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# Convergence of 2D-Schrödinger operators with local scaled short-range interactions to a Hamiltonian with infinitely many $\delta$-point interactions

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We prove, that a Hamiltonian with infinitely many $\delta$-point interactions in the plane can be approximated in the norm resolvent sense by a family of Schrödinger operators with regular, local scaled short-range potentials. Similar well known results from the 1D and the 3D case are complemented thereby.



## 1 Introduction and main result

Let $\Lambda := \{y_l\} \subset \mathbb{R}^2$ be an at most countable set of points satisfying $\inf\{|y_l - y_k| : y_l, y_k \in \Lambda, y_l \neq y_k\} > 0$, let $\alpha : \Lambda \to \mathbb{R} \cup \{\infty\}, y_l \mapsto \alpha_l$ (where $\alpha_l = \infty$ means that there is no interaction at $y_l$) and set $\Lambda_\mathbb{R} := \{y_l \in \Lambda : \alpha_l \in \mathbb{R}\}$. Then, a Schrödinger operator $-\Delta_{\alpha,\Lambda}$ with $\delta$-interactions supported on $\Lambda$ and coupling $\alpha$ can be defined via the resolvent

$$(-\Delta_{\alpha,\Lambda} - \lambda)^{-1} = (-\Delta - \lambda)^{-1} + \sum_{y_k, y_l \in \Lambda_\mathbb{R}} [\Gamma_{\alpha,\Lambda_\mathbb{R}}(\lambda)]^{-1}_{kl} \big(\cdot, \overline{G_\lambda(\cdot - y_l)}\big) G_\lambda(\cdot - y_k), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $-\Delta$ denotes the free Laplacian, $G_\lambda(x) = \frac{i}{4} H_0^{(1)}\big(\sqrt{\lambda}|x|\big)$ is the integral kernel of its resolvent with $\mathrm{Im}\sqrt{\lambda} \geq 0$ and a Hankel function $H_0^{(1)}$ of first kind and order zero, and with the operator $\Gamma_{\alpha,\Lambda_\mathbb{R}}(\lambda)$ acting in $\ell^2(\Lambda_\mathbb{R})$, which is defined as

$$\Gamma_{\alpha,\Lambda_\mathbb{R}}(\lambda) = \left[\left(\alpha_k - \frac{1}{2\pi}\left(\gamma - \ln\frac{\sqrt{\lambda}}{2i}\right)\right)\delta_{kl} - \tilde{G}_\lambda(y_k - y_l)\right]_{y_k, y_l \in \Lambda_\mathbb{R}} \quad \text{with} \quad \tilde{G}_\lambda(x) = \begin{cases} G_\lambda(x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and the Euler-Mascheroni constant $\gamma$, cf. [1, Chapter III.4].

Our main goal in this note is to prove, that $-\Delta_{\alpha,\Lambda}$ can be approximated in the norm resolvent sense by a sequence of Schrödinger operators of the form

$$H_{\varepsilon,\Lambda} := -\Delta + \sum_{y_l \in \Lambda} \frac{\mu_l\big((\ln\varepsilon)^{-1}\big)}{\varepsilon^2} V_l\left(\frac{\cdot - y_l}{\varepsilon}\right), \qquad \varepsilon > 0.$$

Here, $\{V_l\}$ is a set of potentials such that there exists a compactly supported function $V \in L^{1+\beta}(\mathbb{R}^2)$ for some $\beta > 0$ satisfying $|V_l(x)| \leq |V(x)|$ for almost all $x \in \mathbb{R}^2$ and all $l$ and $\{\mu_l\}$ is a set of uniformly bounded real valued functions defined in a neighbourhood of 0 of the form $\mu_l(x) = \mu_{l,1}x + \mu_{l,2}x^2 + o(x^2)$. One can show, that under these assumptions $H_{\varepsilon,\Lambda}$ defines a self-adjoint operator in the form sense for $\varepsilon > 0$ sufficiently small.

Let us write $u_l(x) = |V_l(x)|^{1/2}$ and $v_l(x) = \mathrm{sign}[V_l(x)]|V_l(x)|^{1/2}$ and define the Hilbert-Schmidt operator $T_l$ via its integral kernel $T_l(x,y) = u_l(x)\ln|x-y|v_l(y)$. Now, our main result reads as follows:

**Theorem 1.1** *Define the coupling $\alpha$ as*

$$\alpha_l = \begin{cases} \frac{\mu_{l,2}}{4\pi^2}(u_l, v_l) + \frac{1}{2\pi(u_l,v_l)^2}(T_l u_l, v_l), & \text{if } (u_l, v_l) \neq 0 \text{ and } \mu_{l,1} = \frac{2\pi}{(u_l,v_l)}, \\ \infty, & \text{otherwise.} \end{cases}$$

*Then, the family of operators $H_{\varepsilon,\Lambda}$ converges to $-\Delta_{\alpha,\Lambda}$ in the norm resolvent sense, i.e. $(H_{\varepsilon,\Lambda} - \lambda)^{-1} \to (-\Delta_{\alpha,\Lambda} - \lambda)^{-1}$ in the operator norm for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, as $\varepsilon \to 0+$.*

We would like to mention here, that similar approximation results are known in the one- and in the three-dimensional case for finitely and for infinitely many points $y_l$ (see [1] and the references therein), but in two dimensions this is only known for the one center case [1, 2].

## 2 Proof of Theorem 1.1

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Similar as in [3], where the approximation of a Hamiltonian with infinitely many $\delta$-point potentials in $\mathbb{R}^3$ was investigated, one verifies that the resolvent of $H_{\varepsilon,\Lambda}$ is given by

$$(H_{\varepsilon,\Lambda} - \lambda)^{-1} = (-\Delta - \lambda)^{-1} - A_\varepsilon(\lambda)[1 + (1 + D_\varepsilon(\lambda))^{-1} B_\varepsilon(\lambda) E_\varepsilon(\lambda)]^{-1}(1 + D_\varepsilon(\lambda))^{-1} B_\varepsilon(\lambda) C_\varepsilon(\lambda)$$

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with the bounded linear operators $A_\varepsilon(\lambda) : \bigoplus_{y_l\in\Lambda} L^2(\mathbb{R}^2) =: \mathscr{H} \to L^2(\mathbb{R}^2)$, $B_\varepsilon(\lambda), D_\varepsilon(\lambda), E_\varepsilon(\lambda) : \mathscr{H} \to \mathscr{H}$ and $C_\varepsilon(\lambda) : L^2(\mathbb{R}^2) \to \mathscr{H}$, which are defined as

$$A_\varepsilon(\lambda)[f_l]_l = \sum_{y_l\in\Lambda}\int_{\mathbb{R}^2} G_\lambda(\cdot - \varepsilon y - y_l)v_l(y)f_l(y)\mathrm{d}y,\ [C_\varepsilon(\lambda)f]_k = \left[u_k\int_{\mathbb{R}^2} G_\lambda(\varepsilon\cdot - y + y_k)f(y)\mathrm{d}y\right]_k,$$

$$[B_\varepsilon(\lambda)[f_l]_l]_k = \left[\mu_k\big((\ln\varepsilon)^{-1}\big)f_k\right]_k,\ [E_\varepsilon(\lambda)[f_l]_l]_k = \left[\sum_{y_k\neq y_l\in\Lambda} u_k\int_{\mathbb{R}^2} G_\lambda(\varepsilon(\cdot - y) + y_k - y_l)v_l(y)f_l(y)\mathrm{d}y\right]_k,$$

$$[D_\varepsilon(\lambda)[f_l]_l]_k = \left[\mu_k\big((\ln\varepsilon)^{-1}\big)u_k\int_{\mathbb{R}^2} G_\lambda(\varepsilon(\cdot - y))v_k(y)f_k(y)\mathrm{d}y\right]_k.$$

First, we check convergence of $A_\varepsilon(\lambda)$ and $C_\varepsilon(\lambda)$. The natural candidates for their limits $A(\lambda)$ and $C(\lambda)$ are

$$A(\lambda)[f_l]_l = \sum_{y_l\in\Lambda}\int_{\mathbb{R}^2} G_\lambda(\cdot - y_l)v_l(y)f_l(y)\mathrm{d}y, \quad [C(\lambda)f]_k = \left[u_k\int_{\mathbb{R}^2} G_\lambda(y - y_k)f(y)\mathrm{d}y\right]_k.$$

In order to prove, that $C_\varepsilon(\lambda)$ converges to $C(\lambda)$ in the operator norm, we compute for $f \in L^2(\mathbb{R}^2)$

$$\begin{aligned}\|(C_\varepsilon(\lambda) - C(\lambda))f\|^2 &= \sum_{y_k\in\Lambda}\int_{\mathbb{R}^2}\left|u_k(x)\int_{\mathbb{R}^2}\left(G_\lambda(y-y_k) - G_\lambda(\varepsilon x - y + y_k)\right)f(y)\mathrm{d}y\right|^2\mathrm{d}x\\ &\leq \int_{\mathbb{R}^2}|V(x)|\int_{\mathbb{R}^2}|G_\lambda(y) - G_\lambda(y-\varepsilon x)|^2 e^{\mathrm{Im}\sqrt{\lambda}|y|}\mathrm{d}y\mathrm{d}x\int_{\mathbb{R}^2}\sum_{y_k\in\Lambda} e^{-\mathrm{Im}\sqrt{\lambda}|y-y_k|}|f(y)|^2\mathrm{d}y \leq c(\varepsilon)\|f\|^2,\end{aligned}$$

where we used the Cauchy-Schwarz inequality and then the translation invariance of the Lebesgue measure. Because of $G_\lambda(z) \sim \frac{i}{4}\sqrt{2/(\pi z)}e^{i(z-\frac{\pi}{2})}$ for $|z| \to \infty$, it follows from the dominated convergence theorem that

$$\int_{\mathbb{R}^2}|V(x)|\int_{\mathbb{R}^2}|G_\lambda(y) - G_\lambda(y-\varepsilon x)|^2 e^{\mathrm{Im}\sqrt{\lambda}|y|}\mathrm{d}y\mathrm{d}x \to 0.$$

Moreover, due to our assumptions on $\Lambda$, it holds that $\sum e^{-\mathrm{Im}\sqrt{\lambda}|y-y_k|}$ is uniformly bounded in $y$, as the sum can be estimated by a convergent geometric series. Hence, $c(\varepsilon) \to 0$ and thus $C_\varepsilon(\lambda) \to C(\lambda)$. A similar argument shows $A_\varepsilon(\lambda)^* \to A(\lambda)^*$ and hence $A_\varepsilon(\lambda) \to A(\lambda)$. Next, it follows from the well-known analysis of the one-center case [1,2], that $(1+D_\varepsilon(\lambda))^{-1}B_\varepsilon(\lambda) \to K(\lambda)$, where the components of the diagonal operator $K(\lambda)$ are

$$K_k(\lambda)f = -4\pi^2\left\{2\pi(u_k,v_k)^2\left[\ln\frac{\sqrt{\lambda}}{2i} - \gamma\right] + \mu_{k,2}(u_k,v_k)^3 + 2\pi(T_k u_k, v_k)\right\}^{-1}(f,v_k)u_k,$$

if $(u_k, v_k) \neq 0$ and $\mu_{k,1} = \frac{2\pi}{(u_k,v_k)}$, and $K_k(\lambda) = 0$ otherwise. It remains to analyze the convergence of $E_\varepsilon(\lambda)$. The natural candidate for the limit is

$$[E(\lambda)[f_l]_l]_k = \left[\sum_{y_k\neq y_l\in\Lambda} u_k G_\lambda(y_l - y_k)(f_l, v_l)\right]_k.$$

In order to show $E_\varepsilon(\lambda) \to E(\lambda)$, we use the estimate $\|T\|^2 \leq \sup_l\sum_k\|T_{kl}\|\cdot\sup_k\sum_l\|T_{kl}\|$, which holds for any $T = [T_{kl}]$ which is bounded and everywhere defined in $\mathscr{H}$. Therefore, it is sufficient to prove

$$\sup_l\sum_{y_l\neq y_k}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|V_k(x)|\cdot|G_\lambda(\varepsilon(y-x) + y_l - y_k) - G_\lambda(y_l - y_k)|^2\cdot|V_l(y)|\mathrm{d}y\mathrm{d}x\right)^{1/2} \to 0,$$

which can be shown using the far field behavior $G_\lambda(z) \sim \frac{i}{4}\sqrt{2/(\pi z)}e^{i(z-\frac{\pi}{2})}$ as $|z| \to \infty$ and the dominated convergence theorem. Finally, a straightforward, but tedious computation yields

$$\lim_{\varepsilon\to 0}(H_{\varepsilon,\Lambda} - \lambda)^{-1} = (-\Delta - \lambda)^{-1} - A(\lambda)[1 + K(\lambda)E(\lambda)]^{-1}K(\lambda)C(\lambda) = (-\Delta_{\alpha,\Lambda} - \lambda)^{-1}.$$

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