LIMIT PROPERTIES OF MONOTONE MATRIX FUNCTIONS

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Abstract. The basic objects in this paper are monotonically nondecreasing $n \times n$ matrix functions $D(\cdot)$ defined on some open interval $\imath = (a, b)$ of $\mathbb{R}$ and the limit values $D(a)$ and $D(b)$ at the endpoints a and b which are, in general, selfadjoint relations in $\mathbb{C}^n$. Certain space decompositions induced by the matrix function $D(\cdot)$ are made explicit by means of the limit values $D(a)$ and $D(b)$. They are a consequence of operator inequalities involving these limit values and the notion of strictness (or definiteness) of monotonically nondecreasing matrix functions. The treatment provides a geometric approach to the square-integrability of solutions of definite canonical systems of differential equations.

1. Introduction

An $n \times n$ matrix function $D(\cdot)$ defined on an open interval $\imath = (a, b)$ of $\mathbb{R}$ is called *monotonically nondecreasing* if the values $D(t)$ are selfadjoint matrices for all $t \in \imath$ and $D(t_1) \leq D(t_2)$ when $t_1 \leq t_2$. If the values $D(t)$ are uniformly bounded in the sense that there exist selfadjoint $n \times n$ matrices D_a and D_b for which

$$D_a \leq D(t) \leq D_b, \quad t \in \imath,$$

then the limits $D(a) = \lim_{t \downarrow a} D(t)$ and $D(b) = \lim_{t \uparrow b} D(t)$ exist as selfadjoint matrices. In the general case the limits $D(a)$ and $D(b)$ exist in the graph sense as selfadjoint relations (multivalued operators), which reduce to $n \times n$ matrices only if there are uniform bounds. As for matrices the selfadjoint limit relations $D(a)$ and $D(b)$ satisfy the following two inequalities

$$-D(t) \leq -D(a) \quad \text{and} \quad D(t) \leq D(b), \quad t \in \imath.$$

The importance of these inequalities, for instance, for the study of square-integrable solutions of canonical systems of differential equations is one of the key observations in this paper. To give a precise meaning for these inequalities and to show the role they have in deriving appropriate space decompositions, some necessary facts on selfadjoint relations are needed. To give a full understanding for the main results in the paper a self-contained treatment of selfadjoint relations in finite-dimensional spaces is provided. This includes extensions of some notions, which are familiar for selfadjoint matrices, to the class of selfadjoint relations in a finite-dimensional space, like ordering and inertia.

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The limit values $D(a)$ and $D(b)$ of the matrix function $D(\cdot)$ give rise to the orthogonal decompositions

$$\mathbb{C}^n = \mathcal{A}^+ \oplus \mathcal{A}^- \oplus \mathcal{A}^0 \oplus \mathcal{A}^\infty, \quad \mathbb{C}^n = \mathcal{B}^+ \oplus \mathcal{B}^- \oplus \mathcal{B}^0 \oplus \mathcal{B}^\infty,$$

where the summands in these orthogonal sums stand for the eigenspaces corresponding to positive, negative, zero, and infinite eigenvalues of the selfadjoint relations $D(a)$ and $D(b)$, respectively. One of the aims of this paper is to establish the following direct sum decompositions of $\mathbb{C}^n$ in terms of eigenspaces of $D(a)$ and $D(b)$ simultaneously:

$$\mathbb{C}^n = (\mathcal{A}^+ \oplus \mathcal{A}^0) + (\mathcal{B}^0 \oplus \mathcal{B}^-) = (\mathcal{A}^- \oplus \mathcal{A}^\infty) + (\mathcal{B}^+ \oplus \mathcal{B}^\infty).$$

Such a decomposition result is the essential part for the description of the number of square-integrable solutions of canonical systems of differential equations; cf. Remark 5.4 and [6] and the references therein. It turns out that the above decompositions of $\mathbb{C}^n$ hold if for some $t_0 \in \imath$ the inverse of $D(t_0)$ exists, satisfies the inequalities

$$-D(t_0)^{-1} \leq -D(b)^{-1}, \quad D(t_0)^{-1} \leq D(a)^{-1},$$

and, in addition, the matrix function $D(\cdot)$ is *strict* on $\imath$, that is, for every $\phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b)$, $\phi \neq 0$, one has $\phi^* D(a)_s \phi < \phi^* D(b)_s \phi$, where $D(a)_s$ and $D(b)_s$ stand for the (orthogonal) operator parts of $D(a)$ and $D(b)$, respectively.

Here is a description of the contents of the paper. Section 2 contains an introduction to linear relations in finite-dimensional spaces. The ordering of selfadjoint relations is discussed in Section 3; here also the inertia of selfadjoint relations is introduced and some implications of operator inequalities to the geometric properties of the selfadjoint relations are established. Monotonically nondecreasing matrix functions are treated in Section 4. The notion of strictness for monotone matrix functions is introduced and characterized in various ways. This notion and some of the results given here are motivated by the concept of definiteness appearing in the theory of canonical systems of differential equations. In the special case of so-called matrix Nevanlinna functions this notion of strictness is also connected to the concept of uniform strictness of such functions; in fact, for such functions a stronger form of strictness is shown to hold. Finally, the above mentioned decomposition of $\mathbb{C}^n$ in terms of eigenspaces of the limits $D(a)$ and $D(b)$ is proved and different sufficient conditions are provided. In Section 5 the decomposition results are applied to a class of square-integrable matrix functions. This class contains the square-integrable solutions of definite singular canonical systems of differential equations as appearing in [10, 13, 14, 16, 18].

In a forthcoming paper by the authors (see [5]) some further applications for monotone matrix functions and the inequalities the limit relations satisfy will be given by studying antitonicity of the inverse in the general setting of selfadjoint relations.

2. Selfadjoint relations

This section contains an introduction to selfadjoint linear relations in finite-dimensional spaces. For early work on linear relations in finite-dimensional linear spaces, see [3], [12], [20], and also [1, p. 388].

2.1. **Linear relations.** A *linear relation* H in the finite-dimensional space $\mathbb{C}^n$ is a linear subspace of the product space $\mathbb{C}^n \times \mathbb{C}^n$, so that H is the graph of a multivalued linear operator in $\mathbb{C}^n$. In what follows only linear relations in $\mathbb{C}^n$ are used; hence they are called shortly *relations*. The *domain, range, kernel,* and *multivalued part* of a relation H are defined as follows:

$$\operatorname{dom} H = \{\phi \in \mathbb{C}^n : \{\phi, \psi\} \in H\}, \quad \operatorname{ran} H = \{\psi \in \mathbb{C}^n : \{\phi, \psi\} \in H\},$$
$$\ker H = \{\phi \in \mathbb{C}^n : \{\phi, 0\} \in H\}, \quad \operatorname{mul} H = \{\psi \in \mathbb{C}^n : \{0, \psi\} \in H\}.$$

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of H if $\{\phi, \lambda\phi\} \in H$ for some nontrivial $\phi \in \mathbb{C}^n$, which is then called an *eigenvector*. Similarly, ∞ is said to be an *eigenvalue* of H if $\{0, \psi\} \in H$ or, equivalently, $\psi \in \operatorname{mul} H$, for some nontrivial $\psi \in \mathbb{C}^n$, which is then called an *eigenvector*. The relation H is a singlevalued operator precisely when $\operatorname{mul} H = \{0\}$, i.e., ∞ is not an eigenvalue of H. If, in addition, $\operatorname{dom} H = \mathbb{C}^n$, then the operator H will be called a *matrix*. In the setting of relations inclusions, like $H_1 \subset H_2$, often appear; such an inclusion is expressed by saying that H_2 is an extension of H_1. Of course, for matrices H_1 and H_2 acting on $\mathbb{C}^n$ the inclusion $H_1 \subset H_2$ implies $H_1 = H_2$, since they are singlevalued (i.e. $\operatorname{mul} H_1 = \operatorname{mul} H_2 = \{0\}$) and their domain is $\mathbb{C}^n$. The *operator-like sum* of two linear relations H_1 and H_2 is defined as

$$H_1 + H_2 = \{\, \{\phi, \psi + \varphi\} : \{\phi, \psi\} \in H_1,\ \{\phi, \varphi\} \in H_2 \,\}.$$

Then $H_1 + H_2$ is a relation and

$$\operatorname{dom}(H_1 + H_2) = \operatorname{dom} H_1 \cap \operatorname{dom} H_2, \quad \operatorname{mul}(H_1 + H_2) = \operatorname{mul} H_1 + \operatorname{mul} H_2,$$

as follows directly from the definition.

Each relation H has an *inverse* H^{-1}, which is defined by

$$H^{-1} = \{\{\psi, \phi\} : \{\phi, \psi\} \in H\}. \tag{2.1}$$

Hence, in particular, $\operatorname{dom} H^{-1} = \operatorname{ran} H$ and $\ker H^{-1} = \operatorname{mul} H$. Note that for any $\lambda \in \mathbb{C}$ the inverse relation $(H - \lambda)^{-1} = \{\, \{\psi - \lambda\phi, \phi\} : \{\phi, \psi\} \in H \,\}$ has the following properties:

$$\ker (H - \lambda)^{-1} = \operatorname{mul} H \quad \text{and} \quad \operatorname{mul}(H - \lambda)^{-1} = \ker (H - \lambda). \tag{2.2}$$

If λ is not an eigenvalue of H, then $(H - \lambda)^{-1}$ is an operator. Moreover, if in addition $\operatorname{ran}(H - \lambda) = \mathbb{C}^n$, then λ is said to belong to the resolvent set of H and $(H - \lambda)^{-1}$ is called the *resolvent operator* of H (at λ). If λ and μ belong to the resolvent set of H, then the resolvent identity holds:

$$(H - \lambda)^{-1} - (H - \mu)^{-1} = (\lambda - \mu)(H - \lambda)^{-1}(H - \mu)^{-1}. \tag{2.3}$$

For $\phi, \psi \in \mathbb{C}^n$ the scalar product is denoted by $\psi^*\phi = \sum_{i=1}^n \phi_i \overline{\psi_i}$. The adjoint H^* of a relation H in $\mathbb{C}^n$ is a relation defined by

$$H^* = \{\, \{\phi, \psi\} \in \mathbb{C}^n \times \mathbb{C}^n : \xi^*\psi = \eta^*\phi,\ \{\xi, \eta\} \in H \,\}, \tag{2.4}$$

which coincides with the usual adjoint (conjugate transpose) when H is an $n \times n$ matrix. It follows directly from the definition that

$$\begin{aligned} \mathbb{C}^n &= \operatorname{dom} H \oplus \operatorname{mul} H^* = \operatorname{ran} H \oplus \ker H^*, \\ \mathbb{C}^n &= \operatorname{dom} H^* \oplus \operatorname{mul} H = \operatorname{ran} H^* \oplus \ker H. \end{aligned} \tag{2.5}$$

Observe also that (2.1) combined with (2.4) yields

$$(H^{-1})^* = (H^*)^{-1}. \tag{2.6}$$

2.2. **Selfadjoint relations.** A relation H is said to be *symmetric* if $\psi^*\phi \in \mathbb{R}$ for all $\{\phi, \psi\} \in H$. By the polarization formula H is symmetric precisely when $H \subset H^*$. A relation H is called *selfadjoint* if $H = H^*$; in the literature a selfadjoint matrix is also called Hermitian, but that terminology is not used in the present paper. Obviously, selfadjoint relations are symmetric, but the converse need not hold if H is multivalued.

Lemma 2.1. *Let H be a linear relation in $\mathbb{C}^n$. Then the following statements are equivalent:*
 (i) *H is selfadjoint;*
 (ii) *H^{-1} is selfadjoint;*
 (iii) *H is symmetric and $\mathbb{C}^n = \operatorname{dom} H \oplus \operatorname{mul} H$;*
 (iv) *H is symmetric and $\mathbb{C}^n = \operatorname{ran} H \oplus \ker H$.*

Proof. (i) $\Leftrightarrow$ (ii) This follows directly from (2.6).

(i) $\Rightarrow$ (iii) If (i) holds, then $H \subset H^* = H$ and the space decomposition follows from the first identity in (2.5).

(iii) $\Rightarrow$ (i) It suffices to prove the inclusion $H^* \subset H$. The second condition in (iii) together with the first identities in (2.5) implies that $\operatorname{dom} H = \operatorname{dom} H^*$ and $\operatorname{mul} H = \operatorname{mul} H^*$. Hence, if $\{\phi, \psi\} \in H^*$, then $\{\phi, \varphi\} \in H$ for some $\varphi \in \mathbb{C}^n$, which implies that $\{\phi, \psi\} = \{\phi, \varphi\} + \{0, \psi - \varphi\} \in H$, since $\psi - \varphi \in \operatorname{mul} H^* = \operatorname{mul} H$.

(ii) $\Leftrightarrow$ (iv) Since H is symmetric if and only if H^{-1} is symmetric, see (2.6), this equivalence follows form the equivalence of (i) and (iii) by going over to the inverses. □

Corollary 2.2. *Let H_1 and H_2 be selfadjoint relations in $\mathbb{C}^n$. Then $H_1 + H_2$ is selfadjoint.*

Proof. Since H_1 and H_2 are symmetric the same holds for $H_1 + H_2$. Furthermore, since $(\operatorname{mul} H_1)^\perp = \operatorname{dom} H_1$ and $(\operatorname{mul} H_2)^\perp = \operatorname{dom} H_2$ the identity

$$\begin{aligned}\big(\operatorname{mul}(H_1 + H_2)\big)^\perp &= \big(\operatorname{mul} H_1 + \operatorname{mul} H_2\big)^\perp = (\operatorname{mul} H_1)^\perp \cap (\operatorname{mul} H_2)^\perp \\ &= \operatorname{dom} H_1 \cap \operatorname{dom} H_2 = \operatorname{dom}(H_1 + H_2)\end{aligned}$$

together with Lemma 2.1 (iii) implies the statement. □

Let H be a selfadjoint relation in $\mathbb{C}^n$ and let P be the orthogonal projection onto $\operatorname{dom} H$. Since H is selfadjoint, Lemma 2.1 implies that H induces an orthogonal decomposition of $\mathbb{C}^n$:

$$\mathbb{C}^n = \operatorname{dom} H \oplus \operatorname{mul} H. \tag{2.7}$$

Hence $\operatorname{mul} H = \{ (I - P)\psi : \psi \in \operatorname{ran} H \}$. Therefore H allows the following orthogonal decomposition:

$$H = H_s \mathbin{\widehat{\oplus}} (\{0\} \times \operatorname{mul} H), \tag{2.8}$$

where $H_s = H \cap (\operatorname{dom} H \times \operatorname{dom} H)$, the so-called *orthogonal operator part* of H, is a selfadjoint matrix in $\operatorname{dom} H$ and $\{0\} \times \operatorname{mul} H$ is a selfadjoint relation in $\operatorname{mul} H$. The symbol $\widehat{\oplus}$ in (2.8) indicates the orthogonality of the summands. Note that (2.8) implies that the finite eigenvalues of H and of H_s coincide.

Example 2.3. Let H be a selfadjoint matrix in $\mathbb{C}^n$. In terms of relations one has

$$H = \{ \{\phi, H\phi\} : \phi \in \mathbb{C}^n \}, \quad H^{-1} = \{ \{H\phi, \phi\} : \phi \in \mathbb{C}^n \}.$$

The inverse H^{-1} is a selfadjoint relation in $\mathbb{C}^n$ with $\operatorname{mul} H^{-1} = \ker H$. Hence the orthogonal decomposition (2.8) gives

$$H^{-1} = (H^{-1})_s \,\widehat{\oplus}\, (\{0\} \times \ker H),$$

where the orthogonal operator part of H^{-1} is given by

$$(H^{-1})_s = \{\, \{H\phi, \phi\} \in H^{-1} : \phi \in \mathbb{C}^n \ominus \ker H \,\} = \left(H\!\restriction(\ker H)^{\perp}\right)^{-1}.$$

Note that the Moore-Penrose inverse X of H is given by

$$X = (H^{-1})_s \,\widehat{\oplus}\, (\ker H \times \{0\}).$$

Let H be a selfadjoint relation in $\mathbb{C}^n$ and assume that $\lambda \in \mathbb{C}$ is not an eigenvalue of H. Then λ belongs to the resolvent set of H so that $\operatorname{ran}(H-\lambda) = \mathbb{C}^n$, see (2.5). The resolvent operator $(H-\lambda)^{-1}$ can be used to parametrize H. Therefore observe that by (2.8) the operator $(H-\lambda)^{-1}$ has the following orthogonal decomposition

$$(H-\lambda)^{-1} = (H_s - \lambda)^{-1} \oplus 0_{\operatorname{mul} H},$$

so that the restriction of $(H-\lambda)^{-1}$ to $\operatorname{dom} H$ equals $(H_s-\lambda)^{-1}$. It is straightforward to see that

$$H = \{\, \{(H-\lambda)^{-1}\phi, \phi + \lambda(H-\lambda)^{-1}\phi\} : \phi \in \mathbb{C}^n \,\}. \tag{2.9}$$

Note that every element in $\operatorname{dom} H$ is of the form $(H-\lambda)^{-1}\phi$ for a unique $\phi \in \operatorname{dom} H$; cf. (2.2). Hence it follows from (2.8) and (2.9) that

$$H_s(H-\lambda)^{-1}\phi = \phi + \lambda(H-\lambda)^{-1}\phi, \quad \phi \in \operatorname{dom} H. \tag{2.10}$$

3. Ordering and inertia of selfadjoint relations

In this section the notion of ordering and inertia of selfadjoint relations in finite-dimensional spaces is discussed, and the properties of eigenspaces of a pair of selfadjoint relations with an "intermediate" selfadjoint relation are investigated.

3.1. **Ordering of selfadjoint relations.** Let H be a selfadjoint relation in $\mathbb{C}^n$, then the eigenvalues λ_i of the selfadjoint operator part H_s are real and are assumed to be ordered, i.e. $\lambda_i \le \lambda_{i+1}$. The smallest eigenvalue λ_1 is called the *lower bound* of H; it satisfies

$$\phi^* H_s \phi \ge \lambda_1\, \phi^*\phi \quad \text{for all} \quad \phi \in \operatorname{dom} H = \operatorname{dom} H_s.$$

If the lower bound is nonnegative, then H is said to be *nonnegative*. Note that if H has lower bound m, then $H - x$ has lower bound $m - x$ for any $x \in \mathbb{R}$. Therefore it follows that $H - x$ is nonnegative for $x \le m$ and

$$(H-x)^{-1} \ge 0 \quad \text{for all} \quad x \le m. \tag{3.1}$$

Moreover, if $x < m$, then $(H-x)^{-1}$ is a matrix.

Definition 3.1. Let H_1 and H_2 be selfadjoint relations in $\mathbb{C}^n$ with lower bounds m_1 and m_2, respectively. Then H_1 and H_2 are said to satisfy the inequality $H_1 \le H_2$ if

$$(0 \le)\ (H_2 - x)^{-1} \le (H_1 - x)^{-1} \quad \text{for a fixed } x < \min\{m_1, m_2\}. \tag{3.2}$$

The following proposition gives a characterization of the ordering of selfadjoint relations which is similar to the usual ordering of selfadjoint matrices. The proposition also shows that (3.2) holds automatically for all $x < \min\{m_1, m_2\}$ if it holds for some $x < \min\{m_1, m_2\}$. For the convenience of the reader a simple direct proof based on the Cauchy-Schwarz inequality is included; see [8, Lemma 3.2].

Proposition 3.2. *Let H_1 and H_2 be selfadjoint relations in $\mathbb{C}^n$. Then H_1 and H_2 satisfy $H_1 \le H_2$ if and only if*

$$\operatorname{dom} H_2 \subset \operatorname{dom} H_1 \quad \textit{and} \quad \phi^*(H_1)_s\phi \le \phi^*(H_2)_s\phi \quad \textit{for all} \quad \phi \in \operatorname{dom} H_2. \tag{3.3}$$

Proof. Step 1. Let H_1 and H_2 be selfadjoint relations with $\operatorname{dom} H_2 \subset \operatorname{dom} H_1$. This inclusion implies that an element $\phi \in \operatorname{dom} H_2$ has the following representations with $x < \min\{m_1, m_2\}$:

$$\phi = (H_2 - x)^{-1}\gamma,\ \gamma \in \operatorname{dom} H_2, \quad \text{and} \quad \phi = (H_1 - x)^{-1}\delta,\ \delta \in \operatorname{dom} H_1; \tag{3.4}$$

see (2.9) and the text following it. It follows from (3.4) and (2.10) that

$$\phi^*(H_1)_s\phi - \phi^*(H_2)_s\phi = \delta^*(H_1 - x)^{-1}\delta - \gamma^*(H_2 - x)^{-1}\gamma. \tag{3.5}$$

Step 2. Assume $H_1 \le H_2$ as in Definition 3.1, that is, (3.2) holds. This clearly implies that $\ker\,(H_1 - x)^{-1} \subset \ker\,(H_2 - x)^{-1}$ or, equivalently, $\operatorname{mul} H_1 \subset \operatorname{mul} H_2$; cf. (2.2). By (2.7) this is equivalent to $\operatorname{dom} H_2 \subset \operatorname{dom} H_1$, which is the inclusion in (3.3). To see the inequality in (3.3) let $\phi \in \operatorname{dom} H_2$ and apply Step 1. Let ϕ have the representations in (3.4), then (3.2) implies that

$$\begin{aligned} \delta^*(H_1 - x)^{-1}\delta &= \delta^*(H_2 - x)^{-1}\gamma \\ &\le \sqrt{\delta^*(H_2 - x)^{-1}\delta}\,\sqrt{\gamma^*(H_2 - x)^{-1}\gamma} \\ &\le \sqrt{\delta^*(H_1 - x)^{-1}\delta}\,\sqrt{\gamma^*(H_2 - x)^{-1}\gamma}; \end{aligned}$$

cf. (3.1). These inequalities imply that

$$\delta^*(H_1 - x)^{-1}\delta \le \gamma^*(H_2 - x)^{-1}\gamma.$$

By means of (3.5) this leads to the inequality in (3.3).

Step 3. Assume that (3.3) holds and assume that $x < \min\{m_1, m_2\}$. For $\zeta \in \mathbb{C}^n$ one has the orthogonal decomposition

$$\zeta = \gamma + \eta, \quad \gamma \in \operatorname{dom} H_2, \quad \eta \in \operatorname{mul} H_2.$$

Hence, it follows from (2.2) that

$$(H_2 - x)^{-1}\zeta = (H_2 - x)^{-1}\gamma \quad \text{and} \quad \zeta^*(H_2 - x)^{-1}\zeta = \gamma^*(H_2 - x)^{-1}\gamma.$$

Define the element $\phi \in \mathbb{C}^n$ by $\phi = (H_2 - x)^{-1}\gamma$, so that $\phi \in \operatorname{dom} H_2$. By assumption $\operatorname{dom} H_2 \subset \operatorname{dom} H_1$, so that $\phi = (H_1 - x)^{-1}\delta$ for some $\delta \in \operatorname{dom} H_1$. Therefore $\phi \in \operatorname{dom} H_2$ has the representations as in (3.4). The assumption in (3.3) combined with (3.5) leads to the inequality

$$\delta^*(H_1 - x)^{-1}\delta \le \gamma^*(H_2 - x)^{-1}\gamma.$$

Hence, it follows that

$$\begin{aligned} \zeta^*(H_2 - x)^{-1}\zeta &= \zeta^*(H_2 - x)^{-1}\gamma = \zeta^*(H_1 - x)^{-1}\delta \\ &\le \sqrt{\delta^*(H_1 - x)^{-1}\delta}\,\sqrt{\zeta^*(H_1 - x)^{-1}\zeta} \\ &\le \sqrt{\gamma^*(H_2 - x)^{-1}\gamma}\,\sqrt{\zeta^*(H_1 - x)^{-1}\zeta} \\ &= \sqrt{\zeta^*(H_2 - x)^{-1}\zeta}\,\sqrt{\zeta^*(H_1 - x)^{-1}\zeta}\,; \end{aligned}$$

cf. (3.1). These inequalities imply that

$$\zeta^*(H_2 - x)^{-1}\zeta \leq \zeta^*(H_1 - x)^{-1}\zeta,$$

which via (3.2) shows that $H_1 \leq H_2$. □

Note that by (2.7) the condition $\operatorname{dom} H_2 \subset \operatorname{dom} H_1$ in Proposition 3.2 is equivalent to the condition $\operatorname{mul} H_1 \subset \operatorname{mul} H_2$.

Remark 3.3. Let H_1 and H_2 be selfadjoint relations in $\mathbb{C}^n$. Then the following statements are equivalent:

(i) $H_1 \leq H_2 \Rightarrow -H_2 \leq -H_1$;
(ii) $\operatorname{mul} H_1 = \operatorname{mul} H_2$ or $\operatorname{dom} H_1 = \operatorname{dom} H_2$.

It is clear from Proposition 3.2 that $H_1 \leq H_2$ is equivalent to

$$aH_1 + x \leq aH_2 + x, \quad a > 0, \quad x \in \mathbb{R}.$$

Moreover, $H_1 \leq H_2$ and $H_2 \leq H_3$ imply that $H_1 \leq H_3$ (transitivity). Finally, $H_1 \leq H_2$ implies $H_0 + H_1 \leq H_0 + H_2$ for all $H_0 = H_0^*$; cf. Corollary 2.2.

3.2. **Inertia numbers of selfadjoint relations.** The notion of inertia is well-known for selfadjoint matrices and appears frequently in the matrix literature, see, e.g. [9], [11]. The inertia numbers for a selfadjoint relation are defined in almost the same way, here also the possible eigenvalue ∞ is taken into account.

Definition 3.4. The *inertia* of a selfadjoint relation H in $\mathbb{C}^n$ is an ordered quadruple, consisting of the numbers of positive, negative, zero, and infinite eigenvalues of H; it is denoted by

$$\mathsf{i}(H) = \{\mathsf{i}^+(H), \mathsf{i}^-(H), \mathsf{i}^0(H), \mathsf{i}^\infty(H)\}.$$

If H is a selfadjoint matrix, then $\mathsf{i}^\infty(H) = 0$ and the remaining numbers make up the usual inertia of H; cf. (2.8). Clearly, for the inertia numbers of a selfadjoint relation H one has the following condition

$$\mathsf{i}^+(H) + \mathsf{i}^-(H) + \mathsf{i}^0(H) + \mathsf{i}^\infty(H) = n. \tag{3.6}$$

The following identities are straightforward, but useful:

$$\begin{aligned} \mathsf{i}(-H) &= \{\mathsf{i}^-(H), \mathsf{i}^+(H), \mathsf{i}^0(H), \mathsf{i}^\infty(H)\}, \\ \mathsf{i}(H^{-1}) &= \{\mathsf{i}^+(H), \mathsf{i}^-(H), \mathsf{i}^\infty(H), \mathsf{i}^0(H)\}, \\ \mathsf{i}(-H^{-1}) &= \{\mathsf{i}^-(H), \mathsf{i}^+(H), \mathsf{i}^\infty(H), \mathsf{i}^0(H)\}. \end{aligned} \tag{3.7}$$

A subspace $\mathfrak{L} \subset \operatorname{dom} H$ is said to be *negative* with respect to H if $\phi^* H_s \phi < 0$ for all nontrivial $\phi \in \mathfrak{L}$. The notions *nonpositive, positive, and nonnegative* are defined in a similar way.

Lemma 3.5. *Let H be a selfadjoint relation in $\mathbb{C}^n$ and let $\mathfrak{L}$ be a linear subspace of* $\operatorname{dom} H$*. Then the following statements hold:*

(i) *if $\mathfrak{L}$ is negative with respect to H, then* $\dim \mathfrak{L} \leq \mathsf{i}^-(H)$*;*
(ii) *if $\mathfrak{L}$ is nonpositive with respect to H, then* $\dim \mathfrak{L} \leq \mathsf{i}^-(H) + \mathsf{i}^0(H)$*;*
(iii) *if $\mathfrak{L}$ is positive with respect to H, then* $\dim \mathfrak{L} \leq \mathsf{i}^+(H)$*;*
(iv) *if $\mathfrak{L}$ is nonnegative with respect to H, then* $\dim \mathfrak{L} \leq \mathsf{i}^+(H) + \mathsf{i}^0(H)$.

Proof. (i) Let $\mathcal{L} \subset \operatorname{dom} H$ be a negative subspace with respect to H. Let $\mathcal{H}_-$ be the orthogonal sum of all eigenspaces which correspond to the negative eigenvalues of H_s and let P be the orthogonal projection onto $\mathcal{H}_-$. Then, in particular, $P\phi = 0$ implies that $\phi^* H_s \phi \geq 0$. Hence $\mathcal{L} \cap \ker P = \{0\}$, and the restriction $P{\restriction}\mathcal{L}$ is injective. Therefore,

$$\dim \mathcal{L} = \dim P(\mathcal{L}) \leq \dim \mathcal{H}_- = \mathsf{i}^-(H).$$

(ii) This follows from a similar argument as in (i), when P is taken to be the orthogonal projection onto the orthogonal sum of all eigenspaces which correspond to the nonpositive eigenvalues of H_s.

(iii) & (iv) These statements are obtained by applying items (i) and (ii) to $-H$; see also (3.7). □

As a consequence of Lemma 3.5 the following inertia inequalities hold for two ordered selfadjoint relations.

Proposition 3.6. *Let H_1 and H_2 be selfadjoint relations in $\mathbb{C}^n$ such that $H_1 \leq H_2$. Then their inertia $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$, $j = 1, 2$, satisfy the following inequalities:*

(i) $\mathsf{i}_1^\infty \leq \mathsf{i}_2^\infty$ *or, equivalently,* $\mathsf{i}_1^- + \mathsf{i}_1^0 + \mathsf{i}_1^+ \geq \mathsf{i}_2^- + \mathsf{i}_2^0 + \mathsf{i}_2^+$*;*
(ii) $\mathsf{i}_1^- \geq \mathsf{i}_2^-$ *or, equivalently,* $\mathsf{i}_1^0 + \mathsf{i}_1^+ + \mathsf{i}_1^\infty \leq \mathsf{i}_2^0 + \mathsf{i}_2^+ + \mathsf{i}_2^\infty$*;*
(iii) $\mathsf{i}_1^- + \mathsf{i}_1^0 \geq \mathsf{i}_2^- + \mathsf{i}_2^0$ *or, equivalently,* $\mathsf{i}_1^+ + \mathsf{i}_1^\infty \leq \mathsf{i}_2^+ + \mathsf{i}_2^\infty$*.*

Proof. In each item (i), (ii), and (iii) the equivalence of the two inequalities follows from (3.6). The first mentioned inequalities in (i)–(iii) will be proved.

(i) It follows from (2.7) and Proposition 3.2 that $\operatorname{mul} H_1 \subset \operatorname{mul} H_2$, which gives $\mathsf{i}_1^\infty \leq \mathsf{i}_2^\infty$.

(ii) Let $\mathcal{H}_-$ be the i_2^--dimensional eigenspace which corresponds to the negative eigenvalues of $(H_2)_s$. Then it follows from Proposition 3.2 that $\phi^*(H_1)_s\phi < 0$ for all $\phi \in \mathcal{H}_-$. Now by applying Lemma 3.5 with $\mathcal{L} = \mathcal{H}_-$ and $H = H_1$ yields $\mathsf{i}_2^- = \dim \mathcal{H}_- \leq \mathsf{i}_1^-$.

(iii) This is proved in a similar way as (ii) by using the $(\mathsf{i}_2^- + \mathsf{i}_2^0)$-dimensional eigenspace corresponding to the nonpositive eigenvalues of $(H_2)_s$. □

3.3. **Eigenspaces of a pair of selfadjoint relations.** Let A and B be selfadjoint relations in $\mathbb{C}^n$. Denote the mutually orthogonal eigenspaces of A corresponding to the positive, negative, zero, and infinite eigenvalues by $\mathcal{A}^+$, $\mathcal{A}^-$, $\mathcal{A}^0$, and $\mathcal{A}^\infty$, respectively. Likewise, denote the mutually orthogonal eigenspaces of B corresponding to the positive, negative, zero, and infinite eigenvalues by $\mathcal{B}^+$, $\mathcal{B}^-$, $\mathcal{B}^0$, and $\mathcal{B}^\infty$, respectively. Note that

$$\mathbb{C}^n = \mathcal{A}^+ \oplus \mathcal{A}^- \oplus \mathcal{A}^0 \oplus \mathcal{A}^\infty, \quad \mathbb{C}^n = \mathcal{B}^+ \oplus \mathcal{B}^- \oplus \mathcal{B}^0 \oplus \mathcal{B}^\infty,$$

and

$$\operatorname{dom} A = \mathcal{A}^+ \oplus \mathcal{A}^- \oplus \mathcal{A}^0, \quad \operatorname{dom} B = \mathcal{B}^+ \oplus \mathcal{B}^- \oplus \mathcal{B}^0.$$

The interest will be in decompositions of $\mathbb{C}^n$ in which eigenspaces of A and of B play a role simultaneously by means of an "intermediate" selfadjoint relation H.

Lemma 3.7. *Let H be a selfadjoint relation which satisfies*

$$-H \leq -A \quad \textit{and} \quad H \leq B. \tag{3.8}$$

Then

$$(\mathcal{A}^+ \oplus \mathcal{A}^0) \cap (\mathcal{B}^0 \oplus \mathcal{B}^-) = \mathcal{A}^0 \cap \mathcal{B}^0.$$

Proof. According to Proposition 3.2:

$$
\begin{aligned}
&\operatorname{dom} A \subset \operatorname{dom} H \quad \text{and} \quad \phi^* A_s \phi \leq \phi^* H_s \phi, \quad \phi \in \operatorname{dom} A; \\
&\operatorname{dom} B \subset \operatorname{dom} H \quad \text{and} \quad \phi^* H_s \phi \leq \phi^* B_s \phi, \quad \phi \in \operatorname{dom} B.
\end{aligned} \tag{3.9}
$$

In particular, combining the inequalities in (3.9) gives the inequality

$$
\phi^* A_s \phi \leq \phi^* B_s \phi, \quad \phi \in \operatorname{dom} A \cap \operatorname{dom} B. \tag{3.10}
$$

It suffices to prove the inclusion $(\mathcal{A}^+ \oplus \mathcal{A}^0) \cap (\mathcal{B}^0 \oplus \mathcal{B}^-) \subset \mathcal{A}^0 \cap \mathcal{B}^0$. The operator part A_s restricted to the subspace $\mathcal{A}^+ \oplus \mathcal{A}^0$ defines a nonnegative operator A_s^+ on $\mathcal{A}^+ \oplus \mathcal{A}^0$ and similarly, B_s restricted to the subspace $\mathcal{B}^0 \oplus \mathcal{B}^-$ defines a nonpositive operator B_s^- on $\mathcal{B}^0 \oplus \mathcal{B}^-$. Now the inequality (3.10) shows that if $\phi \in (\mathcal{A}^+ \oplus \mathcal{A}^0) \cap (\mathcal{B}^0 \oplus \mathcal{B}^-)$, then

$$
0 \leq \phi^* A_s^+ \phi = \phi^* A_s \phi \leq \phi^* B_s \phi = \phi^* B_s^- \phi \leq 0.
$$

Thus, $\phi^* A_s^+ \phi = \phi^* B_s^- \phi = 0$ and this implies $A_s \phi = A_s^+ \phi = 0$ and $A_s \phi = A_s^- \phi = 0$, i.e., $\phi \in \mathcal{A}^0 \cap \mathcal{B}^0$. □

More precise information on the above eigenspaces is available, when the selfadjoint relation H in (3.8) is an invertible matrix, so that $\mathsf{i}^0(H) = 0 = \mathsf{i}^\infty(H)$. Then the first inequality in (iii) of Proposition 3.6, when applied to the inequalities (3.8), gives the following inertia inequalities:

$$
\begin{aligned}
&\mathsf{i}^+(A) + \mathsf{i}^0(A) \leq \mathsf{i}^+(H), \\
&\mathsf{i}^-(B) + \mathsf{i}^0(B) \leq \mathsf{i}^-(H).
\end{aligned} \tag{3.11}
$$

The case of equalities in (3.11) is of importance.

Lemma 3.8. *Let H be an invertible selfadjoint matrix such that* (3.8) *holds. Then the following statements are equivalent:*

(i) $(\mathcal{A}^+ \oplus \mathcal{A}^0) + (\mathcal{B}^0 \oplus \mathcal{B}^-) = \mathbb{C}^n$;

(ii) $\mathcal{A}^0 \cap \mathcal{B}^0 = \{0\}$ *and equalities hold in* (3.11).

In this case the sum in (i) is direct, i.e., it gives a decomposition for $\mathbb{C}^n$.

Proof. By the invertibility of H the inequalities (3.11) hold and therefore

$$
\begin{aligned}
\dim \big((\mathcal{A}^+ \oplus \mathcal{A}^0) + (\mathcal{B}^0 \oplus \mathcal{B}^-)\big) &\leq \mathsf{i}^+(A) + \mathsf{i}^0(A) + \mathsf{i}^0(B) + \mathsf{i}^-(B) \\
&\leq \mathsf{i}^+(H) + \mathsf{i}^-(H) = n.
\end{aligned}
$$

Here the first inequality holds as an equality if and only if the sum $(\mathcal{A}^+ \oplus \mathcal{A}^0) + (\mathcal{B}^0 \oplus \mathcal{B}^-)$ is direct and the second inequality holds as an equality if and only if equalities hold in (3.11). Hence, (i) holds precisely when both of the above inequalities hold as equalities. By Lemma 3.7 the sum in (i) direct if and only if $\mathcal{A}^0 \cap \mathcal{B}^0 = \{0\}$. This completes the proof. □

Now assume that H is a selfadjoint relation which satisfies the inequalities

$$
H^{-1} \leq A^{-1} \quad \text{and} \quad -H^{-1} \leq -B^{-1}. \tag{3.12}
$$

Replacing in the above results A, B and H by $-A^{-1}$, $-B^{-1}$ and $-H^{-1}$, respectively, shows that if (3.12) holds, then

$$
(\mathcal{A}^- \oplus \mathcal{A}^\infty) \cap (\mathcal{B}^+ \oplus \mathcal{B}^\infty) = \mathcal{A}^\infty \cap \mathcal{B}^\infty,
$$

and if $-H^{-1}$, or equivalently, H is an invertible matrix, then the following inertia inequalities hold

$$\begin{aligned} \mathsf{i}^-(A)+\mathsf{i}^\infty(A) &\le \mathsf{i}^-(H), \\ \mathsf{i}^+(B)+\mathsf{i}^\infty(B) &\le \mathsf{i}^+(H). \end{aligned} \tag{3.13}$$

Furthermore, if (3.12) holds for an invertible selfadjoint matrix H, then the following statements are equivalent:

(i) $(\mathcal{A}^- \oplus \mathcal{A}^\infty)+(\mathcal{B}^+ \oplus \mathcal{B}^\infty)=\mathbb{C}^n$;
(ii) $\mathcal{A}^\infty \cap \mathcal{B}^\infty=\{0\}$ and equalities hold in (3.13).

A combination of the previous results gives a characterization for an, in general, non-orthogonal space decomposition of $\mathbb{C}^n$, see Proposition 3.10 below. But first a useful lemma will be presented.

Lemma 3.9. *Let H be a selfadjoint relation such that* (3.8) *and* (3.12) *hold. Then H is an invertible matrix if and only if*

$$\begin{aligned} \mathsf{i}^+(A)+\mathsf{i}^0(A)=\mathsf{i}^+(H)=\mathsf{i}^+(B)+\mathsf{i}^\infty(B), \\ \mathsf{i}^-(A)+\mathsf{i}^\infty(A)=\mathsf{i}^-(H)=\mathsf{i}^-(B)+\mathsf{i}^0(B). \end{aligned} \tag{3.14}$$

Proof. If H is an invertible matrix, then the equalities in (3.14) are obtained from the inequalities (3.11) and (3.13) together with (3.6). The converse follows from the fact that the equalities (3.14) together with (3.6) show that $\mathsf{i}^+(H)+\mathsf{i}^-(H)=n$. □

Proposition 3.10. *Let A and B be selfadjoint relations in $\mathbb{C}^n$ and let H be an invertible selfadjoint matrix such that the inequalities* (3.8) *and* (3.12) *are satisfied. Then the following statements are equivalent:*

(i) $\mathcal{A}^0 \cap \mathcal{B}^0=\{0\}$*;*
(ii) $(\mathcal{A}^+ \oplus \mathcal{A}^0)+(\mathcal{B}^0 \oplus \mathcal{B}^-)=\mathbb{C}^n$*;*
(iii) $\mathcal{A}^\infty \cap \mathcal{B}^\infty=\{0\}$*;*
(iv) $(\mathcal{A}^- \oplus \mathcal{A}^\infty)+(\mathcal{B}^+ \oplus \mathcal{B}^\infty)=\mathbb{C}^n$*.*

Proof. Recall that by Lemma 3.9 the assumptions imply that the equalities in (3.14) hold. The items (i) and (ii) are equivalent according to Lemma 3.8. Likewise the items (iii) and (iv) are equivalent by the discussion preceding Lemma 3.9. Finally, the equivalence of (i)-(ii) and (iii)-(iv) follows directly from the fact that if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are subspaces of $\mathbb{C}^n$, then $\mathfrak{L}_1+\mathfrak{L}_2=\mathbb{C}^n$ and $\mathfrak{L}_1 \cap \mathfrak{L}_2=\{0\}$ if and only if $\mathfrak{L}_1^\perp+\mathfrak{L}_2^\perp=\mathbb{C}^n$ and $\mathfrak{L}_1^\perp \cap \mathfrak{L}_2^\perp=\{0\}$. □

Remark 3.11. Let H be an invertible matrix such that (3.8) holds. Then it follows from an antitonicity result for relations, see [5], that H satisfies (3.12) if and only if

$$\begin{aligned} \mathsf{i}^+(H)&=\mathsf{i}^+(A)+\mathsf{i}^0(A), \\ \mathsf{i}^-(H)&=\mathsf{i}^-(B)+\mathsf{i}^0(B). \end{aligned}$$

4. Monotone matrix functions and their limits

In this section the limits of a monotonically nondecreasing matrix function $D(\cdot)$ defined on an open interval of $\mathbb{R}$ are studied. Special attention is paid to so-called strict monotone matrix functions, where it turns out that the eigenspaces of the limit relations lead to certain space decompositions.

4.1. **Graph limits of a monotonically nondecreasing matrix function.** An $n \times n$ matrix function $D(\cdot)$ defined on an open interval $\imath = (a, b)$ of $\mathbb{R}$ is called *monotonically nondecreasing* if its values $D(t)$ are selfadjoint matrices for all $t \in \imath$ and $D(t_1) \le D(t_2)$ when $t_1 \le t_2$, or more explicitly,

$$\phi^* D(t_1)\phi \le \phi^* D(t_2)\phi, \quad \phi \in \mathbb{C}^n, \quad t_1 \le t_2.$$

The limits in graph sense at a and at b of such a matrix function turn out to be selfadjoint relations. A simple direct proof of this fact is provided for the convenience of the reader; see also [4]. For this purpose recall the notion of graph convergence: If H_n is a sequence of matrices or relations in $\mathbb{C}^n$, then the *graph limit* of the sequence H_n is the relation which consists of all $\{\phi, \psi\} \in \mathbb{C}^n \times \mathbb{C}^n$ for which there exist $\{\phi_n, \psi_n\} \in H_n$ such that $\{\phi_n, \psi_n\} \to \{\phi, \psi\}$ in $\mathbb{C}^n \times \mathbb{C}^n$; cf. [4, 17]. Clearly, if Γ is the graph limit of the sequence H_n, then Γ^{-1} is the graph limit of the sequence H_n^{-1}.

Theorem 4.1. *Let $D(\cdot)$ be a monotonically nondecreasing $n \times n$ matrix function on $\imath = (a, b)$. Then the graph limits*

$$D(a) := \lim_{t \downarrow a} D(t) \qquad \text{and} \qquad D(b) := \lim_{t \uparrow b} D(t)$$

exist as selfadjoint relations in $\mathbb{C}^n$ and they satisfy the inequalities

$$-D(t) \le -D(a) \quad \text{and} \quad D(t) \le D(b), \qquad t \in \imath. \tag{4.1}$$

The domains of $D(a)$ and $D(b)$ are given by

$$\begin{aligned} \operatorname{dom} D(a) &= \{\, \phi \in \mathbb{C}^n : \lim_{t \downarrow a} \phi^* D(t)\phi > -\infty \}, \\ \operatorname{dom} D(b) &= \{\, \phi \in \mathbb{C}^n : \lim_{t \uparrow b} \phi^* D(t)\phi < \infty \,\}, \end{aligned} \tag{4.2}$$

and the corresponding orthogonal operator parts $D(a)_s$ and $D(b)_s$ are given by

$$\begin{aligned} \psi^* D(a)_s \phi &= \lim_{t \downarrow a} \psi^* D(t)\phi, \quad \phi, \psi \in \operatorname{dom} D(a), \\ \psi^* D(b)_s \phi &= \lim_{t \uparrow b} \psi^* D(t)\phi, \quad \phi, \psi \in \operatorname{dom} D(b). \end{aligned} \tag{4.3}$$

Proof. Let $c \in (a, b)$ be fixed. Then $D(c)$ is a semibounded matrix and let m_c be its lower bound. Since $D(\cdot)$ is monotonically nondecreasing it follows that $D(t) \ge m_c$ for all $t \in (c, b)$. Hence for $x < m_c$ the selfadjoint matrices $D(t)$ can be written in the form

$$D(t) = \big\{ \{ (D(t) - x)^{-1} f, f + x(D(t) - x)^{-1} f \} : f \in \mathbb{C}^n \big\}; \tag{4.4}$$

see (2.9). The monotonicity of $D(\cdot)$ implies that $D(t_1) - x \le D(t_2) - x$ for $c < t_1 \le t_2$, so that for $x < m_c$

$$0 \le (D(t_2) - x)^{-1} \le (D(t_1) - x)^{-1}, \qquad c < t_1 \le t_2.$$

Hence $(D(\cdot) - x)^{-1}$ is a monotonically nonincreasing matrix function which is nonnegative. Therefore, the limit

$$\lim_{t \uparrow b} (D(t) - x)^{-1} =: L_x$$

exists and is a nonnegative matrix (consider real functions $\phi^*(D(\cdot) - x)^{-1}\psi$ with $\phi, \psi \in \mathbb{C}^n$ and apply the polarization formula). Hence, $L_x = L_x^*$, and Lemma 2.1 (iii) implies that

$$D(b) := \big\{ \{ L_x f, f + x L_x f \} : f \in \mathbb{C}^n \big\} \tag{4.5}$$

is a selfadjoint relation, since clearly $D(b)$ is symmetric, $\operatorname{dom} D(b) = \operatorname{ran} L_x$, $\operatorname{mul} D(b) = \ker L_x$, and $\operatorname{ran} L_x \oplus \ker L_x = \mathbb{C}^n$ by selfadjointness of L_x. Furthermore, the convergence of $(D(\cdot) - x)^{-1}$ to L_x and the equations (4.4) and (4.5) show that $D(b) = \lim_{t\uparrow b} D(t)$ in the sense of graph limits, which shows that $D(b)$ does not depend on the choice of $x < m_c$.

Next define

$$\mathfrak{H}_0 = \left\{ \phi \in \mathbb{C}^n : \lim_{t\uparrow b} \phi^* D(t)\phi < \infty \right\}$$

and note that $\mathfrak{H}_0$ is a linear subspace as follows from the Cauchy-Schwarz inequality. Define the operator $\widetilde{D}_0(b)$ in $\mathfrak{H}_0$ by means of polarization via

$$\phi^* \widetilde{D}_0(b)\phi = \lim_{t\uparrow b} \phi^* D(t)\phi, \quad \phi \in \mathfrak{H}_0.$$

Since $\widetilde{D}_0(b)$ is symmetric and everywhere defined on $\mathfrak{H}_0$, it is a selfadjoint matrix in $\mathfrak{H}_0$. Extend $\widetilde{D}_0(b)$ to a selfadjoint relation in $\mathbb{C}^n$ in the following manner:

$$D_0(b) = \widetilde{D}_0(b) \mathbin{\widehat{\oplus}} \left(\{0\} \times \mathfrak{H}_0^\perp\right).$$

Then, by the construction of $D_0(b)$, Proposition 3.2 gives for $t > c$

$$0 \le (D_0(b) - x)^{-1} \le (D(t) - x)^{-1}, \quad x < m_c.$$

Now by letting t tend to b one obtains

$$0 \le (D_0(b) - x)^{-1} \le L_x = (D(b) - x)^{-1} \le (D(t) - x)^{-1}, \quad x < m_c,$$

which implies that $D(t) \le D(b) \le D_0(b)$. In particular, (4.1) holds (for b) and, moreover, by Proposition 3.2, $\mathfrak{H}_0 \subset \operatorname{dom} D(b)$ and

$$(D(t)\phi, \phi) \le ((D(b))_s\phi, \phi) \le ((D_0(b))_s\phi, \phi) = \lim_{t\uparrow b}(D(t)\phi, \phi), \quad \phi \in \mathfrak{H}_0. \tag{4.6}$$

Furthermore, the inequality $D(t) \le D(b)$ together with Proposition 3.2 yields that

$$(D(t)\phi, \phi) \le ((D(b))_s\phi, \phi), \quad \phi \in \operatorname{dom} D(b).$$

Letting $t \uparrow b$ one concludes that $\phi \in \mathfrak{H}_0$, i.e., $\operatorname{dom} D(b) \subset \mathfrak{H}_0$. Consequently, $\mathfrak{H}_0 = \operatorname{dom} D(b)$ and, hence, it follows from (4.6) by taking the limit as $t \uparrow b$ that $D_0(b) = D(b)$. This proves (4.2) and (4.3) for b.

A similar argument can be given for the limit at the left endpoint a of $\imath$ by considering the behavior of the monotonically nonincreasing function $-D(\cdot)$ when $t \downarrow a$. □

It is emphasized that the inequality $-D(t) \ge -D(a)$ in (4.1) implies the inequality $D(t) \ge D(a)$ if and only if $\operatorname{mul} D(a) = \{0\}$; cf. Remark 3.3.

As an immediate consequence of the inequalities in (4.1) and Proposition 3.2 one obtains the following statement.

Corollary 4.2. *Let $D(\cdot)$ be a monotonically nondecreasing $n \times n$ matrix function on $\imath = (a, b)$ with graph limits $D(a)$ and $D(b)$. Then the inequalities*

$$\begin{aligned} \phi^* D(a)_s\phi &\le \phi^* D(t)\phi, \quad \phi \in \operatorname{dom} D(a), \\ \phi^* D(t)\phi &\le \phi^* D(b)_s\phi, \quad \phi \in \operatorname{dom} D(b), \end{aligned} \tag{4.7}$$

hold for all $t \in \imath$ and, in particular,

$$\phi^* D(a)_s\phi \le \phi^* D(t)\phi \le \phi^* D(b)_s\phi, \quad \phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b).$$

The following lemma is essentially a consequence of the proof of Theorem 4.1. It shows that upper bounds are preserved for the limits of a matrix function.

Lemma 4.3. *Let $D(\cdot)$ be a monotonically nondecreasing $n \times n$ matrix function on $\imath = (a, b)$ and assume that for some selfadjoint relations H_a and H_b the following inequalities hold for all $t \in \imath$:*

$$-D(t) \leq -H_a \quad \text{and} \quad D(t) \leq H_b.$$

Then the graph limits $D(a)$ and $D(b)$ of $D(\cdot)$ satisfy the inequalities

$$-D(a) \leq -H_a \quad \text{and} \quad D(b) \leq H_b.$$

Proof. Let c and m_c be as in the proof of Theorem 4.1 and denote the lower bound of H_b by m_b. Since $D(t) \leq H_b$ the inequality

$$0 \leq (H_b - x)^{-1} \leq (D(t) - x)^{-1} \quad \text{for } x < \min\{m_c, m_b\}, \quad t > c,$$

holds. This inequality remains valid also for $t \uparrow b$, i.e.,

$$0 \leq (H_b - x)^{-1} \leq (D(b) - x)^{-1} \quad \text{for } x < \min\{m_c, m_b\},$$

which implies $D(b) \leq H_b$. A similar argument shows that $-D(t) \leq -H_a$ implies $-D(a) \leq -H_a$. □

Example 4.4. Let H be a selfadjoint matrix or relation in $\mathbb{C}^n$ and let α, β be consecutive eigenvalues of H. For $t \in (\alpha, \beta)$ the function $(H - t)^{-1}$ is monotonically nondecreasing, since

$$\frac{d}{dt}\, \phi^*(H - t)^{-1}\phi = \big((H - t)^{-1}\phi\big)^*(H - t)^{-1}\phi \geq 0, \quad \phi \in \mathbb{C}^n,$$

which follows from the resolvent identity (2.3). Hence by Theorem 4.1 the matrix function $(H - t)^{-1}$, $t \in (\alpha, \beta)$, has graph limits at α and β which are given by

$$\lim_{t \downarrow \alpha} (H - t)^{-1} = (H - \alpha)^{-1} \quad \text{and} \quad \lim_{t \uparrow \beta} (H - t)^{-1} = (H - \beta)^{-1}. \tag{4.8}$$

In fact, to verify the second identity in (4.8) let first $\{\phi, \psi\}$ be in the graph limit of $(H - t)^{-1}$ when $t \uparrow \beta$. Then there exist $\{\phi_t, \psi_t\} \in (H - t)^{-1}$ with $\{\phi_t, \psi_t\} \to \{\phi, \psi\}$ as $t \uparrow \beta$. Since

$$\{\psi_t, \phi_t + (t - \beta)\psi_t\} \in H - \beta \quad \text{and} \quad \{\phi_t + (t - \beta)\psi_t, \psi_t\} \in (H - \beta)^{-1},$$

it follows that $\{\phi, \psi\} \in (H - \beta)^{-1}$. For the converse, let $\{\phi, \psi\} \in (H - \beta)^{-1}$. Then $\{\psi, \phi + (\beta - t)\psi\} \in H - t$, so that $\{\phi + (\beta - t)\psi, \psi\} \in (H - t)^{-1}$ and $\{\phi + (\beta - t)\psi, \phi\} \to \{\phi, \psi\}$ as $t \uparrow \beta$. Hence $\{\phi, \psi\}$ is in the graph limit of $(H - t)^{-1}$. The first identity in (4.8) is proved in a similar way.

4.2. **Nonnegative or nonpositive matrix functions.** Let $D(\cdot)$ be a monotonically nondecreasing $n \times n$ matrix function on $\imath$, and assume that the values of $D(\cdot)$ are all nonnegative matrices. Then for $t_1, t_2 \in \imath$

$$\ker D(t_2) \subset \ker D(t_1), \quad t_1 \leq t_2. \tag{4.9}$$

This fact is used in the following theorem.

Theorem 4.5. *Let $D(\cdot)$ be a monotonically nondecreasing $n \times n$ matrix function on $\imath = (a, b)$ of nonnegative matrices $D(t) \geq 0$, $t \in \imath$, and let $D(b)$ be the graph limit at b as in Theorem 4.1. Then $D(b)$ is a nonnegative relation and the following statements are equivalent:*

(i) ker $D(b) = \{0\}$*;*
(ii) ker $D(t) = \{0\}$ *for all* $t \in [c, b)$ *for some* $c \in \imath$*.*

Proof. Since $D(\cdot)$ is monotonically nondecreasing and nonnegative it follows from Theorem 4.1 that $D(b)$ is a nonnegative relation on $\mathbb{C}^n$ with a nonnegative operator part $D(b)_s$; see (4.3).

(ii) $\Rightarrow$ (i) It follows from (4.7) that ker $D(b) =$ ker $D(b)_s \subset$ ker $D(t)$, $t \in (a, b)$. Hence, the implication (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (ii) Associate with each $t \in (a, b)$ the subset $C_t \subset \mathbb{C}^n$ defined by

$$C_t = \{\, \phi \in \ker D(t) : |\phi| = 1 \,\}.$$

Then C_t is compact and $t \leq \widetilde{t}$ implies $C_{\widetilde{t}} \subset C_t$ as follows from (4.9). Now choose an increasing sequence of numbers t_n, $n \geq 0$, such that $t_n \to b$. Then one has

$$\bigcap_{n \geq 0} C_{t_n} = \emptyset. \tag{4.10}$$

To see this, assume that $\phi \in C_{t_n}$ for all $n \geq 0$. This implies that

$$\phi^* D(t)\phi = 0 \text{ for all } t \in [t_1, b).$$

Now it follows from (4.3) in Theorem 4.1 that $\phi^* D(b)_s \phi = 0$. This implies that $D(b)_s \phi = 0$, and hence $\phi = 0$; a contradiction with $|\phi| = 1$. This proves (4.10). Since each of the sets C_{t_n} in (4.10) is compact it follows that there exists t_n such that $C_{t_n} = \emptyset$. Then $c := t_n$ satisfies the requirements. $\square$

The result in Theorem 4.5 does not hold in infinite-dimensional spaces; the argument in the proof breaks down due to non-compactness of the unit ball and the unit sphere used in the proof. The following simple example illustrates this.

Example 4.6. Consider the Hilbert space $L^2(0, \infty)$ and let P_t be the orthogonal projection onto the subspace $L^2(0, t) \subset L^2(0, \infty)$. Then clearly $t \to P_t$ is a monotonically nondecreasing function on $(0, \infty)$ whose values P_t are nonnegative. Furthermore the graph limit P_∞ satisfies $P_\infty = I$, so that ker $P_\infty = \{0\}$. However, ker $P_t \neq \{0\}$ for any $t \in (0, \infty)$.

At the left endpoint of the interval $\imath$ there is a similar situation. For completeness the corresponding variant of Theorem 4.5 is formulated.

Corollary 4.7. *Let* $D(\cdot)$ *be a monotonically nondecreasing* $n \times n$ *matrix function on* $\imath = (a, b)$ *of nonpositive matrices* $D(t) \leq 0$, $t \in \imath$, *and let* $D(a)$ *be the graph limit at* a *as in Theorem 4.1. Then* $-D(a)$ *is a nonnegative relation and the following statements are equivalent:*

(i) ker $D(a) = \{0\}$*;*
(ii) ker $D(t) = \{0\}$ *for all* $t \in (a, c]$ *for some* $c \in \imath$*.*

4.3. **Strict monotone matrix functions.** The notion of strictness for monotone matrix functions is introduced in the next definition.

Definition 4.8. Let $D(\cdot)$ be a monotonically nondecreasing $n \times n$ matrix function on the interval $\imath = (a, b)$ with graph limits $D(a)$ and $D(b)$ and corresponding operator parts $D(a)_s$ and $D(b)_s$. The function $D(\cdot)$ is said to be *strict on the compact interval* $[\alpha, \beta] \subset \imath$ if

$$\phi^* D(\alpha)\phi < \phi^* D(\beta)\phi, \quad \phi \in \mathbb{C}^n, \quad \phi \neq 0,$$

and $D(\cdot)$ is said to be *strict on* $\imath$ if

$$\phi^* D(a)_s \phi < \phi^* D(b)_s \phi, \quad \phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b), \quad \phi \neq 0.$$

Note that the monotonically nondecreasing matrix function $D(\cdot)$ is strict on $[\alpha, \beta]$ if and only if

$$\phi^* D(\alpha) \phi = \phi^* D(\beta) \phi, \quad \phi \in \mathbb{C}^n \quad \Rightarrow \quad \phi = 0,$$

and that $D(\cdot)$ is strict on $\imath$ if and only if

$$\phi^* D(a)_s \phi = \phi^* D(b)_s \phi, \quad \phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b) \quad \Rightarrow \quad \phi = 0;$$

cf. Corollary 4.2.

Monotone functions which are strict on $\imath$ can be characterized without invoking the graph limits at the endpoint of $\imath$.

Lemma 4.9. *Let $D(\cdot)$ be a monotonically nondecreasing $n \times n$ matrix function on $\imath$ with the graph limits $D(a)$ and $D(b)$ and corresponding operator parts $D(a)_s$ and $D(b)_s$. Then $\phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b)$ satisfies the equality*

$$\phi^* D(a)_s \phi = \phi^* D(b)_s \phi \tag{4.11}$$

if and only if the vector function $D(\cdot)\phi$ is constant on $\imath$.

In particular, $D(\cdot)$ is strict on $\imath$ (strict on a subinterval $\jmath \subset \imath$) if and only if $D(\cdot)\phi$ is constant on $\imath$ (respectively, constant on $\jmath$) implies that $\phi = 0$.

Proof. ($\Rightarrow$) Assume that (4.11) holds for $\phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b)$. According to Corollary 4.2 for $s, t \in \imath$ with $s < t$ one has

$$\phi^* D(a)_s \phi \leq \phi^* D(s) \phi \leq \phi^* D(t) \phi \leq \phi^* D(b)_s \phi \quad \text{for all } s < t,$$

which shows that $\phi^* D(t)\phi - \phi^* D(s)\phi = 0$. Now $D(t) - D(s) \geq 0$ for $s < t$ implies that $D(t)\phi - D(s)\phi = 0$, $s, t \in \imath$, and hence $D(\cdot)\phi$ is constant on $\imath$.

($\Leftarrow$) If $D(t)\phi = D(s)\phi$ for all $s, t \in \imath$, then also $\phi^* D(t)\phi = \phi^* D(s)\phi$ holds for all $s, t \in \imath$. Taking limits the formulas (4.2) in Theorem 4.1 imply that $\phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b)$ and from (4.3) in Theorem 4.1 one gets

$$\phi^* D(a)_s \phi = \phi^* D(s) \phi = \phi^* D(t) \phi = \phi^* D(b)_s \phi \quad \text{for all } s < t.$$

The remaining statements are clear from the above arguments and the definition of strictness. □

Lemma 4.9 implies that if the function $D(\cdot)$ is strict on a compact interval $[\alpha, \beta] \subset \imath$, then $D(\cdot)$ is strict on $\imath$, and it is also strict on every subinterval $\jmath \subset \imath$ for which $[\alpha, \beta] \subset \jmath$. It is a consequence of Theorem 4.5 that the converse is also true.

Theorem 4.10. *Let $D(\cdot)$ be a monotonically nondecreasing $n \times n$ matrix function on $\imath$. Then $D(\cdot)$ is strict on $\imath$ if and only if there exists a compact interval $[\alpha, \beta] \subset \imath$ on which $D(\cdot)$ is strict.*

Proof. ($\Rightarrow$) Let $h : (-1, 1) \to \imath$ be a monotonically increasing continuous function with continuous inverse that maps the open interval $(-1, 1)$ onto $\imath$. Then clearly the matrix function $F(s) = D(h(s))$, $s \in (-1, 1)$, is monotonically nondecreasing on $(-1, 1)$. It is clear that $D(\cdot)$ is strict on $\imath$ if and only if $F(\cdot)$ is strict on $(-1, 1)$. Moreover, $D(\cdot)$ is strict on a compact subinterval $\jmath \subset \imath$ if and only if $F(\cdot)$ is strict on a compact subinterval $\widetilde{\jmath} \subset (-1, 1)$.

Now consider the function

$$E(s) = F(s) - F(-s), \quad s \in [0, 1).$$

Clearly, $E(s) \geq 0$ for all $[0,1)$. Since $F(s_1) \leq F(s_2)$ and $F(-s_1) \geq F(-s_2)$ for $s_1 \leq s_2$, the function $E(\cdot)$ is monotonically nondecreasing on the interval $[0,1)$. Let $E(1) = \lim_{s\uparrow 1} E(s)$ be the graph limit of $E(\cdot)$; cf. Theorem 4.1. Next it will be shown that $\ker E(1) = \{0\}$ holds. In fact, if $E(1)\phi = 0$, then $E(t)\phi = 0$ for all $t \in [0,1)$, which by monotonicity of $F(\cdot)$ implies that for $s < t$, $s,t \in [0,1)$,

$$0 \leq \phi^*(F(s) - F(-t))\phi \leq \phi^*(F(t) - F(-t))\phi = \phi^* E(t)\phi = 0.$$

Hence, $(F(s) - F(-t))\phi = 0$ which implies that $F(\cdot)\phi$ is constant on $(-1,1)$. Since $D(\cdot)$ is assumed to be strict on $\imath$ the function $F(\cdot)$ is strict on $(-1,1)$ and hence Lemma 4.9 implies $\phi = 0$, i.e., $\ker E(1) = \{0\}$. Now Theorem 4.5 yields that $\ker E(s) = \{0\}$ for all $s \geq c$ and some $0 < c < 1$. Then an application of Lemma 4.9 shows that $F(\cdot)$ is strict on the interval $[-c,c]$ and, consequently, $D(\cdot)$ is strict on some compact interval $[\alpha,\beta] \subset \imath$.

($\Leftarrow$) As stated above, this is a direct consequence of Lemma 4.9. $\square$

The above result shows that if $D(\cdot)$ is strict on $\imath$, then there exists a compact subinterval of $\imath$ on which $D(\cdot)$ is strict. In special cases it may happen that $D(\cdot)$ is strict actually on any compact subinterval of $\imath$. The next example shows that in the class of Nevanlinna functions also this stronger strictness property holds.

Example 4.11. Let $D(\cdot)$ be an $n \times n$ matrix Nevanlinna function, so that

$$D(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left(\frac{1}{s-\lambda} - \frac{s}{1+s^2} \right) d\Sigma(s), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.12}$$

where A and B are selfadjoint $n \times n$ matrices with $B \geq 0$ and $d\Sigma$ is a nonnegative $n \times n$ matrix measure such that $\int_{\mathbb{R}} d\Sigma(s)/(1+s^2)$ converges. Assume that D admits a holomorphic continuation to an open interval $\imath \subset \mathbb{R}$, i.e. $d\Sigma = 0$ on $\imath$. Then $D(\cdot)$ is a monotonically nondecreasing $n \times n$ matrix function on $\imath$. Assume, in addition, that $D(\cdot)$ is strict on $\imath$. It will be shown that $D(\cdot)$ is strict on *any* compact subinterval $\Delta \subset \imath$. Let $D(\cdot)\phi$ be constant on Δ for some $\phi \in \mathbb{C}^n$. Then in fact

$$(A\phi,\phi) + (B\phi,\phi)t + \int_{\mathbb{R}\setminus\imath} \left(\frac{1}{s-t} - \frac{s}{1+s^2} \right) d(\Sigma(s)\phi,\phi)$$

is constant for $t \in \Delta$. Differentiation shows that

$$0 = (B\phi,\phi) + \int_{\mathbb{R}\setminus\imath} \frac{1}{(s-t)^2}\, d(\Sigma(s)\phi,\phi), \quad t \in \Delta.$$

The nonnegativity of B and $d\Sigma$ then imply

$$(B\phi,\phi) = 0 \qquad \text{and} \qquad (\Sigma(s)\phi,\phi) = d \text{ for some } d \in \mathbb{R} \text{ and all } s \in \mathbb{R} \setminus \imath.$$

This implies $(D(t)\phi,\phi) = (A\phi,\phi)$, $t \in \imath$, and in particular $D(t)\phi = A\phi$, $t \in \imath$. Hence $D(\cdot)\phi$ is constant on $\imath$, which by assumption gives $\phi = 0$. By Lemma 4.9 it follows that $D(\cdot)$ is strict on Δ.

Recall, that the Nevanlinna function $D(\cdot)$ is said to be *uniformly strict* if the imaginary part $\operatorname{Im} D(\lambda)$ is invertible for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It follows from the integral representation in (4.12) that this property does not hold if and only if there exists $\phi \in \mathbb{C}^n$, $\phi \neq 0$, such that $B\phi = 0$ and $\Sigma(s)\phi = \xi$ for some $\xi \in \mathbb{C}^n$ and all $s \in \mathbb{R} \setminus \imath$. Therefore, $D(\cdot)$ is strict on $\imath$ or, equivalently, on any compact subinterval Δ of $\imath$, if and only if $D(\cdot)$ is uniformly strict.

Denote the mutually orthogonal eigenspaces of the selfadjoint relation $D(a)$ corresponding to the positive, negative, zero, and infinite eigenvalues by $\mathcal{A}^+$, $\mathcal{A}^-$, $\mathcal{A}^0$, and $\mathcal{A}^\infty$, respectively. Likewise, denote the mutually orthogonal eigenspaces of the selfadjoint relation $D(b)$ corresponding to the positive, negative, zero, and infinite eigenvalues by $\mathcal{B}^+$, $\mathcal{B}^-$, $\mathcal{B}^0$, and $\mathcal{B}^\infty$, respectively. Clearly,

$$\mathbb{C}^n = \mathcal{A}^+ \oplus \mathcal{A}^- \oplus \mathcal{A}^0 \oplus \mathcal{A}^\infty, \quad \mathbb{C}^n = \mathcal{B}^+ \oplus \mathcal{B}^- \oplus \mathcal{B}^0 \oplus \mathcal{B}^\infty, \tag{4.13}$$

and

$$\operatorname{dom} D(a) = \mathcal{A}^+ \oplus \mathcal{A}^- \oplus \mathcal{A}^0 \quad \text{and} \quad \operatorname{dom} D(b) = \mathcal{B}^+ \oplus \mathcal{B}^- \oplus \mathcal{B}^0; \tag{4.14}$$

cf. Section 3.3. Under certain conditions the orthogonal decompositions of $\mathbb{C}^n$ in (4.13) can be supplemented with direct sum decompositions involving eigenspaces of $D(a)$ and $D(b)$ simultaneously.

Theorem 4.12. *Let $D(\cdot)$ be a monotonically nondecreasing strict $n \times n$ matrix function on $\imath = (a, b)$ with graph limits $D(a)$ and $D(b)$. Assume that for some $t \in \imath$ the matrix $D(t)$ is invertible and that the inequalities*

$$D(t)^{-1} \leq D(a)^{-1} \quad \textit{and} \quad -D(t)^{-1} \leq -D(b)^{-1} \tag{4.15}$$

hold. Then the space $\mathbb{C}^n$ has the following decompositions:

$$\mathbb{C}^n = (\mathcal{A}^+ \oplus \mathcal{A}^0) + (\mathcal{B}^0 \oplus \mathcal{B}^-) = (\mathcal{A}^- \oplus \mathcal{A}^\infty) + (\mathcal{B}^+ \oplus \mathcal{B}^\infty), \quad \textit{direct sums},$$

and, furthermore,

$$\dim \big(\operatorname{dom} D(a) \cap \operatorname{dom} D(b)\big) = \mathsf{i}^-(D(a)) + \mathsf{i}^+(D(b)).$$

Proof. Observe first that for $\phi \in \mathcal{A}^0 \cap \mathcal{B}^0$ one has $\phi^* D(a)_s \phi = 0 = \phi^* D(b)_s \phi$. Since $\phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b)$, the assumption that $D(\cdot)$ is strict implies that $\phi = 0$ and hence

$$\mathcal{A}^0 \cap \mathcal{B}^0 = \{0\}. \tag{4.16}$$

Next it follows from (4.1) and (4.15) that the inequalitites (3.8) and (3.12) hold with A, B and H replaced by $D(a)$, $D(b)$ and $D(t)$, respectively. Therefore the decompositions of $\mathbb{C}^n$ are implied by Proposition 3.10 and (4.16).

To prove the dimension result, note first that as a consequence of (4.13) and (4.14) one has

$$\operatorname{dom} D(a) \cap \operatorname{dom} D(b) = (\mathcal{A}^\infty)^\perp \cap (\mathcal{B}^\infty)^\perp = (\mathcal{A}^\infty + \mathcal{B}^\infty)^\perp.$$

Furthermore, (4.16) and Proposition 3.10 imply that $\mathcal{A}^\infty \cap \mathcal{B}^\infty = \{0\}$ and hence Lemma 3.9 yields

$$\begin{aligned}
\dim \big(\operatorname{dom} D(a) \cap \operatorname{dom} D(b)\big) &= n - \big(\mathsf{i}^\infty(D(a)) + \mathsf{i}^\infty(D(b))\big) \\
&= \mathsf{i}^+(D(t)) + \mathsf{i}^-(D(t)) - \mathsf{i}^\infty(D(a)) - \mathsf{i}^\infty(D(b)) \\
&= \mathsf{i}^-(D(a)) + \mathsf{i}^+(D(b)).
\end{aligned}$$

This completes the proof of Theorem 4.12. □

Remark 4.13. Assume that $D(\cdot)$ is a monotonically nondecreasing matrix function on $\imath$. Then $-D(\cdot)^{-1}$ is monotonically nondecreasing matrix function on $\imath$ if and only if the inertia numbers of $D(\cdot)$ are of the form $\mathsf{i}(D(t)) = \{\mathsf{i}^+, \mathsf{i}^-, 0, 0\}$, $t \in \imath$. This follows from the so-called antitonicity results for invertible matrices (cf. [7, 15, 19]; see also [5]). In this case the matrix function $-D(\cdot)^{-1}$ has limits $-D(a)^{-1}$ and $-D(b)^{-1}$. Note that $-D(a)^{-1}$ and $-D(b)^{-1}$ are the limits of $-D(\cdot)^{-1}$ since $D(a)$

and $D(b)$ are the limits of $D(\cdot)$ in the graph sense. Hence the conditions in (4.15) are satisfied and the statements in Theorem 4.12 are valid.

In particular, if $D(\cdot)$ is a continuous $n\times n$ matrix function on $\imath$ and $D(t)$ is invertible for each $t\in\imath$, then $\mathsf{i}(D(t))=\{\mathsf{i}^+,\mathsf{i}^-,0,0\}$, $t\in\imath$, holds, and hence the conclusions of Theorem 4.12 hold.

5. An application: Square-integrability of matrix functions

The following situation provides an application of Theorem 4.12. It has a direct consequence in the theory of singular canonical systems of differential equations. Let G be a selfadjoint $n\times n$ matrix, let $Y(\cdot)$ be an $n\times n$ matrix function on an open interval $\imath=(a,b)\subset\mathbb{R}$ and define the $n\times n$ matrix function $D(\cdot)$ on $\imath$ as

$$D(t)=Y(t)^*GY(t),\qquad t\in\imath.$$

Moreover, assume that there exists a locally integrable nonnegative $n\times n$ matrix function $\Delta(\cdot)$ on $\imath$ and some $c>0$ such that for each $\alpha,\beta\in\imath$ with $\alpha<\beta$

$$D(\beta)-D(\alpha)=c\int_\alpha^\beta Y(s)^*\Delta(s)Y(s)\,ds. \tag{5.1}$$

Then the values of $D(\cdot)$ are selfadjoint matrices and the function itself is monotonically nondecreasing. Hence by Theorem 4.1, the selfadjoint limits $D(a)$ and $D(b)$ exist in the graph sense and satisfy

$$-D(t)\le -D(a)\quad\text{and}\quad D(t)\le D(b),\qquad t\in\imath.$$

The identity (5.1) provides a connection between the square-integrable (with respect to Δ) combinations of the columns of the matrix function $Y(\cdot)$ and the limits $D(a)$ and $D(b)$. In fact, the following lemma follows from Theorem 4.1.

Lemma 5.1. *For $\phi\in\mathbb{C}^n$ the function $Y(\cdot)\phi$ is square-integrable with respect to Δ at the left endpoint a or at the right endpoint b if and only if $\phi\in\operatorname{dom}D(a)$ or $\phi\in\operatorname{dom}D(b)$, respectively. Consequently, the number of linearly independent functions $Y(\cdot)\phi$ which are square-integrable with respect to Δ at the left endpoint a or the right endpoint b is*

$$\dim(\operatorname{dom}D(a))=\mathsf{i}^+(D(a))+\mathsf{i}^-(D(a))+\mathsf{i}^0(D(a)),$$

or

$$\dim(\operatorname{dom}D(b))=\mathsf{i}^+(D(b))+\mathsf{i}^-(D(b))+\mathsf{i}^0(D(b)),$$

respectively. In particular, the number of linearly independent functions $Y(\cdot)\phi$ which are square-integrable with respect to Δ on $\imath$ is

$$\dim\big(\operatorname{dom}D(a)\cap\operatorname{dom}D(b)\big). \tag{5.2}$$

It follows directly from (5.1) and Lemma 5.1 that the formula

$$\phi^*D(b)_s\phi-\phi^*D(a)_s\phi=c\int_\imath (Y(s)\phi)^*\Delta(s)Y(s)\phi\,ds \tag{5.3}$$

holds for all $\phi\in\operatorname{dom}D(a)\cap\operatorname{dom}D(b)$. Hence, it is possible to characterize strictness of $D(\cdot)$ in terms of the function $Y(\cdot)$.

Lemma 5.2. *The function $D(\cdot)$ is strict on $\imath$ if and only if the function $Y(\cdot)$ satisfies the following definiteness condition:*

$$\int_\imath (Y(s)\phi)^*\Delta(s)Y(s)\phi\,ds=0,\quad \phi\in\mathbb{C}^n\quad\Rightarrow\quad\phi=0. \tag{5.4}$$

Proof. (⇒) Let $D(\cdot)$ be strict on $\imath$. Assume that for some $\phi \in \mathbb{C}^n$:

$$\int_\imath (Y(s)\phi)^* \Delta(s) Y(s)\phi \, ds = 0.$$

Then, it follows that $\phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b)$, since the integral is finite, see Lemma 5.1. Moreover, (5.3) implies that

$$\phi^* D(b)_s \phi - \phi^* D(a)_s \phi = 0.$$

Since $D(\cdot)$ is strict on $\imath$, it follows that $\phi = 0$.

(⇐) Assume that the condition (5.4) is satisfied. Now let

$$\phi^* D(b)_s \phi - \phi^* D(a)_s \phi = 0, \quad \phi \in \operatorname{dom} D(a) \cap \operatorname{dom} D(b).$$

Then it follows from (5.3) that $\phi = 0$. Hence $D(\cdot)$ is strict on $\imath$. □

If, in addition, $D(\cdot)$ is a strict function with the additional properties in Theorem 4.12, then $\mathbb{C}^n$ can be written as the direct sum of eigenspaces of the limit relations as in Theorem 4.12 and the number (5.2) can be specified.

Theorem 5.3. *Assume that the matrix function $D(\cdot)$ is strict and let $D(a)$ and $D(b)$ be the graph limits at a and b, respectively. Suppose that for some $t \in \imath$ the matrix $D(t)$ is invertible and that the inequalities* (4.15) *hold. Then the statements in Theorem 4.12 are valid and, in particular, the number of linearly independent functions $Y(\cdot)\phi$ which are square-integrable with respect to Δ on $\imath$ is*

$$\mathsf{i}^-(D(a)) + \mathsf{i}^+(D(b)).$$

Remark 5.4. The setting in this section is inspired by the theory of definite canonical systems of differential equations as studied in [10, 13, 14, 16, 18], see also [6] for an application of abstract monotonicity results. In the situation of canonical systems, there exists a matrix valued function $Y_\lambda(\cdot)$ such that

$$JY_\lambda(t)' - H(t)Y_\lambda(t) = \lambda \Delta(t) Y_\lambda(t), \quad t \in \imath, \ \ \lambda \in \mathbb{C},$$

where $J^* = J^{-1} = -J$ and $H(\cdot)$ are $\Delta(\cdot)$ are locally integrable nonnegative $n \times n$ matrix functions on $\imath$, with $\Delta(t)$, $t \in \imath$, being nonnegative almost everywhere. Then $G = -iJ$, the function $D(\cdot) = Y_\lambda(\cdot)^*(-iJ)Y_\lambda(\cdot)$ is continuous and $D(t)$ is invertible for all $t \in \imath$. The relation (5.1) becomes

$$D(\beta) - D(\alpha) = \operatorname{Im} \lambda \int_\alpha^\beta Y_\lambda(s)^* \Delta(s) Y_\lambda(s) \, ds$$

and hence $D(\cdot)$ monotonically nondecreasing on $\imath$ when $\lambda \in \mathbb{C}_+$. The definiteness condition on the canonical system of the form (5.4) then implies strictness of the function $D(\cdot)$, and hence the assumption (4.15) in Theorem 4.12 and Theorem 5.3 is satisfied; cf. Remark 3.11. It is also noted that Theorem 4.10 has a counterpart in the theory of definite canonical systems. The definiteness condition in [2] amounts to strictness on every compact interval of $\imath$.

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