\partial}{\partial x} \Psi_\delta^{(\pm)}(t,x) &= \frac{\mathrm{sgn}(x) c}{2(c \mp ia)} e^{-\frac{x^2}{4it}} \left[\mp ia \Lambda\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right) + \frac{1}{\sqrt{i\pi t}}\right], \\ \frac{\partial^2}{\partial x^2} \Psi_\delta^{(\pm)}(t,x) &= \frac{c}{2(c \mp ia)} e^{-\frac{x^2}{4it}} \left[a^2 \Lambda\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right) - \frac{1}{it\sqrt{\pi}} \left(\frac{|x|}{2\sqrt{it}} \mp ia\sqrt{it}\right)\right].\end{aligned}$$

It follows that the individual functions Ψ_{free}, $\Psi_\delta^{(0)}$, $\Psi_\delta^{(\pm)}$ and hence also their sum Ψ fulfill (1.7). Moreover, the continuity condition (1.8) is satisfied by the individual functions and by Ψ as well. Combining the first spatial derivatives from above, we find the following jump conditions at $x=0$ for the individual functions:

$$\begin{aligned}
&\frac{\partial}{\partial x}\Psi_{\text{free}}(t,0^+) - \frac{\partial}{\partial x}\Psi_{\text{free}}(t,0^-) = 0,\\
&\frac{\partial}{\partial x}\Psi_\delta^{(0)}(t,0^+) - \frac{\partial}{\partial x}\Psi_\delta^{(0)}(t,0^-) = 2c\Psi_\delta^{(0)}(t,0) - \frac{2c^2}{(c^2+a^2)\sqrt{i\pi t}},\\
&\frac{\partial}{\partial x}\Psi_\delta^{(\pm)}(t,0^+) - \frac{\partial}{\partial x}\Psi_\delta^{(\pm)}(t,0^-) = \frac{c}{c\mp ia}\left(\mp ia\Lambda\left(\pm ia\sqrt{it}\right) + \frac{1}{\sqrt{i\pi t}}\right).
\end{aligned}$$

Note that the identity $\sum_\pm \Lambda\left(\pm ia\sqrt{it}\right) = 2e^{-ia^2t}$ in Lemma 3.1 (i) implies

$$\begin{aligned}
\sum_\pm \frac{\mp iac}{c\mp ia}\Lambda\left(\pm ia\sqrt{it}\right) &= 2ce^{-ia^2t} - \sum_\pm \frac{c^2}{c\mp ia}\Lambda\left(\pm ia\sqrt{it}\right)\\
&= 2c\left(\Psi_{\text{free}}(t,0) + \sum_\pm \Psi_\delta^{(\pm)}(t,0)\right).
\end{aligned}$$

The above formulas, combined with $\sum_\pm \frac{1}{c\mp ia} = \frac{2c}{c^2+a^2}$, then give jump condition (1.9),

$$\begin{aligned}
\frac{\partial}{\partial x}\Psi(t,0^+) - \frac{\partial}{\partial x}\Psi(t,0^-) &= 0 + 2c\Psi_\delta^{(0)}(t,0) - \frac{2c^2}{(c^2+a^2)\sqrt{i\pi t}}\\
&\quad + 2c\left(\Psi_{\text{free}}(t,0) + \sum_\pm \Psi_\delta^{(\pm)}(t,0)\right) + \sum_\pm \frac{c}{(c\mp ia)\sqrt{i\pi t}}\\
&= 2c\Psi(t,0).
\end{aligned}$$

Finally, we check the initial condition $\Psi(0,x) = e^{iax}$, for every $x\in\mathbb{R}$. For $x=0$ it follows immediately from $\Lambda(0)=1$, that

$$\Psi(0,0) = 1 + \frac{c^2}{c^2+a^2} - \frac{c}{2(c-ia)} - \frac{c}{2(c+ia)} = 1.$$

Fix now any $x\in\mathbb{R}\setminus\{0\}$ and note that we can choose $t>0$ sufficiently small, such that

$$\operatorname{Re}\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right) \geq 0 \qquad \text{and} \qquad \operatorname{Re}\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right) \geq 0.$$

Hence, for $t\to 0$ we obtain the asympotics

$$\begin{aligned}
\Psi_\delta^{(0)}(t,x) &= \frac{c^2}{c^2+a^2}e^{-\frac{x^2}{4it}}\left(\frac{1}{\sqrt{\pi}\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right)} + \mathcal{O}(t)\right) \to 0,\\
\Psi_\delta^{(\pm)}(t,x) &= -\frac{c}{2(c\mp ia)}e^{-\frac{x^2}{4it}}\left(\frac{1}{\sqrt{\pi}\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right)} + \mathcal{O}(t)\right) \to 0,
\end{aligned}$$

from Lemma 3.1 (iii) and consequently the initial value

$$\Psi(0,x) = \Psi_{\text{free}}(0,x) = e^{iax}, \qquad x \in \mathbb{R} \setminus \{0\}.$$

Hence, the initial condition is fulfilled for all $x \in \mathbb{R}$ and we have shown that the function Ψ in (1.13) is a solution of (1.6) with initial condition e^{iax}. This completes the proof of Theorem 1.3. $\square$

Using the explicit representation of the solution Ψ in Theorem 1.3 we will now verify its long time behaviour in Theorem 1.4.

Proof of Theorem 1.4. For the functions $\Psi_\delta^{(\pm)}$ of (1.14) we note, that for large enough $t > 0$ we get

$$\begin{aligned} &\mathrm{Re}\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right) \geq 0, \qquad \text{if } \pm a < 0, \qquad \text{and} \\ &\mathrm{Re}\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right) \leq 0, \qquad \text{if } \pm a > 0. \end{aligned}$$

Now we can use Lemma 3.1 (iii) and the characteristic functions $\mathbf{1}_{\mathbb{R}^\pm}$ of $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{R}^- = (-\infty, 0)$, to get the expansion

$$\begin{aligned} \Psi_\delta^{(\pm)}(t,x) &= -\frac{c\, e^{-\frac{x^2}{4it}}}{2(c \mp ia)} \left[2\, \mathbf{1}_{\mathbb{R}^\pm}(a) e^{\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right)^2} + \frac{1}{\frac{|x|\sqrt{\pi}}{2\sqrt{it}} \pm ia\sqrt{i\pi t}} + \mathcal{O}\left(\frac{1}{\left|\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right|^2} \right) \right] \\ &= -\frac{c\, \mathbf{1}_{\mathbb{R}^\pm}(a)}{c \mp ia} e^{\pm ia|x| - ia^2 t} \mp \frac{c\, e^{-\frac{x^2}{4it}}}{2ia\sqrt{i\pi t}(c \mp ia)\left(1 \mp \frac{|x|}{2at}\right)} + \mathcal{O}\left(\frac{1}{\left|\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right|^2} \right), \end{aligned}$$

where we were allowed to include the exponential $e^{-\frac{x^2}{4it}}$ into the $\mathcal{O}(\cdot)$ term, since its absolute value is 1. Using $\frac{1}{1+\varepsilon} = 1 + \mathcal{O}(\varepsilon)$ and $\frac{1}{1+\varepsilon} = \mathcal{O}(1)$ for $\varepsilon \to 0$, we can rewrite $\Psi_\delta^{(\pm)}$ asymptotically as

$$\begin{aligned} \Psi_\delta^{(\pm)}(t,x) &= -\frac{c\, \mathbf{1}_{\mathbb{R}^\pm}(a)}{c \mp ia} e^{\pm ia|x| - ia^2 t} \mp \frac{c}{2ia\sqrt{i\pi t}(c \mp ia)} e^{-\frac{x^2}{4it}} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right) + \mathcal{O}\left(\frac{1}{t}\right) \\ &= -\frac{c\, \mathbf{1}_{\mathbb{R}^\pm}(a)}{c \mp ia} e^{\pm ia|x| - ia^2 t} \mp \frac{c}{2ia\sqrt{i\pi t}(c \mp ia)} e^{-\frac{x^2}{4it}} + \mathcal{O}\left(\frac{1}{t}\right). \end{aligned}$$

In order to expand $\Psi_\delta^{(0)}$, we note, that for large enough $t > 0$ we have

$$\begin{aligned} &\mathrm{Re}\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right) \geq 0, \qquad \text{if } c > 0, \qquad \text{and} \\ &\mathrm{Re}\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right) \leq 0, \qquad \text{if } c < 0. \end{aligned}$$

We use again Lemma 3.1 (iii) and obtain in a similar way as above, that

$$\begin{aligned}
\Psi_\delta^{(0)}(t,x) &= \frac{c^2}{c^2+a^2} e^{-\frac{x^2}{4it}} \left[2\,\mathbf{1}_{\mathbb{R}^-}(c) e^{\left(\frac{|x|}{2\sqrt{it}}+c\sqrt{it}\right)^2} + \frac{1}{\frac{|x|\sqrt{\pi}}{2\sqrt{it}}+c\sqrt{i\pi t}} + \mathcal{O}\left(\frac{1}{\left|\frac{|x|}{2\sqrt{it}}+c\sqrt{it}\right|^2} \right) \right] \\
&= \frac{2c^2\mathbf{1}_{\mathbb{R}^-}(c)}{c^2+a^2} e^{c|x|+itc^2} + \frac{c}{\sqrt{i\pi t}(c^2+a^2)\left(1+\frac{|x|}{2itc}\right)} e^{-\frac{x^2}{4it}} + \mathcal{O}\left(\frac{1}{\left|\frac{|x|}{2\sqrt{it}}+c\sqrt{it}\right|^2} \right) \\
&= \frac{2c^2\mathbf{1}_{\mathbb{R}^-}(c)}{c^2+a^2} e^{c|x|+itc^2} + \frac{c}{\sqrt{i\pi t}(c^2+a^2)} e^{-\frac{x^2}{4it}} \left(1+\mathcal{O}\left(\frac{1}{t}\right)\right) + \mathcal{O}\left(\frac{1}{t}\right) \\
&= \frac{2c^2\mathbf{1}_{\mathbb{R}^-}(c)}{c^2+a^2} e^{c|x|+itc^2} + \frac{c}{\sqrt{i\pi t}(c^2+a^2)} e^{-\frac{x^2}{4it}} + \mathcal{O}\left(\frac{1}{t}\right).
\end{aligned}$$

Summing up the terms from the above expansions we obtain

$$\begin{aligned}
\Psi(t,x) &= e^{iax-ia^2t} + \frac{2c^2\mathbf{1}_{\mathbb{R}^-}(c)}{c^2+a^2} e^{c|x|+itc^2} - \sum_{\pm} \frac{c\,\mathbf{1}_{\mathbb{R}^\pm}(a)}{c\mp ia} e^{\pm ia|x|-ia^2t} \\
&\quad + \frac{1}{\sqrt{i\pi t}} \left(\frac{c}{c^2+a^2} - \frac{c}{2ia(c-ia)} + \frac{c}{2ia(c+ia)} \right) e^{-\frac{x^2}{4it}} + \mathcal{O}\left(\frac{1}{t}\right) \\
&= e^{iax-ia^2t} + \frac{2c^2\mathbf{1}_{\mathbb{R}^-}(c)}{c^2+a^2} e^{c|x|+itc^2} - \frac{c}{c-i|a|} e^{i|ax|-ia^2t} + \mathcal{O}\left(\frac{1}{t}\right) \\
&= e^{-ia^2t} \left(e^{iax} - \frac{c}{c-i|a|} e^{i|ax|} \right) + \mathbf{1}_{\mathbb{R}^-}(c) \frac{2c^2}{c^2+a^2} e^{c|x|+itc^2} + \mathcal{O}\left(\frac{1}{t}\right).
\end{aligned}$$

□

Next we define for every fixed $t>0$ and $x\in\mathbb{R}$ the differential expression $U(t,x)$, acting on functions in an auxilary variable ξ, by

$$U(t,x) = U_{\mathrm{free}}(t,x) + U_\delta^{(0)}(t,x) + U_\delta^{(+)}(t,x) + U_\delta^{(-)}(t,x), \tag{3.5}$$

with the components

$$\begin{aligned}
U_{\mathrm{free}}(t,x) &= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{(it)^n x^{k-n}}{n!(k-n)!} \frac{d^{n+k}}{d\xi^{n+k}}, \\
U_\delta^{(0)}(t,x) &= e^{-\frac{x^2}{4it}} \Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it} \right) \sum_{n=0}^{\infty} \frac{1}{c^{2n}} \frac{d^{2n}}{d\xi^{2n}}, \\
U_\delta^{(+)}(t,x) &= e^{-\frac{x^2}{4it}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{n+1}(it)^{\frac{n}{2}-k}|x|^k}{2^{k+1}c^m\Gamma(\frac{n}{2}+1)} \binom{n}{k} \frac{d^{n-k+m}}{d\xi^{n-k+m}}, \\
U_\delta^{(-)}(t,x) &= e^{-\frac{x^2}{4it}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{m-k+1}(it)^{\frac{n}{2}-k}|x|^k}{2^{k+1}c^m\Gamma(\frac{n}{2}+1)} \binom{n}{k} \frac{d^{n-k+m}}{d\xi^{n-k+m}}.
\end{aligned}$$

The next lemma shows how the differential expression $U(t,x)$ can be used to gain the solution of (1.6) out of the initial condition.

Lemma 3.2. *Let $c \in \mathbb{R}\setminus\{0\}$ and consider the differential expression $U(t,x)$ in (3.5) for fixed $t>0$ and $x \in \mathbb{R}$. Then*

$$\Psi_n(t,x) = \big(U(t,x)F_n\big)(0) \tag{3.6}$$

holds for every function F_n of the form (1.4) with $|k_j^{(n)}| < |c|$. Here Ψ_n is the solution of (1.6) with initial condition $\Psi_n(0,x) = F_n(x)$.

Proof. Since the differential expression $U(t,x)$ in (3.5) as well as the Schrödinger equation (1.6) are linear, and because the function F_n is a linear combination of exponentials, it is sufficient to prove (3.6) for $F_n(\xi) = e^{ia\xi}$ with $|a| < |c|$. Since for this initial condition we have already computed the explicit solutions $\Psi_n = \Psi$ in Theorem 1.3, we will compare $\xi \mapsto U(t,x)e^{ia\xi}$ from (3.5) at $\xi = 0$ with Ψ in (1.13). This will now be done for each component seperately. The following identity will be used:

$$\frac{d^n}{d\xi^n} e^{ia\xi}\Big|_{\xi=0} = (ia)^n, \qquad n \in \mathbb{N}_0. \tag{3.7}$$

Step 1. We rewrite the function Ψ_{free} in (1.14) as the power series

$$\begin{aligned}\Psi_{\mathrm{free}}(t,x) &= \sum_{k=0}^{\infty} \frac{(-ita^2 + iax)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} (iax)^{k-n}(-ita^2)^n \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{(it)^n x^{k-n}}{n!(k-n)!}(ia)^{n+k}.\end{aligned}$$

This representation together with the identity (3.7) shows

$$\Psi_{\mathrm{free}}(t,x) = U_{\mathrm{free}}(t,x)e^{ia\xi}\Big|_{\xi=0}.$$

Step 2. Since we have assumed $|a| < |c|$ we can use the geometric series to write $\Psi_\delta^{(0)}$ in (1.14) in the form

$$\begin{aligned}\Psi_\delta^{(0)}(t,x) &= \frac{1}{1-\left(\frac{ia}{c}\right)^2} e^{-\frac{x^2}{4it}} \Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right) \\ &= e^{-\frac{x^2}{4it}} \Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right) \sum_{n=0}^{\infty} \frac{1}{c^{2n}}(ia)^{2n}.\end{aligned}$$

Hence we conclude

$$\Psi_\delta^{(0)}(t,x) = U_\delta^{(0)}(t,x)e^{ia\xi}\Big|_{\xi=0}.$$

Step 3. For the last two terms $U_\delta^{(\pm)}(t,x)$ we use (3.2) to write $\Psi_\delta^{(\pm)}$ in (1.14) as

$$\begin{aligned}\Psi_\delta^{(\pm)}(t,x) &= -\frac{1}{2(1\mp\frac{ia}{c})} e^{-\frac{x^2}{4it}} \Lambda\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right) \\ &= -\frac{1}{2}\sum_{m=0}^{\infty}\left(\pm\frac{ia}{c}\right)^m e^{-\frac{x^2}{4it}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\frac{n}{2}+1)}\left(\frac{|x|}{2\sqrt{it}} \pm ia\sqrt{it}\right)^n \\ &= e^{-\frac{x^2}{4it}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{n+1}(it)^{\frac{n}{2}-k}|x|^k}{2^{k+1}c^m\Gamma(\frac{n}{2}+1)}\binom{n}{k}(\pm ia)^{n-k+m},\end{aligned}$$

which shows

$$\Psi_\delta^{(\pm)}(t,x) = U_\delta^{(\pm)}(t,x) e^{ia\xi}\Big|_{\xi=0}.$$

Summing up the identities in the above Steps 1 – 3 we conclude (3.6). $\square$

In the next lemma we study the continuity of $U(t,x)$ in the space $A_{1,|c|}(\mathbb{C})$.

Lemma 3.3. *Let $c \in \mathbb{R} \setminus \{0\}$ and consider the differential expression $U(t,x)$ from (3.5) for fixed $t > 0$ and $x \in \mathbb{R}$. Then for every $B \in [0,|c|)$ there exists some $S_B(t,x) \geq 0$ such that for any function $f \in A_{1,|c|}(\mathbb{C})$, whose power series coefficients $f^{(k)}$ from (2.4) satisfy $|f^{(k)}| \leq A\frac{B^k}{k!}$ for some $A \geq 0$, the estimate*

$$|U(t,x)f(\xi)| \leq AS_B(t,x)e^{B|\xi|}, \qquad \xi \in \mathbb{C}, \tag{3.8}$$

holds. In particular, the differential expression $U(t,x)$ gives rise to an everywhere defined continuous operator in $A_{1,|c|}(\mathbb{C})$. Furthermore, the function S_B can be chosen to be continuous in the variables $(t,x) \in (0,\infty) \times \mathbb{R}$.

Remark 3.4. *In Lemma 3.3 the differential operator $U(t,x)$ is studied in the space $A_{1,|c|}(\mathbb{C})$, introduced in Definition 2.1. We emphasize, that it is not possible to deal with the larger space $A_1(\mathbb{C}) = A_{1,\infty}(\mathbb{C})$, which appears in [7, 8, 9, 11, 14], in a similar context. In fact, the finite constant $c \in \mathbb{R}\setminus\{0\}$ models the strength of the δ-interaction in the Schrödinger equation* (1.6) *and the estimate* (3.8)*, which is the key ingredient in the proof of Theorem 1.5, does not hold for an arbitrary function $f \in A_1(\mathbb{C})$. Furthermore, even the expression $(U(t,x)f)(\xi)$ itself is in general not defined for $f \in A_1(\mathbb{C})$, since the appearing sums do not converge.*

Proof of Lemma 3.3. Recall from Lemma 2.2 that for $f \in A_{1,|c|}(\mathbb{C})$ there exists $A \geq 0$ and $B \in [0,|c|)$ such that $|f^{(k)}| \leq A\frac{B^k}{k!}$ holds for the coefficients $f^{(k)}$ in the power series (2.4). In particular, for such f we have the useful estimate

$$\left|\frac{d^n}{d\xi^n}f(\xi)\right| \leq \sum_{k=n}^{\infty} |f^{(k)}|\frac{k!}{(k-n)!}|\xi|^{k-n} \leq A\sum_{k=n}^{\infty}\frac{B^k}{(k-n)!}|\xi|^{k-n} = AB^n e^{B|\xi|}, \qquad \xi \in \mathbb{C}. \tag{3.9}$$

In order to verify (3.8), the components $U_{\mathrm{free}}(t,x)$, $U_\delta^{(0)}(t,x)$ and $U_\delta^{(\pm)}(t,x)$ of $U(t,x)$ in (3.5) will be discussed in separate steps in a similar way as in the proof of Lemma 3.2.

Step 1. Using (3.9) in the representation of $U_{\mathrm{free}}(t,x)$ leads to the estimate

$$\begin{aligned}
\left|U_{\mathrm{free}}(t,x)f(\xi)\right| &\leq Ae^{B|\xi|}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{t^n|x|^{k-n}}{n!(k-n)!}B^{n+k} \\
&= Ae^{B|\xi|}\sum_{k=0}^{\infty}\frac{1}{k!}\left(B^2t + B|x|\right)^k \\
&= AS_{B,\mathrm{free}}(t,x)e^{B|\xi|},
\end{aligned}$$

where $S_{B,\mathrm{free}}(t,x) = e^{B^2t+B|x|}$.

Step 2. For the component $U_\delta^{(0)}(t,x)$ we obtain

$$\left|U_\delta^{(0)}(t,x)f(\xi)\right| \leq Ae^{B|\xi|}\left|\Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right)\right|\sum_{n=0}^{\infty}\frac{B^{2n}}{c^{2n}} = AS_{B,\delta}^{(0)}(t,x)e^{B|\xi|},$$

with $S^{(0)}_{B,\delta}(t,x) = \left|\Lambda\left(\frac{|x|}{2\sqrt{it}} + c\sqrt{it}\right)\right| \frac{c^2}{c^2-B^2}$, which is a positive finite number since we assumed $B < |c|$.

Step 3. For the components $U^{(\pm)}_\delta(t,x)$ we again use (3.9) and estimate

$$\begin{aligned}\left|U^{(\pm)}_\delta(t,x)f(\xi)\right| &\leq Ae^{B|\xi|}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{t^{\frac{n}{2}-k}|x|^k}{2^{k+1}|c|^m\Gamma(\frac{n}{2}+1)}\binom{n}{k}B^{n-k+m} \\ &= \frac{A}{2}e^{B|\xi|}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{B}{|c|}\right)^m\frac{1}{\Gamma(\frac{n}{2}+1)}\left(\frac{|x|}{2\sqrt{t}}+\sqrt{t}B\right)^n \\ &= Ae^{B|\xi|}\frac{|c|}{2(|c|-B)}\Lambda\left(-\frac{|x|}{2\sqrt{t}}-\sqrt{t}B\right),\end{aligned}$$

where $S^{(\pm)}_{B,\delta}(t,x) = \frac{|c|}{2(|c|-B)}\Lambda\left(-\frac{|x|}{2\sqrt{t}}-\sqrt{t}B\right)$.

The above estimates, together with (3.5), show that (3.8) holds for $t > 0$ and $x \in \mathbb{R}$ with

$$S_B(t,x) \coloneqq S_{B,\mathrm{free}}(t,x) + S^{(0)}_{B,\delta}(t,x) + S^{(+)}_{B,\delta}(t,x) + S^{(-)}_{B,\delta}(t,x),$$

and it is also clear that S_B is a continuous nonnegative function on $(0,\infty)\times\mathbb{R}$. Note also that Lemma 2.2 and (3.8) imply that $U(t,x)$ is an everywhere defined operator in $A_{1,|c|}(\mathbb{C})$, which maps continuously into $A_{1,|c|}(\mathbb{C})$. □

We will now use the representation in Lemma 3.2 and the estimate in Lemma 3.3 to finally prove Theorem 1.5.

Proof of Theorem 1.5. Since $F_n \overset{A_{1,|c|}}{\longrightarrow} e^{ia\,\cdot}$, there exist a sequence $(A_n) \geq 0$ and $B \in [0,|c|)$ such that the coefficients of the power series of $(F_n - e^{ia\,\cdot})$ satisfy

$$\left|F_n^{(k)} - \frac{(ia)^k}{k!}\right| \leq A_n\frac{B^k}{k!}, \qquad \text{and} \qquad \lim_{n\to\infty}A_n = 0;$$

cf. Lemma 2.2. Let $K \subseteq (0,\infty)\times\mathbb{R}$ be an arbitrary compact set, denote the solutions of (1.7)–(1.9) with initial condition $F_n(x)$ by Ψ_n, and let Ψ be the solution in (1.13) with initial condition $\Psi(0,x) = e^{iax}$. Using the representation in Lemma 3.2 and the estimate in Lemma 3.3 for $\xi = 0$, we conclude

$$\begin{aligned}\lim_{n\to\infty}\sup_{(t,x)\in K}\left|\Psi_n(t,x)-\Psi_0(t,x)\right| &= \lim_{n\to\infty}\sup_{(t,x)\in K}\left|\left(U(t,x)(F_n - e^{ia\,\cdot})\right)(0)\right| \\ &\leq \lim_{n\to\infty}\sup_{(t,x)\in K}A_nS_B(t,x)e^{B\,0} \\ &= \sup_{(t,x)\in K}S_B(t,x)\lim_{n\to\infty}A_n = 0,\end{aligned}$$

where $\sup_{(t,x)\in K}S_B(t,x) < \infty$ since S_B is continuous on $(0,\infty)\times\mathbb{R}$ by Lemma 3.3. □

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