Boundary Value Problems with Local Generalized Nevanlinna Functions in the Boundary Condition

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Abstract. For a class of abstract λ-dependent boundary value problems where a local variant of generalized Nevanlinna functions appears in the boundary condition, linearizations are constructed and their local spectral properties are investigated.

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1. Introduction

In this paper we study a class of abstract λ-dependent boundary value problems with a local variant of generalized Nevanlinna functions appearing in the boundary condition. For this let A be a closed symmetric operator or relation with defect one in a separable Krein space $\mathcal{H}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for the adjoint relation A^+. We assume that the selfadjoint extension $A_0 := \ker \Gamma_0$ of A admits a spectral decomposition into two relations one of which acts in a Pontryagin space. A selfadjoint relation with this property is called locally of type π_+ (see Definition 3.3). Let τ be a function which can be written as a sum of a generalized Nevanlinna function and a locally holomorphic function; a so-called local generalized Nevanlinna function (see Definition 3.1). In Theorem 4.1 we investigate boundary value problems of the following form: For a given $h \in \mathcal{H}$ find a vector $\hat{f} = \binom{f}{f'} \in A^+$ such that

$$f' - \lambda f = h \quad \text{and} \quad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0 \tag{1.1}$$

holds. For a suitable $\lambda \in \mathbb{C}$ a solution of this boundary value problem can be obtained with the help of the compressed resolvent of a selfadjoint extension $\widetilde{A}$ of A which acts in a larger Krein space $\mathcal{H} \times \mathcal{K}$, i.e.

$$f = P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{H}}\, h \quad \text{and} \quad f' = \lambda f + h$$

fulfil (1.1). The relation $\widetilde{A}$ is called a *linearization* of (1.1). We construct $\widetilde{A}$ and investigate its local spectral properties, which are closely connected with the solvability of (1.1), with the help of the coupling method from [8, §5.2] and a perturbation result from [3]. Here we obtain that $\widetilde{A}$ is locally of type π_+.

We briefly describe the contents of this paper. In Section 2 we recall some basic facts on boundary value spaces and Weyl functions associated with symmetric relations in Krein spaces. In Section 3 it is shown that a local generalized Nevanlinna function can be expressed as the Weyl function corresponding to a symmetric relation and a suitable boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ where the selfadjoint relation $\ker \Gamma_0'$ is locally of type π_+. Section 4 contains our main result. Based on the approach in [8] we construct a linearization of the boundary value problem (1.1) which again is locally of type π_+. Under an additional assumption this linearization fulfils a minimality condition. In this case the linearization is, roughly speaking, locally uniquely determined up to unitary equivalence (Remark 4.4). As an example we consider in Section 5 a singular Sturm-Liouville operator with the indefinite weight $\operatorname{sgn} x$ and a λ-dependent interface condition.

2. Boundary value spaces and Weyl functions associated with a symmetric relation in a Krein space

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space with a corresponding fundamental symmetry J. The linear space of bounded linear operators defined on a Krein space $\mathcal{K}_1$ with values in a Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we simply write $\mathcal{L}(\mathcal{K})$. We study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K}^2$. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators in $\mathcal{K}$ are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [10]. The sum and the direct sum of subspaces in $\mathcal{K}^2$ is denoted by $\boldsymbol{+}$ and $\dot{\boldsymbol{+}}$. We define an indefinite inner product on $\mathcal{K}^2$ by

$$[\![\hat{f}, \hat{g}]\!] = i([f, g'] - [f', g]), \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix},\ \hat{g} = \begin{pmatrix} g \\ g' \end{pmatrix} \in \mathcal{K}^2. \tag{2.1}$$

Then $(\mathcal{K}^2, [\![\cdot,\cdot]\!])$ is a Krein space and $\mathcal{J} = \left(\begin{smallmatrix} 0 & -iJ \\ iJ & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{K}^2)$ is a corresponding fundamental symmetry. Observe that also in the special case when $(\mathcal{K}, [\cdot,\cdot])$ is a Hilbert space, $[\![\cdot,\cdot]\!]$ is an indefinite metric. In the following we shall use at the same time inner products $[\![\cdot,\cdot]\!]$ arising from different Krein and Hilbert spaces as in (2.1). Then we shall indicate these forms by subscripts, for example, $[\![\cdot,\cdot]\!]_{\mathcal{K}^2}$, $[\![\cdot,\cdot]\!]_{\mathcal{G}^2}$.

Let S be a linear relation in $\mathcal{K}$. The adjoint relation S^+ is defined as

$$S^{[\![\perp]\!]} = \{\hat{h} \in \mathcal{K}^2 \,|\, [\![\hat{h}, \hat{f}]\!] = 0 \text{ for all } \hat{f} \in S\}.$$

S is said to be *symmetric* (*selfadjoint*) if $S \subset S^+$ (resp. $S = S^+$). The resolvent set $\rho(S)$ of $S \in \widetilde{\mathcal{C}}(\mathcal{K})$ is the set of all $\lambda \in \mathbb{C}$ such that $(S-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, the spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. A point $\lambda \in \mathbb{C}$ is of regular type, $\lambda \in r(S)$, if $(S-\lambda)^{-1}$ is a bounded operator. For the definition of the point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ we refer to [10] and [11]. The extended spectrum $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise.

We say that a closed symmetric relation A has *defect* $m \in \mathbb{N} \cup \{\infty\}$, if both deficiency indices

$$n_\pm(JA) = \dim\ker\bigl((JA)^* - \overline{\lambda}\bigr), \qquad \lambda \in \mathbb{C}^\pm,$$

of the symmetric relation JA in the Hilbert space $(\mathcal{K}, [J\cdot,\cdot])$ are equal to m. With the help of the von Neumann formulas for a closed symmetric relation in a Hilbert space (see e.g. [7, §2.3]) one can verify without difficulty that this is equivalent to the fact that there exists a selfadjoint extension of A in $\mathcal{K}$ and that each selfadjoint extension $\hat{A}$ of A in $\mathcal{K}$ satisfies $\dim\bigl(\hat{A}/A\bigr) = m$.

We shall use the so-called boundary value spaces for the description of the selfadjoint extensions of closed symmetric relations in Krein spaces. The following definition is taken from [5].

Definition 2.1. Let A be a closed symmetric relation in the Krein space $\mathcal{K}$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *boundary value space* for A^+ if $\mathcal{G}$ is a Hilbert space and there exist mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$ such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : A^+ \to \mathcal{G}^2$ is surjective, and the relation

$$[\![\Gamma\hat{f}, \Gamma\hat{g}]\!]_{\mathcal{G}^2} = [\![\hat{f}, \hat{g}]\!]_{\mathcal{K}^2}$$

holds for all $\hat{f}, \hat{g} \in A^+$.

In the following we recall some basic facts on boundary value spaces which can be found in e.g. [4] and [5]. For the Hilbert space case we refer to [12], [6] and [7]. Let A, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and Γ be as in Definition 2.1. It follows that the mappings Γ_0 and Γ_1 are continuous. The selfadjoint extensions

$$A_0 := \ker\Gamma_0 \quad \text{and} \quad A_1 := \ker\Gamma_1$$

of A are transversal, that is $A_0 \cap A_1 = A$ and $A_0 \dotplus A_1 = A^+$. The mapping Γ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \{\hat{f} \in A^+ \,|\, \Gamma\hat{f} \in \Theta\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{2.2}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between the set of all closed linear relations $\widetilde{\mathcal{C}}(\mathcal{G})$ in $\mathcal{G}$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (2.2) gives a one-to-one correspondence between the symmetric (selfadjoint) extensions of A

and the symmetric (resp. selfadjoint) relations in $\mathcal{G}$. If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker\bigl(\Gamma_1 - \Theta\Gamma_0\bigr). \tag{2.3}$$

Let again A be a closed symmetric relation in $\mathcal{K}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ and assume that $A_0 = \ker \Gamma_0$ has a nonempty resolvent set. Let $\mathcal{N}_{\lambda,A^+} := \ker(A^+ - \lambda) = \operatorname{ran}(A - \overline{\lambda})^{[\perp]}$, $\lambda \in r(A)$, be the defect subspace of A and let

$$\hat{\mathcal{N}}_{\lambda,A^+} = \bigl\{\bigl(\begin{smallmatrix} f_\lambda \\ \lambda f_\lambda \end{smallmatrix}\bigr) \big| f_\lambda \in \mathcal{N}_{\lambda,A^+}\bigr\}. \tag{2.4}$$

When no confusion can arise we will simply write $\mathcal{N}_\lambda$ and $\hat{\mathcal{N}}_\lambda$ instead of $\mathcal{N}_{\lambda,A^+}$ and $\hat{\mathcal{N}}_{\lambda,A^+}$. We have

$$A^+ = A_0 \dot{+} \hat{\mathcal{N}}_\lambda \quad \text{for all} \quad \lambda \in \rho(A_0) \tag{2.5}$$

(see e.g. [5]). By π_1 we denote the orthogonal projection onto the first component of $\mathcal{K}^2$. For every $\lambda \in \rho(A_0)$ we define the operators

$$\gamma(\lambda) = \pi_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G},\mathcal{K}) \quad \text{and} \quad M(\lambda) = \Gamma_1(\Gamma_0|\hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}). \tag{2.6}$$

The functions $\lambda \mapsto \gamma(\lambda)$ and $\lambda \mapsto M(\lambda)$ are called the *γ-field* and *Weyl function* corresponding to A and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. γ and M are holomorphic on $\rho(A_0)$ and the relations

$$\gamma(\zeta) = (1 + (\zeta - \lambda)(A_0 - \zeta)^{-1})\gamma(\lambda)$$

and

$$M(\lambda) - M(\zeta)^* = (\lambda - \overline{\zeta})\gamma(\zeta)^+\gamma(\lambda)$$

hold for $\lambda, \zeta \in \rho(A_0)$ (see e.g. [5]). A little calculation yields

$$\begin{aligned} M(\lambda) = \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^+\bigl((\lambda - \operatorname{Re}\lambda_0) \\ + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\bigr)\gamma(\lambda_0) \end{aligned} \tag{2.7}$$

for all $\lambda \in \rho(A_0)$ and a fixed $\lambda_0 \in \rho(A_0)$.

The following well-known theorem shows how the spectra of closed extensions of A can be described with the help of the Weyl function. For a proof see e.g. [5].

Theorem 2.2. *Let A be a closed symmetric relation in a Krein space $\mathcal{K}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ where $A_0 = \ker \Gamma_0$ has a nonempty resolvent set. Denote by γ and M the corresponding γ-field and Weyl function, let $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and let A_Θ be the corresponding extension. For $\lambda \in \rho(A_0)$ the following assertions are true.*

(i) $\lambda \in \sigma_i(A_\Theta)$ *if and only if* $0 \in \sigma_i(\Theta - M(\lambda))$, $i = p, c, r$.

(ii) $\lambda \in \rho(A_\Theta)$ *if and only if* $0 \in \rho(\Theta - M(\lambda))$.

(iii) *For all* $\lambda \in \rho(A_\Theta) \cap \rho(A_0)$

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\bigl(\Theta - M(\lambda)\bigr)^{-1}\gamma(\overline{\lambda})^+.$$

3. Local generalized Nevanlinna functions as Weyl functions of symmetric relations

Recall that a piecewise meromorphic function τ in $\mathbb{C}\backslash\mathbb{R}$ which is symmetric with respect to the real axis (that is $\tau(\overline{\lambda}) = \overline{\tau(\lambda)}$ for all points λ where τ is holomorphic) is a generalized Nevanlinna function if the kernel

$$N_\tau(\lambda, \mu) := \frac{\tau(\lambda) - \tau(\overline{\mu})}{\lambda - \overline{\mu}}$$

has a finite number of negative squares. Here we consider a local variant of generalized Nevanlinna functions. We recall the definition of the class of local generalized Nevanlinna functions, which is a subclass of the class of the so-called locally definitizable functions (see [17]).

Let Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected.

Definition 3.1. Let τ be a piecewise meromorphic function in $\Omega\backslash\overline{\mathbb{R}}$ which is symmetric with respect to the real axis. We say that τ is a *local generalized Nevanlinna function in* Ω, if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, τ can be written in the form

$$\tau = \tau_0 + \tau_{(0)},$$

where τ_0 is a generalized Nevanlinna function and $\tau_{(0)}$ is a holomorphic function in Ω'. The class of local generalized Nevanlinna functions in Ω is denoted by $N(\Omega)$.

The class $N(\overline{\mathbb{C}})$ coincides with the class of generalized Nevanlinna functions (see [17]). Note, that for $\tau \in N(\Omega)$ the nonreal poles of τ in Ω do not accumulate to $\Omega \cap \overline{\mathbb{R}}$. The set of the points of holomorphy of τ in $\Omega\backslash\overline{\mathbb{R}}$ and all points $\lambda \in \Omega \cap \mathbb{R}$ such τ can be analytically continued to λ and the continuations from $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ coincide, is denoted by $\mathfrak{h}(\tau)$.

Below we shall make use of the following lemma.

Lemma 3.2. *Let* Δ *be a connected open subset of* $\overline{\mathbb{R}}$ *such that* $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$*, and let* $\tau \in N(\Omega)$ *be locally holomorphic on* Δ*. If* τ *is not the zero function, then the zeros of* τ *in* Δ *do not accumulate to the endpoints of* Δ*.*

Proof. It is no restriction to assume that Δ is a bounded open interval (a, b) such that $[a, b] \subset \Omega \cap \mathbb{R}$. The general case can be reduced to this case by a linear fractional transformation of the variable.

Suppose that for some $\varepsilon > 0$ the set of all zeros of τ in $(a, a + \varepsilon)$ consists of the elements of a sequence $(a_i)_{i=1}^{\infty}$ with $a_1 > a_2 > \ldots$, $\lim_{i\to\infty} a_i = a$. Since $-\tau^{-1} \in N(\Omega)$ (see [1]) there exists an N_κ function ν_0, $\kappa \in \mathbb{N} \cup \{0\}$, and a function $\nu_{(0)}$ locally holomorphic on $[a, b]$ such that

$$-\tau(\lambda)^{-1} = \nu_0(\lambda) + \nu_{(0)}(\lambda)$$

for all points of holomorphy of ν_0 and $\nu_{(0)}$. Then ν_0 is meromorphic on some neighbourhood (in $\mathbb{C}$) of $(a, a+\varepsilon)$ and the points a_i, $i = 1, 2, \dots$, are just the poles of ν_0 in $(a, a+\varepsilon)$. By the well-known product representation of generalized Nevanlinna functions (see [9]) there exist an $\varepsilon_0 \in (0, \varepsilon)$, a positive function χ on $(a, a+\varepsilon_0)$ and a Nevanlinna function ν such that $\nu_0(\lambda) = \chi(\lambda)\nu(\lambda)$ for all points of holomorphy of ν_0 in $(a, a+\varepsilon_0)$. The poles of ν_0 in $(a, a+\varepsilon_0)$ coincide with the poles of ν in $(a, a+\varepsilon_0)$. Between two neighbouring poles of the Nevanlinna function ν the function $\chi\nu + \nu_{(0)} = -\tau^{-1}$ has a zero. Therefore, τ has a pole in every interval (a_{i+1}, a_i) with $a_i < a+\varepsilon_0$, which contradicts the fact that τ is locally holomorphic on Δ. □

In Section 4 we will make use of the fact that every local generalized Nevanlinna function coincides with the Weyl function of some boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for a closed symmetric relation where the selfadjoint relation $\ker \Gamma_0'$ has special spectral properties. For this representation we need the following subclass of locally definitizable selfadjoint relations in a Krein space (see [16]).

Definition 3.3. Let Ω be a domain as in the beginning of this section and let A_0 be a selfadjoint relation in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. A_0 is said to be of *type* π_+ *(positive type) over* Ω if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a selfadjoint projection E in $\mathcal{K}$ such that A_0 can be decomposed in

$$A_0 = \big(A_0 \cap (E\mathcal{K})^2\big) \dotplus \big(A_0 \cap ((1-E)\mathcal{K})^2\big)$$

and the following holds.

(i) $(E\mathcal{K}, [\cdot,\cdot])$ is a Pontryagin space (resp. Hilbert space), $\rho(A_0 \cap (E\mathcal{K})^2) \neq \emptyset$.
(ii) $\widetilde{\sigma}\big(A_0 \cap ((1-E)\mathcal{K})^2\big) \cap \Omega' = \emptyset$.

The selfadjoint relation A_0 is said to be of *type* π_- *(negative type) over* Ω if A_0 is of type π_+ (resp. positive type) in the Krein space $(\mathcal{K}, -[\cdot,\cdot])$.

If A_0 is a selfadjoint relation in the Krein space $\mathcal{K}$ we shall say that an open subset $\Delta \subset \overline{\mathbb{R}}$ is of *positive type* (*negative type*, *type* π_+, *type* π_-) with respect to A_0 if there exists a domain Ω as above, $\Omega \cap \overline{\mathbb{R}} = \Delta$, such that A_0 is of positive type (resp. negative type, type π_+, type π_-) over Ω.

Let now A_0 be a selfadjoint relation in $\mathcal{K}$ which is of type π_+ over some domain Ω. Then the set $\widetilde{\sigma}(A_0) \cap (\Omega \backslash \overline{\mathbb{R}})$ is discrete and the nonreal spectrum of A_0 in Ω does not accumulate to $\Omega \cap \overline{\mathbb{R}}$. If A_0 is of positive type over Ω then $\widetilde{\sigma}(A_0) \cap (\Omega \backslash \overline{\mathbb{R}})$ is empty. We remark that the spectral points in $\Omega \cap \overline{\mathbb{R}}$ can also be characterized with the help of approximative eigensequences (see e.g. [2], [18], [16]). Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let E be a selfadjoint projection with the properties as in Definition 3.3. If E' is the spectral function of the selfadjoint relation $A_0 \cap (E\mathcal{K})^2$ in the Pontryagin space $E\mathcal{K}$, then the mapping

$$\delta \mapsto E'(\delta)E =: E_{A_0}(\delta) \qquad (3.1)$$

defined for all finite unions δ of connected subsets of $\Omega' \cap \overline{\mathbb{R}}$ the endpoints of which belong to $\Omega' \cap \overline{\mathbb{R}}$ and are not critical points of $A_0 \cap (E\mathcal{K})^2$, is the spectral function of A_0 on $\Omega' \cap \overline{\mathbb{R}}$ (see [16, Section 3.4, Remark 4.9]). $E_{A_0}(\cdot)$ does not depend on the choice of E.

Let $A \subset A_0$ be a closed symmetric relation with defect one and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ with $\ker \Gamma_0 = A_0$. We denote the corresponding γ-field and Weyl function by γ and M, respectively. Here $\gamma(\lambda) \in \mathcal{L}(\mathbb{C}, \mathcal{K})$ for $\lambda \in \rho(A_0)$, and M is a scalar function. From (2.7) and the assumption on A_0 we conclude that the Weyl function M can be written as the sum of the generalized Nevanlinna function

$$M_0(\lambda) := \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^+ \big((\lambda - \operatorname{Re}\lambda_0) \\ + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big) E \gamma(\lambda_0)$$

and the function

$$M_{(0)}(\lambda) := \gamma(\lambda_0)^+ \big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)(1 - E)\gamma(\lambda_0)$$

which is holomorphic in Ω'. Therefore, $M \in N(\Omega)$.

Assume now that a function $\tau \in N(\Omega)$ is given. In [17] it was shown that for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a Krein space $(\mathcal{K}, [\cdot,\cdot])$, a selfadjoint relation T_0 in $\mathcal{K}$ of type π_+ over Ω' with

$$\rho(T_0) \cap \Omega' = \mathfrak{h}(\tau) \cap \Omega', \tag{3.2}$$

and an element $e \in \mathcal{K}$ such that for a fixed $\lambda_0 \in \Omega' \cap \mathfrak{h}(\tau)$ and every $\lambda \in \Omega' \cap \mathfrak{h}(\tau)$ the relation

$$\tau(\lambda) = \operatorname{Re}\tau(\lambda_0) + (\lambda - \operatorname{Re}\lambda_0)[e,e] + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)\big[(T_0 - \lambda)^{-1} e, e\big] \tag{3.3}$$

holds.

The representation (3.3) is called *minimal* if

$$\mathcal{K} = \operatorname{clsp}\big\{(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1})e \,|\, \lambda \in \rho(T_0) \cap \Omega'\big\} \tag{3.4}$$

holds for some $\lambda_0 \in \rho(T_0) \cap \Omega'$. Such a minimal representation of τ exists e.g. if, in addition, τ is the restriction of a generalized Nevanlinna function or a so-called definitizable function (see [14], [15]) to Ω' or if, in addition, the boundary of Ω' is contained in $\mathfrak{h}(\tau)$.

Making use of the representation (3.3) we construct in the following theorem a boundary value space such that $\tau \in N(\Omega)$ is its Weyl function. The idea of the proof is the same as in the proof of [6, Theorem 1].

Theorem 3.4. *Let Ω be as in the beginning of this section and let $\tau \in N(\Omega)$ be nonconstant. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let τ be represented with a selfadjoint relation T_0 of type π_+ over Ω' in a Krein space $\mathcal{K}$ as in* (3.2)-(3.3). *Then there exists a closed symmetric relation $T \subset T_0$ of defect one and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+ such that $\ker \Gamma_0' = T_0$ and τ coincides with the corresponding Weyl function on Ω'.*

In the case $\Omega = \overline{\mathbb{C}}$ Theorem 3.4 reads as follows.

Corollary 3.5. *Let τ be a nonconstant generalized Nevanlinna function. Then there exists a closed symmetric relation T in a Pontryagin space $\mathcal{K}$ with finite rank of negativity and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ is the corresponding Weyl function.*

Proof of Theorem 3.4. The assumption that τ is not constant implies that the vector $e \in \mathcal{K}$ in the representation (3.3) is not zero. For every $\lambda \in \Omega' \cap \mathfrak{h}(\tau)$ and a fixed $\lambda_0 \in \Omega' \cap \mathfrak{h}(\tau)$ we define

$$\gamma'(\lambda) := \big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)e, \tag{3.5}$$

which implies $\gamma'(\lambda_0) = e$, $\gamma'(\zeta) = (1 + (\zeta - \lambda)(T_0 - \zeta)^{-1})\gamma'(\lambda)$ and $\gamma'(\lambda) \neq 0$ for all $\lambda, \zeta \in \Omega' \cap \mathfrak{h}(\tau)$. For some $\mu \in \Omega' \cap \mathfrak{h}(\tau)$ we define the closed symmetric relation

$$T := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in T_0 \,\middle|\, [g - \overline{\mu} f, \gamma'(\mu)] = 0 \right\} \tag{3.6}$$

in $\mathcal{K}$. As

$$\begin{aligned} [g - \overline{\mu}' f, \gamma'(\mu')] &= \big[g - \overline{\mu}' f, (1 + (\mu' - \mu)(T_0 - \mu')^{-1})\gamma'(\mu)\big] \\ &= [g - \overline{\mu} f, \gamma'(\mu)] \end{aligned}$$

for all $\binom{f}{g} \in T_0$ and $\mu' \in \Omega' \cap \mathfrak{h}(\tau)$, the relation T does not depend on the choice of μ. By (3.6) we have $\mathcal{N}_\mu = \operatorname{ran}(T - \overline{\mu})^{[\perp]} = \operatorname{sp}\gamma'(\mu)$.

Now we regard $\gamma'(\lambda)$, $\lambda \in \Omega' \cap \mathfrak{h}(\tau)$, as the linear mapping $\mathbb{C} \ni c \mapsto c\,\gamma'(\lambda) \in \mathcal{K}$ and denote the linear functional $c\,\gamma'(\lambda) \mapsto c$ defined on $\mathcal{N}_\lambda = \operatorname{sp}\gamma'(\lambda)$ by $\gamma'(\lambda)^{(-1)}$.

We write the elements $\hat{f} \in T^+$, for every $\lambda \in \Omega' \cap \mathfrak{h}(\tau)$, in the form

$$\hat{f} = \begin{pmatrix} f_0 \\ f_0' \end{pmatrix} + \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix},$$

where $\binom{f_0}{f_0'} \in T_0$ and $f_\lambda \in \mathcal{N}_\lambda$ (see (2.5)). Let $\Gamma_0', \Gamma_1' : T^+ \to \mathbb{C}$ be the linear functionals defined by

$$\begin{aligned} \Gamma_0' \hat{f} &:= \gamma'(\lambda)^{(-1)} f_\lambda, \\ \Gamma_1' \hat{f} &:= \gamma'(\lambda)^+ (f_0' - \overline{\lambda} f_0) + \tau(\lambda)\gamma'(\lambda)^{(-1)} f_\lambda. \end{aligned} \tag{3.7}$$

The mapping $\Gamma' := \binom{\Gamma_0'}{\Gamma_1'} : T^+ \to \mathbb{C}^2$ is surjective. Indeed, let $\binom{h_1}{h_2} \in \mathbb{C}^2$ and set $f_\lambda := \gamma'(\lambda) h_1 \in \mathcal{N}_\lambda$. Since, by the relation $\{0\} = \ker\gamma'(\lambda) = (\operatorname{ran}\gamma'(\lambda)^+)^\perp$, $\gamma'(\lambda)^+$ is surjective, there exists $\binom{f_0}{f_0'} \in T_0$ such that $\gamma'(\lambda)^+(f_0' - \overline{\lambda} f_0) = h_2 - \tau(\lambda) h_1$. Then

$$\Gamma' \left(\begin{pmatrix} f_0 \\ f_0' \end{pmatrix} + \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} \right) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Making use of the relation $\tau(\lambda)-\tau(\overline{\zeta})=(\lambda-\overline{\zeta})\gamma'(\zeta)^+\gamma'(\lambda)$, which can be verified by a straightforward calculation, we obtain

$$\begin{aligned}
[\![\hat f,\hat g]\!]_{\mathcal{K}^2} &= \Big[\!\Big[\begin{pmatrix} f_0\\ f_0'\end{pmatrix}+\begin{pmatrix} f_\lambda\\ \lambda f_\lambda\end{pmatrix},\begin{pmatrix} g_0\\ g_0'\end{pmatrix}+\begin{pmatrix} g_\lambda\\ \lambda g_\lambda\end{pmatrix}\Big]\!\Big]_{\mathcal{K}^2}\\
&= i\big([f_\lambda, g_0'-\overline{\lambda}g_0]-[f_0'-\overline{\lambda}f_0, g_\lambda]-[(\lambda-\overline{\lambda})f_\lambda, g_\lambda]\big)\\
&= i\Big(\big(\gamma'(\lambda)^{(-1)}f_\lambda,\gamma'(\lambda)^+(g_0'-\overline{\lambda}g_0)\big)-\big(\gamma'(\lambda)^+(f_0'-\overline{\lambda}f_0),\gamma'(\lambda)^{(-1)}g_\lambda\big)\\
&\qquad -\big((\tau(\lambda)-\tau(\overline{\lambda}))\gamma'(\lambda)^{(-1)}f_\lambda,\gamma'(\lambda)^{(-1)}g_\lambda\big)\Big)\\
&= i\Big(\big(\gamma'(\lambda)^{(-1)}f_\lambda,\gamma'(\lambda)^+(g_0'-\overline{\lambda}g_0)+\tau(\lambda)\gamma'(\lambda)^{(-1)}g_\lambda\big)\\
&\qquad -\big(\gamma'(\lambda)^+(f_0'-\overline{\lambda}f_0)+\tau(\lambda)\gamma'(\lambda)^{(-1)}f_\lambda,\gamma'(\lambda)^{(-1)}g_\lambda\big)\Big)\\
&= [\![\Gamma'\hat f,\Gamma'\hat g]\!]_{\mathbb{C}^2}.
\end{aligned}$$

Hence $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ is a boundary value space for T^+. Moreover, we have $\ker\Gamma_0'=T_0$ and the corresponding γ-field coincides with γ'. For $h=\begin{pmatrix} h_\lambda\\ \lambda h_\lambda\end{pmatrix}\in\hat{\mathcal{N}}_\lambda$ we obtain

$$\tau(\lambda)\Gamma_0'h=\tau(\lambda)\gamma'(\lambda)^{(-1)}h_\lambda=\Gamma_1'h.$$

Therefore τ coincides with the Weyl function of T on Ω' corresponding to the boundary value space $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ defined in (3.7). □

4. Boundary value problems with spectral parameter in the boundary condition

In this section we consider a class of abstract boundary value problems of the form (1.1) where the spectral parameter appears nonlinearly in the boundary condition. Theorem 4.1 and Corollary 4.2 extend results obtained with the help of the coupling method in [8] for a symmetric operator A in a Hilbert space and a Nevanlinna function τ in the boundary condition. In contrast to [8] we consider only the case where τ is a scalar function.

Let $(\mathcal{H},[\cdot,\cdot]_{\mathcal{H}})$ and $(\mathcal{K},[\cdot,\cdot]_{\mathcal{K}})$ be Krein spaces. The elements of $\mathcal{H}\times\mathcal{K}$ will be written in the form $\{h,k\}$, $h\in\mathcal{H}$, $k\in\mathcal{K}$. $\mathcal{H}\times\mathcal{K}$ equipped with the inner product $[\cdot,\cdot]$ defined by

$$[\{h_1,k_1\},\{h_2,k_2\}]:=[h_1,h_2]_{\mathcal{H}}+[k_1,k_2]_{\mathcal{K}},\quad h_1,h_2\in\mathcal{H},\ \ k_1,k_2\in\mathcal{K},$$

is a Krein space. If A is a relation in $\mathcal{H}$ and T is a relation in $\mathcal{K}$ we shall write $A\times T$ for the direct product of A and T which is a relation in $\mathcal{H}\times\mathcal{K}$,

$$A\times T=\left\{\begin{pmatrix}\{a,t\}\\ \{a',t'\}\end{pmatrix}\ \Big|\ \begin{pmatrix} a\\ a'\end{pmatrix}\in A,\ \begin{pmatrix} t\\ t'\end{pmatrix}\in T\right\}. \tag{4.1}$$

For the pair $\begin{pmatrix}\{a,t\}\\ \{a',t'\}\end{pmatrix}$ on the right hand side of (4.1) we shall also write $\{\hat a,\hat t\}$, where $\hat a=\begin{pmatrix} a\\ a'\end{pmatrix}$, $\hat t=\begin{pmatrix} t\\ t'\end{pmatrix}$.

Let, as in Section 3, Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected and let Ω' be a domain with the same properties as Ω such that $\Omega' \subset \Omega$. Theorem 3.4 and the remarks preceding this theorem show that under the additional assumption $\overline{\Omega'} \subset \Omega$ the condition $(\mathcal{T})$ in the following theorem is always fulfilled.

Theorem 4.1. *Let A be a closed symmetric relation of defect one in the Krein space $\mathcal{H}$ and assume that there exists a selfadjoint extension A_0 of A which is of type π_+ over Ω. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+, $A_0 = \ker \Gamma_0$, and denote by γ and M the corresponding γ-field and the Weyl function, respectively. Let $\tau \in N(\Omega)$ be nonconstant, assume that $M + \tau$ is not identically equal to zero in $\Omega \backslash \overline{\mathbb{R}}$ and that the following condition $(\mathcal{T})$ is fulfilled.*

$(\mathcal{T})$ There exist a closed symmetric relation T in a Krein space $\mathcal{K}$ and a boundary value space $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ coincides with the corresponding Weyl function on Ω', $T_0 = \ker \Gamma_0'$ is of type π_+ over Ω' and $\mathfrak{h}(\tau) \cap \Omega' = \rho(T_0) \cap \Omega'$ holds.

Then the relation

$$\widetilde{A} = \left\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,|\, \Gamma_1 \hat{f}_1 - \Gamma_1' \hat{f}_2 = \Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = 0 \right\} \tag{4.2}$$

is a selfadjoint extension of A in $\mathcal{H} \times \mathcal{K}$ which is of type π_+ over Ω'. For every $h \in \mathcal{H}$ and every $\lambda \in \rho(\widetilde{A}) \cap \mathfrak{h}(\tau) \cap \Omega'$ a solution of the λ-dependent boundary value problem

$$f_1' - \lambda f_1 = h, \quad \tau(\lambda)\Gamma_0 \hat{f}_1 + \Gamma_1 \hat{f}_1 = 0, \quad \hat{f}_1 = \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in A^+, \tag{4.3}$$

is given by

$$f_1 = P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\{h, 0\} \quad \textit{and} \quad f_1' = \lambda f_1 + h. \tag{4.4}$$

The open set $\rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega'$ with the exception of the points of the discrete subset

$$\Sigma := \left\{ \mu \in \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega' \,|\, M(\mu) + \tau(\mu) = 0 \right\}$$

is contained in $\rho(\widetilde{A})$. For every open connected subset $\Delta \subset \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega' \cap \overline{\mathbb{R}}$ the set $\Sigma \cap \Delta$ is finite and the points in $\Sigma \cap (\Omega' \backslash \overline{\mathbb{R}})$ do not accumulate to $\Omega' \cap \overline{\mathbb{R}}$. For all $\lambda \in \big(\rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega'\big) \backslash \Sigma$ we have

$$P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{H}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+. \tag{4.5}$$

If, in addition, the representation (3.3) of τ on Ω' is minimal, then $\widetilde{A}$ satisfies the minimality condition

$$\operatorname{clsp}\left\{ (1 + (\lambda - \lambda_0)(\widetilde{A} - \lambda)^{-1})\{h, 0\} \,|\, h \in \mathcal{H}, \, \lambda \in \rho(\widetilde{A}) \cap \Omega' \right\} = \mathcal{H} \times \mathcal{K}, \tag{4.6}$$

for some $\lambda_0 \in \rho(\widetilde{A}) \cap \Omega'$.

An analogous statement holds if A_0 is of type π_- over Ω and $-\tau$ belongs to the class $N(\Omega)$.

In the next corollary we consider the special case that $\mathcal{H}$ is a Pontryagin space and τ is a generalized Nevanlinna function.

Corollary 4.2. *Let A be a closed symmetric relation of defect one in the Pontryagin space $\mathcal{H}$ with finite rank of negativity and assume that there exists a selfadjoint extension A_0 which has a nonempty resolvent set. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+, $A_0 = \ker \Gamma_0$, and denote by γ and M the corresponding γ-field and the Weyl function, respectively.*
Let $\tau \in N(\overline{\mathbb{C}})$ be nonconstant and assume that $M + \tau$ is not identically equal to zero. Let T_0 be a minimal representing relation for τ in a Pontryagin space $\mathcal{K}$, let $T \subset T_0$ be a closed symmetric relation of defect one and let $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ be a boundary value space for T^+ such that τ is the corresponding Weyl function and $T_0 = \ker \Gamma_0'$ (see Corollary 3.5).

Then the relation $\widetilde{A}$ in (4.2) is a selfadjoint extension of A in the Pontryagin space $\mathcal{H} \times \mathcal{K}$, and $\widetilde{A}$ is minimal, that is (4.6) holds with $\rho(\widetilde{A}) \cap \Omega'$ replaced by $\rho(\widetilde{A})$. For every $h \in \mathcal{H}$ and every $\lambda \in \rho(\widetilde{A}) \cap \mathfrak{h}(\tau)$ a solution of the λ-dependent boundary value problem (4.3) is given by (4.4).

The open set $\rho(A_0) \cap \mathfrak{h}(\tau)$ with the exception of the points of the discrete subset $\Sigma = \{\mu \in \rho(A_0) \cap \mathfrak{h}(\tau) \mid M(\mu) + \tau(\mu) = 0\}$ is contained in $\rho(\widetilde{A})$. $\Sigma \cap (\mathbb{C} \backslash \mathbb{R})$ is finite and for every open connected subset $\Delta \subset \rho(A_0) \cap \mathfrak{h}(\tau) \cap \overline{\mathbb{R}}$ the set $\Sigma \cap \Delta$ is finite. For all $\lambda \in (\rho(A_0) \cap \mathfrak{h}(\tau)) \backslash \Sigma$ the compressed resolvent of $\widetilde{A}$ onto $\mathcal{H}$ is given by (4.5).

An analogous statement holds if $\mathcal{H}$ is a Pontryagin space with finite rank of positivity and $-\tau$ belongs to the class $N(\overline{\mathbb{C}})$

Proof of Theorem 4.1. As was shown below Definition 3.3 the Weyl function M corresponding to the boundary value space $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a local generalized Nevanlinna function in Ω. Since τ and $M + \tau$ belong to the class $N(\Omega)$ the function $-(M + \tau)^{-1}$ belongs also to $N(\Omega)$ (see [1]). Therefore its nonreal poles in Ω do not accumulate to $\Omega \cap \overline{\mathbb{R}}$ and we conclude that the set

$$\Sigma := \bigl\{\mu \in \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega' \mid M(\mu) + \tau(\mu) = 0\bigr\}$$

is discrete in $\rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega'$ and the nonreal points of Σ do not accumulate to points in $\Omega' \cap \overline{\mathbb{R}}$. In the case $\overline{\Omega'} \subset \Omega$ the set $\Sigma \cap (\Omega' \backslash \overline{\mathbb{R}})$ is finite. If Δ is an open connected subset of $\rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega' \cap \overline{\mathbb{R}}$ then Lemma 3.2 applied to the function $M + \tau$ implies that $\Sigma \cap \Delta$ is finite. We define the set

$$\mathfrak{h}_0 := \rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\bigl((M + \tau)^{-1}\bigr) \cap \Omega'.$$

Then $(\rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega') \backslash \Sigma = \mathfrak{h}_0$.

Let $\mathcal{K}$, $T \subset T_0 \subset T^+$ and $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ be as in the assumptions of the theorem and let γ' be the γ-field corresponding to $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$. We define the mappings $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1 : A^+ \times T^+ \to \mathbb{C}^2$ by

$$\widetilde{\Gamma}_0 = \begin{pmatrix} \Gamma_0 & 0 \\ 0 & \Gamma_0' \end{pmatrix} \quad \text{and} \quad \widetilde{\Gamma}_1 = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_1' \end{pmatrix}.$$

It is easy to see that $\{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary value space for $A^+ \times T^+$. On account of $\hat{\mathcal{N}}_{\lambda,A^+\times T^+} = \hat{\mathcal{N}}_{\lambda,A^+} \times \hat{\mathcal{N}}_{\lambda,T^+}$ (see (2.4)), it follows that the γ-field $\widetilde{\gamma}$ corresponding to $\{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is given by

$$\widetilde{\gamma}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma'(\lambda) \end{pmatrix}, \quad \lambda \in \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega', \tag{4.7}$$

and the corresponding Weyl function $\widetilde{M}$ is

$$\widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}, \quad \lambda \in \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega',$$

(see (2.6)).

The selfadjoint relation $\widetilde{A}$ in $\mathcal{H} \times \mathcal{K}$ corresponding to the selfadjoint relation

$$\Theta := \left\{ \begin{pmatrix} \{u, -u\} \\ \{v, v\} \end{pmatrix} \,\Big|\, u, v \in \mathbb{C} \right\} \in \widetilde{\mathcal{C}}(\mathbb{C}^2)$$

via (2.2) is given by

$$\widetilde{A} = \big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,|\, \Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = \Gamma_1 \hat{f}_1 - \Gamma_1' \hat{f}_2 = 0 \big\}. \tag{4.8}$$

For $\lambda \in \mathfrak{h}_0$ we have

$$\begin{aligned} \big(\Theta - \widetilde{M}(\lambda)\big)^{-1} &= \left\{ \begin{pmatrix} \{v - M(\lambda)u, v + \tau(\lambda)u\} \\ \{u, -u\} \end{pmatrix} \,\Big|\, u, v \in \mathbb{C} \right\} \\ &= \begin{pmatrix} -\big(M(\lambda) + \tau(\lambda)\big)^{-1} & \big(M(\lambda) + \tau(\lambda)\big)^{-1} \\ \big(M(\lambda) + \tau(\lambda)\big)^{-1} & -\big(M(\lambda) + \tau(\lambda)\big)^{-1} \end{pmatrix}. \end{aligned} \tag{4.9}$$

By Theorem 2.2 a point $\lambda \in \rho(A_0) \cap \mathfrak{h}(\tau) \cap \Omega'$ belongs to $\rho(\widetilde{A})$ if and only if $0 \in \rho(\Theta - \widetilde{M}(\lambda))$, that is, $M(\lambda) + \tau(\lambda) \neq 0$. Hence for $\lambda \in \mathfrak{h}_0$ Theorem 2.2 implies

$$(\widetilde{A} - \lambda)^{-1} = \begin{pmatrix} (A_0 - \lambda)^{-1} & 0 \\ 0 & (T_0 - \lambda)^{-1} \end{pmatrix} + \widetilde{\gamma}(\lambda)\big(\Theta - \widetilde{M}(\lambda)\big)^{-1} \widetilde{\gamma}(\overline{\lambda})^+ \tag{4.10}$$

and we obtain from (4.7), (4.9) and (4.10) that the compressed resolvent is given by

$$P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{H}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1} \gamma(\overline{\lambda})^+, \quad \lambda \in \mathfrak{h}_0.$$

By our assumptions and the properties of T_0 the selfadjoint extension $A_0 \times T_0$ of $A \times T$ in $\mathcal{H} \times \mathcal{K}$ is of type π_+ over Ω'. Since the defect of A and T is one

$$(A_0 \times T_0 - \lambda)^{-1} - (\widetilde{A} - \lambda)^{-1}, \qquad \lambda \in \mathfrak{h}_0,$$

is a rank two operator. Making use of [3, Theorem 2.4] we conclude that $\widetilde{A}$ is of type π_+ over Ω'.

Let us show that for $\lambda \in \rho(\widetilde{A}) \cap \mathfrak{h}(\tau) \cap \Omega'$ the compressed resolvent of $\widetilde{A}$ onto $\mathcal{H}$ is a solution of (4.3). For a given $h \in \mathcal{H}$ we define

$$f_1 := P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\{h, 0\} \quad \text{and} \quad f_2 := P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}\{h, 0\}.$$

Then

$$\begin{pmatrix} \{f_1, f_2\} \\ \{\lambda f_1 + h, \lambda f_2\} \end{pmatrix} \in \widetilde{A}.$$

Since $\widetilde{A} \subset A^+ \times T^+$ we have $\hat{f}_1 := \left(\begin{smallmatrix} f_1 \\ \lambda f_1 + h \end{smallmatrix}\right) \in A^+$ and $\hat{f}_2 := \left(\begin{smallmatrix} f_2 \\ \lambda f_2 \end{smallmatrix}\right) \in \hat{\mathcal{N}}_{\lambda, T^+}$. By (4.8), and since τ is the Weyl function of $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$, we obtain

$$\Gamma_1 \hat{f}_1 = \Gamma_1' \hat{f}_2 = \tau(\lambda)\Gamma_0' \hat{f}_2 = -\tau(\lambda)\Gamma_0 \hat{f}_1$$

for $\lambda \in \mathfrak{h}(\tau)$. Hence for $h \in \mathcal{H}$ and $\lambda \in \rho(\widetilde{A}) \cap \mathfrak{h}(\tau) \cap \Omega'$ the vector

$$\hat{f}_1 = \begin{pmatrix} f_1 \\ \lambda f_1 + h \end{pmatrix} \in A^+$$

is a solution of (4.3).

It remains to verify (4.6). Assume that the representation (3.3) is minimal (see (3.4)). Then, by (3.5),

$$\mathcal{K} = \operatorname{clsp}\{\gamma'(\lambda) \mid \lambda \in \rho(T_0) \cap \Omega'\}, \tag{4.11}$$

and the set $\rho(T_0) \cap \Omega'$ in (4.11) can be replaced by $\rho(\widetilde{A}) \cap \Omega'$. From (4.7), (4.9) and (4.10) we obtain

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}\{h, 0\} = \gamma'(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ h$$

for $h \in \mathcal{H}$ and $\lambda \in \mathfrak{h}_0$. If $h \notin \mathcal{N}^{[\perp]}_{\overline{\lambda}, A^+}$ we have $\gamma(\overline{\lambda})^+ h \neq 0$. Making use of (4.11) we obtain

$$\mathcal{K} = \operatorname{clsp}\{P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}\{h, 0\} \mid h \in \mathcal{H},\ \lambda \in \rho(\widetilde{A}) \cap \Omega'\},$$

and therefore (4.6) holds. Theorem 4.1 is proved. □

Remark 4.3. Let $A \subset A_0$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, γ and M be as in the assumptions of Theorem 4.1. The case that $\tau \in N(\Omega)$ is a real constant is excluded in Theorem 4.1. In this case the boundary value problem (4.3) has the form

$$f_1' - \lambda f_1 = h, \quad \tau\Gamma_0 \hat{f}_1 + \Gamma_1 \hat{f}_1 = 0, \quad \hat{f}_1 = \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in A^+. \tag{4.12}$$

The relation $\widetilde{A}_{-\tau} = \ker(\tau\Gamma_0 + \Gamma_1) \in \widetilde{\mathcal{C}}(\mathcal{H})$ (see (2.2), (2.3)) is a selfadjoint extension of A in $\mathcal{H}$. By Theorem 2.2 we have

$$(\widetilde{A}_{-\tau} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(\tau + M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+$$

for $\lambda \in \mathfrak{h}(M) \cap \mathfrak{h}\big((\tau + M)^{-1}\big)$. Therefore, making use of the assumption that A_0 is of type π_+ over Ω and [3, Theorem 2.4] we conclude that $\widetilde{A}_{-\tau}$ is also of type π_+ over Ω. Setting $f_1 := (\widetilde{A}_{-\tau} - \lambda)^{-1} h$ it follows that

$$\hat{f}_1 := \begin{pmatrix} f_1 \\ \lambda f_1 + h \end{pmatrix} \in \widetilde{A}_{-\tau}$$

is a solution of (4.12).

Remark 4.4. Let the assumptions be as in Theorem 4.1 and assume that $\widetilde{A}$ fulfils the minimality condition (4.6). Let $\widetilde{B}$ be a selfadjoint extension of A in some Krein space $\mathcal{H} \times \widetilde{\mathcal{K}}$ which is of type π_+ over Ω' such that the compression of the resolvent of $\widetilde{B}$ onto $\mathcal{H}$ yields a solution of (4.3). Assume that $\widetilde{B}$ fulfils the minimality condition (4.6) with $\rho(\widetilde{A}) \cap \Omega'$ replaced by $\rho(\widetilde{B}) \cap \Omega'$. We denote the local spectral functions of $\widetilde{A}$ and $\widetilde{B}$ by $E_{\widetilde{A}}$ and $E_{\widetilde{B}}$, respectively (see (3.1)). Let $\Delta \subset \Omega' \cap \overline{\mathbb{R}}$ be a closed connected set such that $E_{\widetilde{A}}(\Delta)$ is defined. Then also $E_{\widetilde{B}}(\Delta)$ is defined, the Pontryagin spaces $E_{\widetilde{A}}(\Delta)(\mathcal{H} \times \mathcal{K})$ and $E_{\widetilde{B}}(\Delta)(\mathcal{H} \times \widetilde{\mathcal{K}})$ have the same finite rank of negativity and the relations

$$\widetilde{A}_1 := \widetilde{A} \cap \big(E_{\widetilde{A}}(\Delta)(\mathcal{H} \times \mathcal{K})\big)^2 \quad \text{and} \quad \widetilde{B}_1 := \widetilde{B} \cap \big(E_{\widetilde{B}}(\Delta)(\mathcal{H} \times \widetilde{\mathcal{K}})\big)^2$$

are unitarily equivalent (see [17]), that is, there exists an isometric isomorphism V which maps $E_{\widetilde{A}}(\Delta)(\mathcal{H} \times \mathcal{K})$ onto $E_{\widetilde{B}}(\Delta)(\mathcal{H} \times \widetilde{\mathcal{K}})$ such that

$$\left\{ \begin{pmatrix} V\{h,k\} \\ V\{h',k'\} \end{pmatrix} \,\Big|\, \begin{pmatrix} \{h,k\} \\ \{h',k'\} \end{pmatrix} \in \widetilde{A}_1 \right\} = \widetilde{B}_1.$$

5. An example

5.1. A Sturm-Liouville differential expression with an indefinite weight and the spectra of its locally definitizable realizations

In this section we investigate the spectral properties of the selfadjoint extensions of a symmetric singular Sturm-Liouville operator with the signum function as indefinite weight and a simple potential V. We assume that V is a real function on $\mathbb{R}$ which is constant on $\mathbb{R}^+ := (0,\infty)$ and $\mathbb{R}^- := (-\infty,0)$,

$$V(x) := \begin{cases} V_+ & \text{if } x \in \mathbb{R}^+, \\ V_- & \text{if } x \in \mathbb{R}^-. \end{cases}$$

Let $L^2(\mathbb{R}, \operatorname{sgn})$ be the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$, where

$$[f,g] := \int_{-\infty}^{\infty} f(x)\overline{g(x)} \operatorname{sgn} x \, dx, \qquad f,g \in L^2(\mathbb{R}),$$

and denote by J the fundamental symmetry of $L^2(\mathbb{R}, \operatorname{sgn})$ defined by $(Jf)(x) := (\operatorname{sgn} x) f(x)$, $x \in \mathbb{R}$. Then $[J\cdot,\cdot] =: (\cdot,\cdot)$ is the usual scalar product of $L^2(\mathbb{R})$. In the following the elements f of $L^2(\mathbb{R})$ will often be identified with the elements $\langle f_+, f_- \rangle$, $f_\pm := f|\mathbb{R}^\pm$, of $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^-)$.

Let A be the operator in $L^2(\mathbb{R}, \operatorname{sgn})$ defined by

$$\begin{aligned} \operatorname{dom} A :=& \big\{ f \in W^{2,2}(\mathbb{R}) \mid f(0) = 0 \big\} \\ =& \big\{ \langle f_+, f_- \rangle \in W^{2,2}(\mathbb{R}^+) \times W^{2,2}(\mathbb{R}^-) \mid f_+'(0+) = f_-'(0-), \\ & \qquad f_+(0+) = f_-(0-) = 0 \big\}, \\ (Af)(x) :=& (\operatorname{sgn} x)\big(-f''(x) + V(x) f(x)\big), \quad f \in \operatorname{dom} A. \end{aligned} \tag{5.1}$$

By $\sqrt[+]{\cdot}$ ($\sqrt[-]{\cdot}$) we denote the branch of $\sqrt{\cdot}$ defined in $\mathbb{C}$ with a cut along $[0,\infty)$ ($(-\infty,0]$) and fixed by $\operatorname{Im}\sqrt{\lambda}>0$ for $\lambda\notin[0,\infty)$ and $\sqrt{\lambda}\ge 0$ for $\lambda\in[0,\infty)$ (resp. $\operatorname{Re}\sqrt{\lambda}>0$ for $\lambda\notin(-\infty,0]$ and $\operatorname{Im}\sqrt{\lambda}\ge 0$ for $\lambda\in(-\infty,0]$).

Claim 5.1. *The operator A is a densely defined closed symmetric operator of defect one in the Krein space $L^2(\mathbb{R},\operatorname{sgn})$. Every nonreal λ is a point of regular type of A and the corresponding defect space $\mathcal{N}_{\lambda,A^+}=\ker(A^+-\lambda)$ coincides with $\operatorname{sp}\{f_\lambda\}$, where*

$$f_\lambda(x):=\begin{cases}\exp\big(i\sqrt[+]{\lambda-V_+}\,x\big) & \text{if } x>0,\\ \exp\big(\sqrt[-]{\lambda+V_-}\,x\big) & \text{if } x<0.\end{cases}\tag{5.2}$$

We have

$$\begin{aligned}\operatorname{dom}A^+ =&\big\{\langle f_+,f_-\rangle\in W^{2,2}(\mathbb{R}^+)\times W^{2,2}(\mathbb{R}^-)\,|\,f_+(0+)=f_-(0-)\big\},\\ A^+\langle f_+,f_-\rangle =&\big\langle -f_+''+V_+f_+,f_-''-V_-f_-\big\rangle,\quad \langle f_+,f_-\rangle\in\operatorname{dom}A^+.\end{aligned}\tag{5.3}$$

Indeed, let $A_{0,+}$ and $A_{0,-}$ be the selfadjoint operators in $(L^2(\mathbb{R}^+),(\cdot,\cdot))$ and $(L^2(\mathbb{R}^-),(\cdot,\cdot))$, respectively, defined by

$$\begin{aligned}\operatorname{dom}A_{0,\pm} :=&\big\{f_\pm\in W^{2,2}(\mathbb{R}^\pm)\,|\,f_\pm(0\pm)=0\big\},\\ (A_{0,\pm}f_\pm)(x):=&\pm\big(-f_\pm''(x)+V_\pm f_\pm(x)\big),\quad f_\pm\in\operatorname{dom}A_{0,\pm}.\end{aligned}$$

Then we have

$$\sigma(A_{0,+})=[V_+,\infty),\quad \sigma(A_{0,-})=(-\infty,-V_-]$$

and

$$A_0:=A_{0,+}\times A_{0,-}\tag{5.4}$$

regarded as an operator in the Krein space $L^2(\mathbb{R},\operatorname{sgn})$ is a selfadjoint extension of A with $\sigma(A_0)\subseteq\mathbb{R}$. This implies that A is a closed symmetric operator in $L^2(\mathbb{R},\operatorname{sgn})$ of defect one, that all nonreal points are of regular type with respect to A and that the corresponding defect spaces are one-dimensional.

With the function $f_\lambda\in L^2(\mathbb{R})$, $\lambda\in\mathbb{C}\backslash\mathbb{R}$, from (5.2) and for all $g\in\operatorname{dom}A$ integration by parts gives $[Ag,f_\lambda]=[g,\lambda f_\lambda]$ which implies

$$f_\lambda\in\ker(A^+-\lambda),\quad \lambda\in\mathbb{C}\backslash\mathbb{R}.$$

Then for every nonreal λ we have $\operatorname{dom}A^+=\operatorname{dom}A_0\dot{+}\operatorname{sp}\{f_\lambda\}$ which implies (5.3).

Claim 5.2. *If*

$$\Gamma_0\hat f:=f(0),\quad \Gamma_1\hat f:=f_+'(0+)-f_-'(0-),\quad \hat f=\begin{pmatrix}f\\A^+f\end{pmatrix},$$

then $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ is a boundary value space for A^+ and the corresponding γ-field γ and Weyl function M are

$$\gamma(\lambda)c=cf_\lambda\quad\text{and}\quad M(\lambda)=i\sqrt[+]{\lambda-V_+}-\sqrt[-]{\lambda+V_-},\qquad \lambda\in\rho(A_0),\tag{5.5}$$

where A_0 is the selfadjoint extension of A defined by $A_0=\ker\Gamma_0$.

For the spectra of the selfadjoint extensions of A in $L^2(\mathbb{R},\mathrm{sgn})$, i.e. of the selfadjoint operators $A_{(\alpha)}$ defined by

$$A_{(\alpha)} := \ker\bigl(\Gamma_1 + \alpha\Gamma_0\bigr), \quad \alpha \in \mathbb{R}, \qquad A_{(\infty)} := A_0,$$

we have

$$\bigl((-\infty,-V_-] \cup [V_+,\infty)\bigr) = \sigma_c(A_{(\alpha)}) \subset \sigma(A_{(\alpha)}) \subset \bigl(\mathbb{R} \cup \{V_0 + it \,|\, t \in \mathbb{R}\}\bigr),$$

where $V_0 := \frac{1}{2}(V_+ - V_-)$, and for every $\alpha \in \overline{\mathbb{R}}$

$$\sigma(A_{(\alpha)})\backslash\bigl((-\infty,-V_-] \cup [V_+,\infty)\bigr)$$

is empty or consists of two eigenvalues (multiplicities counted). More precisely:

(i) *Assume that $V_+ \le -V_-$. Then $\sigma_c(A_{(\alpha)}) = \mathbb{R}$, $\alpha \in \overline{\mathbb{R}}$, and there is a one-to-one increasing continuous mapping $(0,\infty) \ni \alpha \mapsto t_\alpha \in (0,\infty)$ such that*

$$\sigma_p(A_{(\alpha)}) = \begin{cases} \emptyset & \text{if } \alpha \in \overline{\mathbb{R}}\backslash(0,\infty), \\ \{V_0 + it_\alpha, V_0 - it_\alpha\} & \text{if } \alpha \in (0,\infty). \end{cases}$$

(ii) *Assume that $-V_- < V_+$. Then $\sigma_c(A_{(\alpha)}) = (-\infty,-V_-] \cup [V_+,\infty)$, $\alpha \in \overline{\mathbb{R}}$, and there is a one-to-one decreasing continuous mapping*

$$\bigl(\sqrt{V_+ + V_-}, \sqrt{2(V_+ + V_-)}\,\bigr] \ni \alpha \mapsto s_\alpha \in \bigl[0, \tfrac{1}{2}(V_+ + V_-)\bigr)$$

and a one-to-one increasing continuous mapping

$$\bigl[\sqrt{2(V_+ + V_-)}, \infty\bigr) \ni \alpha \mapsto u_\alpha \in \bigl[0,\infty\bigr)$$

such that

$$\sigma_p(A_{(\alpha)}) = \begin{cases} \emptyset & \text{if } \alpha \in \bigl(-\infty, \sqrt{V_+ + V_-}\,\bigr], \\ \{V_0 - s_\alpha\} \cup \{V_0 + s_\alpha\} & \text{if } \alpha \in \bigl(\sqrt{V_+ + V_-}, \sqrt{2(V_+ + V_-)}\,\bigr], \\ \{V_0 + iu_\alpha\} \cup \{V_0 - iu_\alpha\} & \text{if } \alpha \in \bigl[\sqrt{2(V_+ + V_-)}, \infty\bigr). \end{cases}$$

We remark that $A_{(0)} = \ker\Gamma_1$ is the only selfadjoint extension of A with a domain consisting of C^1-functions:

$$\begin{aligned} \operatorname{dom} A_{(0)} &= W^{2,2}(\mathbb{R}), \\ (A_{(0)}f)(x) &= (\operatorname{sgn} x)\bigl(-f''(x) + V(x)f(x)\bigr), \quad f \in \operatorname{dom} A_{(0)}. \end{aligned}$$

Let us verify Claim 5.2. It is not difficult to verify that

$$\begin{aligned} &\bigl[A^+\langle f_+, f_-\rangle, \langle g_+, g_-\rangle\bigr] - \bigl[\langle f_+, f_-\rangle, A^+\langle g_+, g_-\rangle\bigr] \\ &\quad = (f_+'(0+) - f_-'(0-))\,\overline{g(0)} - f(0)\,\overline{(g_+'(0+) - g_-'(0-))} \\ &\quad = \Gamma_1 f\,\overline{\Gamma_0 g} - \Gamma_0 f\,\overline{\Gamma_1 g} \end{aligned}$$

holds. Therefore $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary value space for A^+. $A_0 = \ker\Gamma_0$ coincides with the operator defined by (5.4). Then (5.5) follows from (5.2).

If $-V_- < V_+$ we have $\sigma(A_0) = \mathbb{R}\backslash(-V_-, V_+)$; if $V_+ \le -V_-$, then $\sigma(A_0) = \mathbb{R}$. In both of these cases for $\lambda \in \mathbb{C}\backslash\mathbb{R}$ a straightforward calculation gives

$$\operatorname{Im} M(\lambda) > 0 \ (= 0,\ < 0) \quad \Leftrightarrow \quad \operatorname{Re}\lambda > V_0 \ (= V_0,\ < V_0).$$

Now it follows from Theorem 2.2 that the spectra of all selfadjoint extensions $A_{(\alpha)}$, $\alpha \in \overline{\mathbb{R}}$, are contained in $\mathbb{R} \cup (V_0 + i\mathbb{R})$.

Let us show that none of the extensions $A_{(\alpha)}$, $\alpha \in \overline{\mathbb{R}}$, has an eigenvalue in $(-\infty, -V_-] \cup [V_+, \infty)$. Evidently, this is true for $A_{(\infty)} = A_0$. Suppose that, for some $\alpha \in \mathbb{R}$, $\mu \in (\infty, -V_-] \cup [V_+, \infty)$ is an eigenvalue of $A_{(\alpha)}$ and f is a corresponding eigenelement. Then $(\operatorname{sgn} x)(-f''(x) + V(x)f(x)) = \mu f(x)$ implies

$$f(x) = \begin{cases} 0 & \text{if } \mu \le -V_-, \\ c\exp\big(\sqrt{\mu + V_-}\, x\big) & \text{if } \mu > -V_-, \end{cases} \qquad x \in \mathbb{R}^-,$$

and

$$f(x) = \begin{cases} 0 & \text{if } \mu \ge V_+, \\ d\exp\big(-\sqrt{V_+ - \mu}\, x\big) & \text{if } \mu < V_+, \end{cases} \qquad x \in \mathbb{R}^+,$$

where c and d are some constants. If $\mu \ge V_+$ or $\mu \le -V_-$, then the continuity of the functions in $\operatorname{dom} A_{(\alpha)}$ yields $f = 0$.

The last statements of Claim 5.2 are consequences of the following facts: A point $\lambda \in \mathfrak{h}(M)$ is an eigenvalue of $A_{(\alpha)}$ if and only if $M(\lambda) = -\alpha$, the mapping

$$(0, \infty) \ni t \mapsto M(V_0 + it) \in (-\infty, 0)$$

is continuous and decreasing such that

$$\lim_{t\uparrow\infty} M(V_0 + it) = -\infty$$

and

$$\lim_{t\downarrow 0} M(V_0 + it) = \begin{cases} 0 & \text{if } V_+ \le -V_-, \\ -\sqrt{2(V_+ + V_-)} & \text{if} - V_- < V_+. \end{cases}$$

If $-V_- < V_+$ the mapping

$$\big[0, \tfrac{1}{2}(V_+ + V_-)\big) \ni t \mapsto M(V_0 + t) = M(V_0 - t) \in \big[-\sqrt{2(V_+ + V_-)}, -\sqrt{V_+ + V_-}\,\big)$$

is one-to-one, continuous and increasing, $M(V_0) = -\sqrt{2(V_+ + V_-)}$ and

$$\lim_{\varepsilon\downarrow 0} M(V_+ - \varepsilon) = \lim_{\varepsilon\downarrow 0} M(-V_- + \varepsilon) = -\sqrt{V_+ + V_-}.$$

Claim 5.3. *For all the selfadjoint extensions $A_{(\alpha)}$, $\alpha \in \overline{\mathbb{R}}$, of A the following holds.*

(i) $\Omega_+ := \mathbb{C}\backslash(-\infty, -V_-]$ *is of type* π_+ *and* $\Omega_- := \mathbb{C}\backslash[V_+, \infty)$ *is of type* π_-.

(ii) *If $V_+ \le -V_-$ the interval $(-V_-, \infty)$ is of positive type and the interval $(-\infty, V_+)$ is of negative type. If $-V_- < V_+$ the interval (V_0, ∞) is of positive type and the interval $(-\infty, V_0)$ is of negative type.*

Indeed, since (i) is true for the fundamentally reducible operator $A_0 = A_{(\infty)}$ and the differences of the resolvents of $A_{(\alpha)}$, $\alpha \in \mathbb{R}$, and $A_{(\infty)}$ have rank one it follows by [3, Theorem 2.4] that (i) holds for every extension $A_{(\alpha)}$ of A.

Assume that $V_+ \leq -V_-$. We show that $(-V_-, \infty)$ is of positive type with respect to $A_{(\alpha)}$. By (i) each operator $A_{(\alpha)}$, $\alpha \in \overline{\mathbb{R}}$, is also of type π_+ over the domain

$$\Omega := \big\{ t + is \,|\, t \in (-V_-, \infty),\ s \in (-1, 1) \big\}$$

and we have $\sigma_p(A_{(\alpha)}) \cap \Omega = \emptyset$. Let Ω' and Ω'' be subdomains of Ω satisfying the conditions mentioned before Definition 3.1 such that $\overline{\Omega''} \subset \Omega$ and $\overline{\Omega'} \subset \Omega''$. We are going to show that there exists a projection E as in Definition 3.3 with $\mathcal{K}$ and A_0 replaced by $L^2(\mathbb{R}, \mathrm{sgn})$ and $A_{(\alpha)}$, respectively, such that the range of E is a Hilbert space.

Since Ω is of type π_+ with respect to $A_{(\alpha)}$ there exists a projection E'' in $L^2(\mathbb{R}, \mathrm{sgn})$ as in Definition 3.3 with Ω' replaced by Ω'' such that $\mathrm{ran}\, E''$ is a Pontryagin space and the intersection of the spectrum of the selfadjoint operator $A_{(\alpha)}|\mathrm{ran}\, E''$ and Ω'' is real and contains no eigenvalues. If F is the spectral function of $A_{(\alpha)}|\mathrm{ran}\, E''$, then by a well known result for selfadjoint operators in Pontryagin spaces the ranges of the spectral projections $F([a, b])$ corresponding to the intervals $[a, b] \subset \Omega''$ are Hilbert spaces. Assume, in addition, that $[a, b] \subset \Omega''$ contains $\overline{\Omega'} \cap \mathbb{R}$. Then $E = F([a, b])E''$ is a projection in $L^2(\mathbb{R}, \mathrm{sgn})$ with the required properties and $(-V_-, \infty)$ is of positive type with respect to $A_{(\alpha)}$. An analogous argument applies for the interval $(-\infty, V_+)$.

If $-V_- < V_+$ and $\mu \in (V_0, V_+)$ ($\mu \in (-V_-, V_0)$) is an eigenvalue of some $A_{(\alpha)}$ a simple calculation shows that the corresponding eigenelement f is positive (negative) in $L^2(\mathbb{R}, \mathrm{sgn})$, i.e. $[f, f] > 0$ (resp. $[f, f] < 0$). Now it follows as above that (V_0, ∞) is of positive type and $(-\infty, V_0)$ is of negative type with respect to the operators $A_{(\alpha)}$, $\alpha \in \overline{\mathbb{R}}$.

5.2. λ-dependent boundary conditions

In this section we consider the following boundary value problem with λ-dependent boundary conditions: For a given function $h \in L^2(\mathbb{R})$ find an element $f = \langle f_+, f_- \rangle$ in $W^{2,2}(\mathbb{R}^+) \times W^{2,2}(\mathbb{R}^-)$ such that

$$(\mathrm{sgn}\, x) \left(-\frac{d^2}{dx^2} + V(x) \right) f(x) - \lambda f(x) = h(x), \quad x \in \mathbb{R}^+ \cup \mathbb{R}^-, \tag{5.6}$$

holds, where V is as in Section 5.1, and the boundary conditions

$$\tau(\lambda) f(0) + f'_+(0+) - f'_-(0-) = 0 \quad \text{and} \quad f(0+) = f(0-) \tag{5.7}$$

are satisfied. Here τ is assumed to be a meromorphic function in $\mathbb{C}$ (which implies $\tau, -\tau \in N(\mathbb{C})$) from a special class described below. It will be shown that the meromorphic functions τ of that class possess a minimal representation of the form (3.3).

Let

$$\big(\mu_j\big)_{j=1}^{j_\infty} \subset \overline{\mathbb{C}^+}, \quad \big(a_j\big)_{j=0}^{j_\infty} \subset \mathbb{C}\backslash\{0\}, \quad \big(k_j\big)_{j=1}^{j_\infty} \subset \mathbb{N}$$

be finite ($j_\infty < \infty$) or infinite ($j_\infty = \infty$) sequences such that

1. $\mu_j \neq \mu_k$ for $j \neq k$, $0 \neq |\mu_1| \leq |\mu_2| \leq \dots$,
   $\sup_j |\mu_j| = \infty$ if $j_\infty = \infty$, $\mu_j \neq i$, $j = 1, \dots, j_\infty$,
2. $a_j \in \mathbb{R}$ if $\mu_j \in \mathbb{R}$, $j = 1, \dots, j_\infty$, $a_0 \in \mathbb{R}$,
3. $\sup_j k_j < \infty$,
4. $\sum_{j=1}^{j_\infty} |a_j||\mu_j|^{-1} < \infty$.

Then we define

$$\tau(\lambda) := a_0 + \frac{1}{2}\sum_{j=1}^{j_\infty}\big(a_j(\lambda-\mu_j)^{-k_j} + \overline{a}_j(\lambda-\overline{\mu}_j)^{-k_j}\big).$$

For $j_\infty = \infty$ the series converges absolutely and uniformly on every compact subset L of $\mathbb{C}$ such that $\mu_j \notin L$, $j = 1, 2, \dots, j_\infty$. We have $\tau(\overline{\lambda}) = \overline{\tau(\lambda)}$.

We denote by $[\cdot,\cdot]_k$, $k \in \mathbb{N}$, the inner product in $\mathbb{C}^k$ defined by

$$\big[(x_1,\dots,x_k)^\top,(y_1,\dots,y_k)^\top\big]_k := \big((x_k,\dots,x_1)^\top,(y_1,\dots,y_k)^\top\big)_{\mathbb{C}^k};$$

the sip matrix (see [13, Chapter 1]) is a fundamental symmetry of $(\mathbb{C}^k, [\cdot,\cdot]_k)$. The $k \times k$ Jordan block corresponding to $\mu \in \mathbb{C}$ is denoted by $J_k(\mu)$:

$$J_k(\mu) := \begin{pmatrix} \mu & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{pmatrix}.$$

If $\mu_j \in \mathbb{R}$ we set $\mathcal{K}_j := (\mathbb{C}^{k_j}, -(\operatorname{sgn} a_j)[\cdot,\cdot]_{k_j})$, $T_{0,j} := J_{k_j}(\mu_j)$ and

$$\begin{aligned} f_j :=& |a_j|^{\frac{1}{2}}\big(T_{0,j}-i\big)^{-1}(0,\dots,0,1)^\top \\ =& -|a_j|^{\frac{1}{2}}\big((i-\mu_j)^{-k_j},\dots,(i-\mu_j)^{-1}\big)^\top. \end{aligned}$$

If $\mu_j \notin \mathbb{R}$ we set $\mathcal{K}_j := (\mathbb{C}^{2k_j}, [\cdot,\cdot]_{2k_j})$,

$$T_{0,j} := \operatorname{diag}\big(J_{k_j}(\mu_j), J_{k_j}(\overline{\mu}_j)\big)$$

and

$$\begin{aligned} f_j :=& \left|\frac{a_j}{2}\right|^{\frac{1}{2}}(T_{0,j}-i)^{-1}(0,\dots,0,1,0,\dots,0,-e^{-i\arg a_j})^\top \\ =& -\left|\frac{a_j}{2}\right|^{\frac{1}{2}}\big((i-\mu_j)^{-k_j},\dots,(i-\mu_j)^{-1}, \\ & \qquad -(i-\overline{\mu}_j)^{-k_j}e^{-i\arg a_j},\dots,-(i-\overline{\mu}_j)^{-1}e^{-i\arg a_j}\big)^\top. \end{aligned}$$

Then

$$\frac{1}{2}\big(a_j(\lambda-\mu_j)^{-k_j}+\overline{a}_j(\lambda-\overline{\mu}_j)^{-k_j}\big)$$
$$=\big[T_{0,j}f_j,f_j\big]_{k_j}+\lambda\big[f_j,f_j\big]_{k_j}+(\lambda^2+1)\big[(T_{0,j}-\lambda)^{-1}f_j,f_j\big]_{k_j}.$$

Let $(\ell^2_\tau,[\cdot,\cdot])$ be the direct product of the Krein spaces $\mathcal{K}_j$, $j=1,\dots,j_\infty$. By the definition of the vectors f_j there exists an M such that

$$\|f_j\|^2_{\mathbb{C}^{k_j}}\leq M|a_j||\mu_j|^{-2}. \tag{5.8}$$

Then the assumption 4 above implies $f:=(f_j)_{j=1}^{j_\infty}\in\ell^2_\tau$.

The family of operators $V_\lambda\in\mathcal{L}(\ell^2_\tau)$, $\lambda\neq\mu_j$ for all $j=1,\dots,j_\infty$, defined by

$$V_\lambda\big((x_j)_{j=1}^{j_\infty}\big):=\big((T_{0,j}-\lambda)^{-1}x_j\big)_{j=1}^{j_\infty}$$

fulfils the resolvent equation, we have $V_{\overline{\lambda}}=V_\lambda^+$ and $\ker V_\lambda=\{0\}$. If T_0 denotes the selfadjoint operator in ℓ^2_τ the resolvent of which coincides with V_λ then for any j the space $\mathcal{K}_j$ regarded as a subspace of ℓ^2_τ is contained in $\operatorname{dom}T_0$ and $T_0|\mathcal{K}_j=T_{0,j}$. By (5.8) and assumption 4 the series

$$t_0:=\sum_{j=1}^{j_\infty}\big[T_{0,j}f_j,f_j\big]$$

converges and we have

$$\tau(\lambda)=a_0+t_0+\lambda[f,f]+(\lambda^2+1)\big[(T_0-\lambda)^{-1}f,f\big]. \tag{5.9}$$

Making use of the fact that $\mu_j\neq\mu_i$ for $i\neq j$ it is not difficult to verify that the representation (5.9) of τ is minimal. There exist a closed symmetric operator $T\subset T_0$ with defect one and a boundary value space $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ for T^+ such that τ is the corresponding Weyl function (cf. Theorem 3.4). Since τ and $-\tau$ belong to the class $N(\mathbb{C})$ here the selfadjoint operator $T_0=\ker\Gamma_0'$ is of type π_+ as well as of type π_- over $\mathbb{C}$. The minimality of the representation implies that

$$\operatorname{sp}\big\{\ker(T^+-\lambda),|\,\lambda\neq\mu_j,\,j=1,\dots,j_\infty\big\}$$

is dense in ℓ^2_τ and therefore T has no eigenvalues.

Claim 5.4. *Let A be the symmetric operator from* (5.1) *and let $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ be the boundary value space from Claim* 5.2. *Then*

$$\widetilde{A}=\big\{\{\hat{f},\hat{k}\}\in A^+\times T^+|\,\Gamma_1\hat{f}-\Gamma_1'\hat{k}=\Gamma_0\hat{f}+\Gamma_0'\hat{k}=0\big\} \tag{5.10}$$

is a selfadjoint extension of A in the Krein space $L^2(\mathbb{R},\operatorname{sgn})\times\ell^2_\tau$ and the following holds.

(i) $\sigma(\widetilde{A})\backslash\mathbb{R}$ *is either finite or the only accumulation point of* $\sigma(\widetilde{A})\backslash\mathbb{R}$ *is* ∞.

(ii) *If $V_+\leq -V_-$ then $\sigma_c(\widetilde{A})=\mathbb{R}$, the interval $(-V_-,\infty)$ is of positive type and the interval $(-\infty,V_+)$ is of negative type with respect to $\widetilde{A}$.*

(iii) *If* $-V_- < V_+$ *then* $\sigma_c(\widetilde{A}) = \mathbb{R}\backslash(-V_-, V_+)$ *and* $\sigma_p(\widetilde{A}) \cap (-V_-, V_+)$ *is finite. The interval* (V_+, ∞) *is of positive type and the interval* $(-\infty, -V_-)$ *is of negative type with respect to* $\widetilde{A}$*.*

(iv) *If* P *denotes the orthogonal projection of* $L^2(\mathbb{R}, \operatorname{sgn}) \times \ell^2_\tau$ *onto* $L^2(\mathbb{R}, \operatorname{sgn})$ *then for every* $\lambda \in \rho(\widetilde{A})\backslash\{\mu_j \,|\, j \in 1, \dots, j_\infty\}$ *the function*

$$f = P(\widetilde{A} - \lambda)^{-1}\{h, 0\}$$

is a solution of the boundary value problem (5.6)-(5.7).

In fact, let Ω_+ and Ω_- be as in Claim 5.3. Then $A_0 = \ker \Gamma_0$ is of type $\pi_\pm$ over $\Omega_\pm$ and it follows from Theorem 4.1 (with $\Omega = \Omega' = \Omega_\pm$) that the selfadjoint extension $\widetilde{A}$ in (5.10) of A in $L^2(\mathbb{R}, \operatorname{sgn}) \times \ell^2_\tau$ is also of type $\pi_\pm$ over $\Omega_\pm$.

(i) If $-V_- < V_+$ then the fact that $\widetilde{A}$ is of type $\pi_\pm$ over $\Omega_\pm$ implies that ∞ is the only possible accumulation point of $\sigma(\widetilde{A})\backslash\mathbb{R}$. It remains to show that in the case $V_+ \leq -V_-$ the nonreal spectrum of $\widetilde{A}$ does not accumulate to points in $[V_+, -V_-]$. Recall that the Weyl function M corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is given by

$$M(\lambda) = i\sqrt[+]{\lambda - V_+} - \sqrt[-]{\lambda + V_-},$$

(cf. (5.5)). If $\mu \in (V_+, -V_-)\backslash\{\mu_j \,|\, j \in 1, \dots, j_\infty\}$ then the function $M + \tau$ can be continued analytically from the upper half plane into an open neighbourhood $\mathcal{U}_\mu$ of μ and this implies that the zeros of $M + \tau$ in $\mathbb{C}^+$ can not accumulate to μ. A similar argument applies in the case that μ is a pole of τ. Let $\mu = V_+$ or $\mu = -V_-$. Then the limit $\lim_{\lambda \to \mu} M(\lambda)$, $\lambda \in \mathbb{C}^+$, from the upper half plane exists and is finite. Therefore the zeros of $M + \tau$ cannot accumulate to μ if μ is a pole of τ. Now assume that $\mu \neq \mu_j$, $j = 1, \dots, j_\infty$. Then

$$\inf\left\{\Big|\frac{d}{d\lambda}\big(M(\lambda) + \tau(\lambda)\big)\Big| : 0 < |\lambda - \mu| \leq r,\, \lambda \in \mathbb{C}^+\right\} \to \infty \text{ if } \quad r \downarrow 0$$

implies that the zeros of $M + \tau$ do not accumulate to μ.

Hence it follows from Theorem 2.2 that for every $\mu \in [V_+, -V_-]$ there exists an open neighbourhood $\mathcal{U}_\mu$ such that $\mathcal{U}_\mu \cap \mathbb{C}^+$ belongs to the resolvent set of the closed extensions

$$A_{(\tau(\lambda))} = \ker\big(\Gamma_1 + \tau(\lambda)\Gamma_0\big), \qquad \lambda \in \mathcal{U}_\mu \cap \mathbb{C}^+,$$

of A in $L^2(\mathbb{R}, \operatorname{sgn})$. It is no restriction to assume that $\mathcal{U}_\mu \cap \mathbb{C}^+ \cap \{\mu_j \,|\, j = 1, \dots, j_\infty\}$ is empty. If some $\lambda \in \mathcal{U}_\mu \cap \mathbb{C}^+$ would be an eigenvalue of $\widetilde{A}$, then there would exist $f \in L^2(\mathbb{R}, \operatorname{sgn})$, $k \in \ell^2_\tau$, $\{f, k\} \neq \{0, 0\}$, such that

$$\{\hat{f}, \hat{k}\} = \begin{pmatrix} \{f, k\} \\ \{\lambda f, \lambda k\} \end{pmatrix} \in \widetilde{A} \subset A^+ \times T^+. \tag{5.11}$$

In particular $f \neq 0$, as otherwise (5.10) would imply $\Gamma_0' \hat{k} = \Gamma_1' \hat{k} = 0$ and $\hat{k} \in T$ what is impossible as T has no eigenvalues. Hence $A^+ f = \lambda f$ and, by (5.10),

$$\Gamma_1 \hat{f} = \Gamma_1' \hat{k} = \tau(\lambda)\Gamma_0' \hat{k} = -\tau(\lambda)\Gamma_0 \hat{f}$$

would imply $\lambda \in \sigma_p(A_{(\tau(\lambda))})$, a contradiction. Therefore the only possible accumulation point of $\sigma(\widetilde{A})\backslash\mathbb{R}$ is ∞.

(ii) From Claim 5.2 and the fact that $\widetilde{A}$ is a two-dimensional perturbation in resolvent sense of $A_0 \times T_0$ we conclude $\mathbb{R} \subseteq \sigma(\widetilde{A})$ if $V_+ \leq -V_-$. If $\lambda \in \mathbb{R}$, $\lambda \neq \mu_j$, $j = 1, \dots, j_\infty$, would be an eigenvalue of $\widetilde{A}$ then the same argument as in the proof of (i) would imply that λ is an eigenvalue of the selfadjoint operator $A_{(\tau(\lambda))}$, $\tau(\lambda) \in \mathbb{R}$, which contradicts Claim 5.2. For $\lambda \in \{\mu_j \,|\, j = 1, \dots, j_\infty\}$ we have $\lambda \in \sigma_p(T_0)$ and since λ is a normal eigenvalue of T_0 we conclude that $\operatorname{ran}(T_0 - \lambda)$ and, therefore, also $\operatorname{ran}(T - \lambda)$ is closed. Hence the defect subspace $\ker(T^+ - \lambda)$ has dimension one and $\ker(T^+ - \lambda) = \ker(T_0 - \lambda)$. If λ would be an eigenvalue of $\widetilde{A}$ then there would exist $f \in L^2(\mathbb{R}, \mathrm{sgn})$, $k \in \ell^2_\tau$, $\{f, k\} \neq \{0, 0\}$, such that $\{\hat{f}, \hat{k}\} \in \widetilde{A} \subset A^+ \times T^+$ (cf. (5.11)). From $\hat{k} \in T_0$ and (5.10) we conclude $\Gamma_0 \hat{f} = 0$, i.e. $\lambda \in \sigma_p(A_{(\infty)})$, which again contradicts Claim 5.2. Therefore we have $\mathbb{R} = \sigma_c(\widetilde{A})$. The same argument as in the proof of Claim 5.3 shows that $(-V_-, \infty)$ is of positive type and $(-\infty, V_+)$ is of negative type with respect to $\widetilde{A}$.

(iii) In the case $-V_- < V_+$ the same arguments as in the proof of (ii) show $\sigma_c(\widetilde{A}) = \mathbb{R}\backslash(-V_-, V_+)$ and that (V_+, ∞) is of positive type and $(-\infty, -V_-)$ is of negative type with respect to $\widetilde{A}$. The interval $(-V_-, V_+)$ with the possible exception of finitely many points μ_j, $j \in 1, \dots, j_\infty$, is contained in $\rho(A_0) \cap \mathfrak{h}(\tau)$. Hence by Theorem 4.1 the set $\sigma_p(\widetilde{A}) \cap (-V_-, V_+)$ is finite.

(iv) The λ-dependent boundary value problem (5.6)-(5.7) is equivalent to

$$(A^+ - \lambda) f = h, \quad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ A^+ f \end{pmatrix}.$$

Hence for all $\lambda \in \rho(\widetilde{A})\backslash\{\mu_j \,|\, j \in 1, \dots, j_\infty\}$ Theorem 4.1 implies that the function $f = P(\widetilde{A} - \lambda)^{-1}\{h, 0\}$ is a solution of (5.6)-(5.7). Moreover the formula

$$P(\widetilde{A} - \lambda)^{-1}|_{L^2(\mathbb{R},\mathrm{sgn})} = (A_0 - \lambda)^{-1} - \big(M(\lambda) + \tau(\lambda)\big)^{-1}[\cdot, f_{\overline{\lambda}}] f_\lambda$$

holds for all $\lambda \in \rho(\widetilde{A})\backslash\widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is some discrete subset of $\rho(\widetilde{A})$ with ∞ as only possible accumulation point and f_λ is the defect element from (5.2).

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