A remark on Schatten-von Neumann properties of resolvent differences of generalized Robin Laplacians on bounded domains

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In memory of M. Š. Birman (1928–2009)

Abstract

In this note we investigate the asymptotic behaviour of the s-numbers of the resolvent difference of two generalized self-adjoint, maximal dissipative or maximal accumulative Robin Laplacians on a bounded domain Ω with smooth boundary $\partial\Omega$. For this we apply the recently introduced abstract notion of quasi boundary triples and Weyl functions from extension theory of symmetric operators together with Krein type resolvent formulae and well-known eigenvalue asymptotics of the Laplace–Beltrami operator on $\partial\Omega$. It will be shown that the resolvent difference of two generalized Robin Laplacians belongs to the Schatten–von Neumann class of any order p for which

$$p > \frac{\dim\Omega - 1}{3}\,.$$

Moreover, we also give a simple sufficient condition for the resolvent difference of two generalized Robin Laplacians to belong to a Schatten–von Neumann class of arbitrary small order. Our results extend and complement classical theorems due

to M. Š. Birman on Schatten–von Neumann properties of the resolvent differences of Dirichlet, Neumann and self-adjoint Robin Laplacians.

Key words: Laplacian, self-adjoint extension, quasi boundary triple, Weyl function, Krein's formula, non-local boundary condition, Schatten–von Neumann class, singular numbers

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1 Introduction

It is well known that the difference of the resolvents of two self-adjoint extensions of a symmetric operator (with equal infinite deficiency numbers) usually behaves 'better' than the resolvents themselves, e.g. even if the resolvents are non-compact operators, the difference may belong to a Schatten–von Neumann class, or if the resolvents are from a Schatten–von Neumann class, the difference may lie in one of smaller order. In particular, according to classical results due to M. Š. Birman [6] the resolvent difference of the Dirichlet and Neumann Laplacian in a bounded or unbounded domain Ω with compact C^∞ boundary $\partial\Omega$ satisfies

$$(-\Delta_D^\Omega - \lambda)^{-1} - (-\Delta_N^\Omega - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega)), \qquad \forall\, p > \frac{\dim\Omega - 1}{2},$$

where $\mathcal{S}_p(L^2(\Omega))$ is the Schatten–von Neumann class of order p and Δ_D^Ω, Δ_N^Ω are the Dirichlet and Neumann Laplacians on Ω, respectively. Analogous estimates were also obtained for the difference of the resolvents of self-adjoint Laplacians with (ordinary) Robin boundary conditions $\beta f|_{\partial\Omega} = \frac{\partial f}{\partial\nu}$, where β is a real-valued function on $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the outer normal derivative. Later such results on spectral asymptotics were refined and generalized by, e.g. M. Š. Birman and M. Z. Solomjak in [7] and G. Grubb in [19]. Recently some new Schatten–von Neumann properties of resolvent differences of differential operators were announced by F. Gesztesy and M. M. Malamud in [13], and in the paper by G. Grubb [23] the influence of generalized Robin boundary conditions on the essential spectrum in exterior domains was studied.

The main objective of the present paper is to extend and complement some results on Schatten–von Neumann properties for the resolvent difference of self-adjoint Laplacians from [6]. Instead of Dirichlet, Neumann and self-adjoint

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Robin Laplacians we study so-called *generalized* Robin Laplacians which are self-adjoint, maximal dissipative or maximal accumulative. More precisely, we study self-adjoint, maximal dissipative and maximal accumulative realizations $-\Delta_{\Theta_1}^\Omega$ and $-\Delta_{\Theta_2}^\Omega$ of the Laplacian corresponding to the generalized (or non-local) Robin boundary conditions

$$\Theta_1 \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = f\big|_{\partial\Omega} \qquad \text{and} \qquad \Theta_2 \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = f\big|_{\partial\Omega},$$

respectively, where Θ_1 and Θ_2 are self-adjoint, maximal dissipative or maximal accumulative operators in $L^2(\partial\Omega)$ such that $0 \notin \sigma_{\mathrm{ess}}(\Theta_i)$, $i = 1, 2$. We note that generalized self-adjoint Robin Laplacians were recently also considered by F. Gesztesy and M. Mitrea in [14–17]. It is shown in Theorem 3.5 and Corollary 3.6 that

$$(-\Delta_{\Theta_1}^\Omega - \lambda)^{-1} - (-\Delta_{\Theta_2}^\Omega - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega)), \qquad \forall p > \frac{\dim\Omega - 1}{3}, \quad (1.1)$$

holds for all $\lambda \in \rho(-\Delta_{\Theta_1}^\Omega) \cap \rho(-\Delta_{\Theta_2}^\Omega)$. Moreover, if $\Theta_1 - \Theta_2 \in \mathcal{S}_{p_0}(L^2(\Omega))$ for some $p_0 \in (0, \infty)$, then

$$(-\Delta_{\Theta_1}^\Omega - \lambda)^{-1} - (-\Delta_{\Theta_2}^\Omega - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega)), \quad \forall p > \frac{(\dim\Omega - 1)p_0}{(\dim\Omega - 1) + 3p_0}; \tag{1.2}$$

see Theorem 3.11. The proofs of these estimates are quite elementary and short when applying the abstract concept of quasi boundary triples and Weyl functions from extension theory of symmetric operators together with Krein type resolvent formulae from [5] and well-known eigenvalue asymptotics of the Laplace–Beltrami operator on $\partial\Omega$; see, e.g. [2]. We note that our main results (1.1) and (1.2) can be proved in the same way for generalized Robin Schrödinger operators $-\Delta_{\Theta_i}^\Omega + V$ with a real valued L^∞ potential V or for more general uniformly elliptic differential operators with coefficients satisfying appropriate conditions.

2 Quasi boundary triples

In this section we briefly recall the abstract notion of quasi boundary triples and Weyl functions in extension theory of symmetric operators, some of their properties and how they can be applied to the Laplacian on bounded domains. This concept was introduced in connection with elliptic boundary value problems by the first two authors in [5] as a generalization of the notion of ordinary and generalized boundary triples from [9–12,24]. The following definition is a variant of [5, Definition 2.1] for densely defined, closed, symmetric operators.

Definition 2.1 *Let A be a densely defined, closed, symmetric operator in a Hilbert space $\mathcal{H}$. We say that $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is a* quasi boundary triple *for A^* if $\mathcal{G}$ is a Hilbert space, Γ_0 and Γ_1 are linear mappings defined on the same subset $\operatorname{dom} \Gamma_0 = \operatorname{dom} \Gamma_1$ of $\operatorname{dom} A^*$ with values in $\mathcal{G}$ such that $T := A^*|_{\operatorname{dom} \Gamma_0}$ satisfies $\overline{T} = A^*$, that $\binom{\Gamma_0}{\Gamma_1} \colon \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ has dense range, that $A_0 := T|_{\ker \Gamma_0}$ is self-adjoint and that the identity*

$$(Tf, g)_{\mathcal{H}} - (f, Tg)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$$

holds for all $f, g \in \operatorname{dom} T$.

From the definition it follows that both $\operatorname{ran} \Gamma_0$ and $\operatorname{ran} \Gamma_1$ are dense in $\mathcal{G}$. Moreover, one can easily show that $\Gamma_0|_{\ker(T-\lambda)}$ is bijective from $\ker(T - \lambda)$ onto $\operatorname{ran} \Gamma_0$ for $\lambda \in \rho(A_0)$. Next we recall the definition of the γ-field, the Weyl function and the parameterization of certain extensions of the symmetric operator A.

Definition 2.2 *Let A be a densely defined, closed, symmetric operator in a Hilbert space, $(\mathcal{G}, \Gamma_0, \Gamma_1)$ a quasi boundary triple for A^* and T as above.*

(i) *The bijective mapping*

$$\gamma(\lambda) := (\Gamma_0|_{\ker(T-\lambda)})^{-1} \colon \operatorname{ran} \Gamma_0 \to \ker(T - \lambda), \qquad \lambda \in \rho(A_0),$$

is called γ-field.

(ii) *The mapping*

$$M(\lambda) := \Gamma_1 \gamma(\lambda), \qquad \lambda \in \rho(A_0),$$

is called Weyl function.

(iii) *For a linear operator Θ in $\mathcal{G}$, let A_Θ be the restriction of T to the set*

$$\operatorname{dom} A_\Theta := \left\{ f \in \operatorname{dom} T \colon \Gamma_1 f = \Theta \Gamma_0 f \right\}.$$

We gather in one proposition some facts about the γ-field, the Weyl function and A_Θ which were proved in [5, Proposition 2.6 and Theorem 2.8].

Proposition 2.3 *Let A be a densely defined, closed, symmetric operator in a Hilbert space and let $(\mathcal{G}, \Gamma_0, \Gamma_1)$ be a quasi boundary triple for A^* with γ-field γ and Weyl function M. For $\lambda \in \rho(A_0)$ the following assertions hold.*

(i) *$\gamma(\lambda)$ is a densely defined bounded operator from $\mathcal{G}$ to $\mathcal{H}$ with $\operatorname{dom} \gamma(\lambda) = \operatorname{ran} \Gamma_0$.*

(ii) *$\gamma(\overline{\lambda})^*$ is a bounded mapping defined on $\mathcal{H}$ with values in $\operatorname{ran} \Gamma_1 \subset \mathcal{G}$, and*

$$\gamma(\overline{\lambda})^* = \Gamma_1 (A_0 - \lambda)^{-1} \tag{2.1}$$

holds.

(iii) $M(\lambda)$ *maps* $\operatorname{ran}\Gamma_0$ *into* $\operatorname{ran}\Gamma_1$. *If, in addition,* $T|_{\ker\Gamma_1}$ *is self-adjoint in* $\mathcal{H}$ *and* $\lambda \in \rho(T|_{\ker\Gamma_1})$, *then* $M(\lambda)$ *maps* $\operatorname{ran}\Gamma_0$ *onto* $\operatorname{ran}\Gamma_1$.

(iv) *For* $\lambda \in \mathbb{C}^+$ *(or* $\mathbb{C}^-$*), where* $\mathbb{C}^{\pm} := \{z \in \mathbb{C}\colon \pm\operatorname{Im} z > 0\}$, *the operator*

$$\operatorname{Im} M(\lambda) := \frac{1}{2i}\big(M(\lambda) - M(\lambda)^*\big)$$

is bounded and positive (negative, respectively).

(v) *Let* Θ *be a linear operator in* $\mathcal{G}$. *Then* λ *is an eigenvalue of* A_Θ *if and only if* 0 *is an eigenvalue of* $\Theta - M(\lambda)$. *If* λ *is not an eigenvalue of* A_Θ, *then Krein's formula*

$$(A_\Theta - \lambda)^{-1}f = (A_0 - \lambda)^{-1}f + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^*f \qquad (2.2)$$

holds for every $f \in \mathcal{H}$ *for which* $\gamma(\overline{\lambda})^*f \in \operatorname{ran}(\Theta - M(\lambda))$.

In the following we recall how the concept of quasi boundary triples can be applied to the Laplace operator on a bounded domain with C^∞ boundary; cf. [5, Section 4.2]. We refer the reader to [14–16,22] for recent work on the Laplacian and elliptic operators in non-smooth domains, and to [8,13,19] for a different approach that leads to an ordinary boundary triple. Let $\Omega \subseteq \mathbb{R}^n$, $n > 1$, be a bounded domain with C^∞ boundary $\partial\Omega$, let $\nu(x)$ be the normal vector at the point $x \in \partial\Omega$ pointing outwards and consider the differential expression $-\Delta$ on Ω. The operator A defined by

$$Af = -\Delta f, \qquad \operatorname{dom} A = H_0^2(\Omega) = \left\{f \in H^2(\Omega)\colon f|_{\partial\Omega} = \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = 0\right\},$$

where $f|_{\partial\Omega}$ is the trace of f and

$$\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = \sum_{i=1}^{n} \nu_i \frac{\partial f}{\partial x_i}\Big|_{\partial\Omega}$$

is the outer normal derivative, is a densely defined, closed, symmetric operator with equal infinite deficiency indices in $L^2(\Omega)$. The adjoint of A is

$$A^*f = -\Delta f, \qquad \operatorname{dom} A^* = \left\{f \in L^2(\Omega)\colon -\Delta f \in L^2(\Omega)\right\}.$$

We consider a restriction T of A^* so that we can define boundary mappings on $\operatorname{dom} T$. As in [5] we use as domain of T a Beals space, which turns out to be very convenient. Let us recall its definition; for further details see, e.g. [4]. Since $\partial\Omega$ is a C^∞ boundary of Ω, there exists $\varepsilon_0 > 0$ such that for all $0 \le \varepsilon < \varepsilon_0$ the mapping $x \mapsto x - \varepsilon\nu(x)$ is a homeomorphism from $\partial\Omega$ onto $\{x - \varepsilon\nu(x)\colon x \in \partial\Omega\}$. If $f \in L^2(\Omega)$ and $-\Delta f \in L^2(\Omega)$, then $f \in H^2_{\text{loc}}(\Omega)$. Hence f_ε defined by $f_\varepsilon(x) := f(x - \varepsilon\nu(x))$ is in $L^2(\partial\Omega)$. We say that f has L^2

boundary value on $\partial\Omega$ *if* $\lim_{\varepsilon\to 0+} f_\varepsilon$ exists as a limit in $L^2(\partial\Omega)$. In this case we write $f|_{\partial\Omega} := \lim_{\varepsilon\to 0+} f_\varepsilon$.

Definition 2.4 *The* Beals space of first order *is defined as*

$$\mathcal{D}_1(\Omega) := \Big\{ f \in L^2(\Omega) \colon\ -\Delta f \in L^2(\Omega),\ \textit{and}$$

$$f, \frac{\partial f}{\partial x_i}\ \textit{have } L^2 \textit{ boundary values on } \partial\Omega \textit{ for all } i = 1,\dots,n \Big\}.$$

It is known (see [4]) that $H^2(\Omega) \subset \mathcal{D}_1(\Omega) \subset H^{\frac{3}{2}}(\Omega)$. We define the operator T,

$$Tf = -\Delta f, \quad \operatorname{dom} T = \mathcal{D}_1(\Omega),$$

and the boundary mappings

$$\Gamma_0 \colon \operatorname{dom} T \mapsto L^2(\partial\Omega), \qquad \Gamma_0 f = \frac{\partial f}{\partial\nu}\Big|_{\partial\Omega},$$

$$\Gamma_1 \colon \operatorname{dom} T \mapsto L^2(\partial\Omega), \qquad \Gamma_1 f = f|_{\partial\Omega}.$$

The restrictions

$$-\Delta_N^\Omega := T|_{\ker\Gamma_0}, \qquad -\Delta_D^\Omega := T|_{\ker\Gamma_1}$$

are the usual Neumann and Dirichlet Laplacians whose domains are both contained in $H^2(\Omega)$; moreover, $T|_{\ker\Gamma_0\cap\ker\Gamma_1} = A$. Fundamental properties of Beals spaces imply that

$$\operatorname{ran}\Gamma_0 = L^2(\partial\Omega), \qquad \operatorname{ran}\Gamma_1 = H^1(\partial\Omega).$$

In [5] it was shown that the triple $(L^2(\partial\Omega), \Gamma_0, \Gamma_1)$ is a quasi boundary triple for A^*.

In the next proposition Krein's formula is recalled, and a class of self-adjoint, maximal dissipative and maximal accumulative generalized Robin Laplacians is parameterized with the help of the quasi boundary triple $(L^2(\partial\Omega), \Gamma_0, \Gamma_1)$. Recall that a linear operator Θ in a Hilbert space is said to be *dissipative* (*accumulative*) if $\operatorname{Im}(\Theta f, f) \geq 0$ ($\operatorname{Im}(\Theta f, f) \leq 0$, respectively) for all $f \in \operatorname{dom}\Theta$, and Θ is said to be *maximal dissipative* (*maximal accumulative*) if Θ is dissipative (accumulative, respectively) and has no proper dissipative (accumulative, respectively) extension. A dissipative (accumulative) operator Θ is maximal dissipative (maximal accumulative, respectively) if and only if $\Theta - \lambda_-$ ($\Theta - \lambda_+$, respectively) is surjective for some (and hence for all) $\lambda_- \in \mathbb{C}^-$ ($\lambda_+ \in \mathbb{C}^+$, respectively).

Proposition 2.5 *Let* $T = -\Delta|_{\mathcal{D}_1(\Omega)}$, $(L^2(\partial\Omega), \Gamma_0, \Gamma_1)$, Δ_N^Ω, Δ_D^Ω *be as above and denote by γ and M the corresponding γ-field and Weyl function. Then the following assertions hold.*

(i) *For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the operator $M(\lambda)$ is compact in $L^2(\partial\Omega)$ and $M(\lambda)^{-1}$ is a bounded operator from $H^1(\partial\Omega)$ onto $L^2(\partial\Omega)$.*

(ii) *Krein's formula*

$$(-\Delta_D^\Omega - \lambda)^{-1} - (-\Delta_N^\Omega - \lambda)^{-1} = -\gamma(\lambda)M(\lambda)^{-1}\gamma(\overline{\lambda})^* \qquad (2.3)$$

holds for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Further, let Θ be a self-adjoint (maximal dissipative, maximal accumulative) operator in $L^2(\partial\Omega)$ such that $0 \notin \sigma_{\mathrm{ess}}(\Theta)$. Then also the following statements are true.

(iii) *For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ($\lambda \in \mathbb{C}^-$, $\lambda \in \mathbb{C}^+$, respectively) the operator*

$$\bigl(\Theta - M(\lambda)\bigr)^{-1}$$

is bounded and everywhere defined in $L^2(\partial\Omega)$.

(iv) *Denote by $-\Delta_\Theta^\Omega$ the restriction of T to*

$$\mathrm{dom}\,(-\Delta_\Theta^\Omega) = \bigl\{ f \in \mathcal{D}_1(\Omega) \colon \Gamma_1 f = \Theta\Gamma_0 f \bigr\}.$$

Then $-\Delta_\Theta^\Omega$ is self-adjoint (maximal dissipative, maximal accumulative, respectively) in $L^2(\Omega)$, and Krein's formula

$$(-\Delta_\Theta^\Omega - \lambda)^{-1} - (-\Delta_N^\Omega - \lambda)^{-1} = \gamma(\lambda)\bigl(\Theta - M(\lambda)\bigr)^{-1}\gamma(\overline{\lambda})^*$$

holds for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ($\lambda \in \mathbb{C}^-$, $\lambda \in \mathbb{C}^+$, respectively).

Proof. (i) Without loss of generality let $\lambda \in \mathbb{C}^+$. That $M(\lambda)$ is compact in $L^2(\partial\Omega)$ was proved in [5, Proposition 4.6]. Since

$$\mathrm{Im}\,\bigl(M(\lambda)x, x\bigr) = \bigl(\mathrm{Im}\,M(\lambda)x, x\bigr) > 0$$

for every $x \in L^2(\partial\Omega)$, $x \neq 0$, by Proposition 2.3 (iv), we have $\ker M(\lambda) = \{0\}$. It follows from the proof of [5, Proposition 4.6] that $M(\lambda)$ is closed from $L^2(\partial\Omega)$ onto $H^1(\partial\Omega)$. Hence its inverse $M(\lambda)^{-1}$ is also closed and by the closed graph theorem bounded from $H^1(\partial\Omega)$ onto $L^2(\partial\Omega)$.

(ii) In (2.2) we can choose $\Theta = 0$, which yields (2.3) applied to all f for which $\gamma(\overline{\lambda})^* f \in \mathrm{ran}\, M(\lambda)$. It follows from (2.1) that

$$\mathrm{ran}\, \gamma(\overline{\lambda})^* \subset \mathrm{ran}\, \Gamma_1 = H^1(\partial\Omega) = \mathrm{ran}\, M(\lambda),$$

and hence Krein's formula (2.3) holds on the whole space $L^2(\Omega)$.

(iii) and (iv) were shown in [5, Theorems 4.8 and 4.10]. $\qquad\square$

3   Schatten–von Neumann classes and resolvent differences

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. We denote by $\mathcal{S}_\infty(\mathcal{H}, \mathcal{K})$ the class of compact operators from $\mathcal{H}$ to $\mathcal{K}$. For $T \in \mathcal{S}_\infty(\mathcal{H}, \mathcal{K})$ the eigenvalues $s_k(T)$ of the non-negative compact operator $(T^*T)^{\frac{1}{2}}$, ordered non-increasingly and counted with multiplicites, are called *s-numbers* of T.

Definition 3.1 *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. For $p > 0$, the Schatten–von Neumann class is defined by*

$$\mathcal{S}_p(\mathcal{H}, \mathcal{K}) := \left\{ T \in \mathcal{S}_\infty(\mathcal{H}, \mathcal{K}) \colon \sum_{k=1}^{\infty} (s_k(T))^p < \infty \right\}.$$

If $\mathcal{K} = \mathcal{H}$, we write $\mathcal{S}_p(\mathcal{H})$ for $\mathcal{S}_p(\mathcal{H}, \mathcal{K})$, $0 < p \le \infty$.

The set $\mathcal{S}_p(\mathcal{H}, \mathcal{K})$ is an ideal for every p with $0 < p \le \infty$ and a normed ideal if $1 \le p \le \infty$. In the following two lemmas we recall some well-known facts about s-numbers and Schatten–von Neumann classes. For the proofs see, e.g. Sections II.§2.1, II.§2.2, III.§7.2 in [18].

Lemma 3.2 *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and let $T \in \mathcal{S}_\infty(\mathcal{H}, \mathcal{K})$. Then the following hold:*

(i) *If B, C are bounded operators, then*

$$s_k(BTC) \le \|B\| \, \|C\| s_k(T) \qquad \text{for all } k \in \mathbb{N}.$$

(ii) *$s_k(T) = s_k(T^*)$ for all $k \in \mathbb{N}$.*

(iii) *If $s_k(T) = O(k^{-\alpha})$ as $k \to \infty$ for some $\alpha > 0$, then*

$$T \in \mathcal{S}_p(\mathcal{H}, \mathcal{K}) \quad \text{for all } p > \frac{1}{\alpha}.$$

Lemma 3.3   *Let $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_n$ be separable Hilbert spaces, let $p, p_1, \ldots, p_n > 0$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n},$$

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and assume that T_i are compact operators in $\mathcal{S}_{p_i}(\mathcal{H}_{i-1}, \mathcal{H}_i)$, $i = 1, \ldots, n$. Then

$$T_n \cdots T_1 \in \mathcal{S}_p(\mathcal{H}_0, \mathcal{H}_n).$$

The next lemma will be used in the proofs of our main results.

Lemma 3.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a compact domain with C^∞ boundary $\partial\Omega$. Further, let B be an everywhere defined, bounded operator from $L^2(\Omega)$ to $H^{r_1}(\partial\Omega)$ with $\operatorname{ran} B \subseteq H^{r_2}(\partial\Omega)$ for $r_2 > r_1 \geq 0$. Then*

$$B \in \mathcal{S}_p\big(L^2(\Omega), H^{r_1}(\partial\Omega)\big) \quad \text{for all } p > \frac{n-1}{r_2 - r_1}.$$

Proof. As in [2, Proposition 5.4.1] we can define

$$\Lambda_{r_1,r_2} := (I - \Delta_{\mathrm{LB}}^{\partial\Omega})^{\frac{r_2-r_1}{2}},$$

where $\Delta_{\mathrm{LB}}^{\partial\Omega}$ is the Laplace–Beltrami operator on $\partial\Omega$. The operator Λ_{r_1,r_2} is an isometric isomorphism from $H^{r_2}(\partial\Omega)$ onto $H^{r_1}(\partial\Omega)$. The asymptotics of the eigenvalues of the Laplace–Beltrami operator, $\lambda_k(I - \Delta_{\mathrm{LB}}^{\partial\Omega}) \sim Ck^{\frac{2}{n-1}}$ with some constant C, imply that

$$s_k(\Lambda_{r_1,r_2}^{-1}) = O\big(k^{-\frac{r_2-r_1}{n-1}}\big), \qquad k \to \infty,$$

where Λ_{r_1,r_2}^{-1} is considered as an operator in $H^{r_1}(\partial\Omega)$. We can write B in the form

$$B = \Lambda_{r_1,r_2}^{-1}(\Lambda_{r_1,r_2} B).$$

The operator B is closed as an operator from $L^2(\Omega)$ to $H^{r_1}(\partial\Omega)$, hence also closed as an operator from $L^2(\Omega)$ to $H^{r_2}(\partial\Omega)$, which implies that it is bounded from $L^2(\Omega)$ to $H^{r_2}(\partial\Omega)$. Therefore the operator $\Lambda_{r_1,r_2} B$ is bounded from $L^2(\partial\Omega)$ to $H^{r_1}(\partial\Omega)$, and hence Lemma 3.2 (i) implies

$$s_k(B) \leq \|\Lambda_{r_1,r_2} B\| s_k(\Lambda_{r_1,r_2}^{-1}) = O(k^{-\frac{r_2-r_1}{n-1}}), \qquad k \to \infty,$$

from which the assertion follows by Lemma 3.2 (iii). $\qquad\square$

The next theorem, our first main result, is about Schatten–von Neumann properties of differences of resolvents of the Neumann Laplacian and a Laplacian determined by some boundary operator Θ. For similar results involving the Dirichlet, Neumann and Robin Laplacian we refer the reader to [3,6,7,13,20–22] and references therein.

Theorem 3.5 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$ and let Θ be a self-adjoint (maximal dissipative, maximal accumulative) operator in $L^2(\partial\Omega)$ such that $0 \notin \sigma_{\mathrm{ess}}(\Theta)$. Denote by $-\Delta_N^\Omega$ the Neumann Laplacian on Ω and by $-\Delta_\Theta^\Omega$ the generalized Robin Laplacian from Proposition 2.5 (iv). Then*

$$(-\Delta_\Theta^\Omega - \lambda)^{-1} - (-\Delta_N^\Omega - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega)) \quad \text{for all } p > \frac{n-1}{3} \qquad (3.1)$$

and all $\lambda \in \rho(-\Delta_\Theta^\Omega) \cap \rho(-\Delta_N^\Omega)$. In particular, for $n = 2$ and $n = 3$ the resolvent difference is a trace class operator.

Proof. According to Proposition 2.5 (iv) we have Krein's formula

$$(-\Delta_\Theta^\Omega - \lambda)^{-1} - (-\Delta_N^\Omega - \lambda)^{-1} = \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\bar\lambda)^* \qquad (3.2)$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ($\lambda \in \mathbb{C}^-$, $\lambda \in \mathbb{C}^+$, respectively). Equation (2.1), the inclusion $\mathrm{dom}\,(\Delta_N^\Omega) \subseteq H^2(\Omega)$ and the trace theorem (see, e.g. [1,25]) imply that

$$\mathrm{ran}\left(\gamma(\bar\lambda)^*\right) \subseteq H^{\frac{3}{2}}(\partial\Omega).$$

Because the operator $\gamma(\bar\lambda)^*$ is bounded from $L^2(\Omega)$ to $L^2(\partial\Omega)$ by Proposition 2.3 (ii), it is closed from $L^2(\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega)$ and hence bounded by the closed graph theorem. Now Lemma 3.4 yields $\gamma(\bar\lambda)^* \in \mathcal{S}_p(L^2(\Omega), L^2(\partial\Omega))$ for all $p > \frac{2(n-1)}{3}$.

The same is true for $\gamma(\lambda)^*$, and hence the adjoint $\gamma(\lambda) = \gamma(\lambda)^{**}$ is in $\mathcal{S}_p(L^2(\partial\Omega), L^2(\Omega))$ for all $p > \frac{2(n-1)}{3}$. The operator $(\Theta - M(\lambda))^{-1}$ is bounded by Proposition 2.5 (iii). Therefore Lemma 3.3 implies that the right-hand side of (3.2) is in $\mathcal{S}_p(L^2(\Omega))$ for all $p > \frac{n-1}{3}$ and all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ($\lambda \in \mathbb{C}^-$, $\lambda \in \mathbb{C}^+$, respectively). The fact that (3.1) holds for all points in $\rho(-\Delta_\Theta^\Omega) \cap \rho(-\Delta_N^\Omega)$ follows from the formula

$$(-\Delta_\Theta^\Omega - \mu)^{-1} - (-\Delta_N^\Omega - \mu)^{-1} = \Big(I + (\mu - \lambda)(-\Delta_N^\Omega - \mu)^{-1}\Big)$$
$$\times \Big((-\Delta_\Theta^\Omega - \lambda)^{-1} - (-\Delta_N^\Omega - \lambda)^{-1}\Big)\Big(I + (\mu - \lambda)(-\Delta_\Theta^\Omega - \mu)^{-1}\Big)$$

which is true for all $\lambda, \mu \in \rho(-\Delta_\Theta^\Omega) \cap \rho(-\Delta_N^\Omega)$. $\qquad\qquad\square$

Note that the resolvent of the Neumann Laplacian on a bounded domain itself is a compact operator, so that the same holds true for the resolvent of the generalized Robin Laplacian $-\Delta_\Theta^\Omega$. In other words, the spectrum of any self-adjoint (maximal dissipative, maximal accumulative) Robin Laplacian $-\Delta_\Theta^\Omega$ in Theorem 3.5 consists only of normal eigenvalues. Therefore, the intersections of the resolvent sets $\rho(-\Delta_{\Theta_1}^\Omega) \cap \rho(-\Delta_{\Theta_2}^\Omega)$ of two such Laplacians is always non-empty and by taking the difference of the expressions in (3.1) we obtain the following corollary.

Corollary 3.6 *Let Θ_1 and Θ_2 be self-adjoint, maximal dissipative or maximal accumulative operators in $L^2(\partial\Omega)$ such that $0 \notin \sigma_{\mathrm{ess}}(\Theta_i)$, $i = 1, 2$. Then*

$$(-\Delta^{\Omega}_{\Theta_1} - \lambda)^{-1} - (-\Delta^{\Omega}_{\Theta_2} - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega)) \quad \text{for all } p > \frac{n-1}{3}$$

and all $\lambda \in \rho(-\Delta^{\Omega}_{\Theta_1}) \cap \rho(-\Delta^{\Omega}_{\Theta_2})$.

Remark 3.7 *Proposition 2.5 (iii), (iv) and hence Theorem 3.5 are still valid if Θ is a self-adjoint (maximal dissipative, maximal accumulative) linear relation (i.e. a multi-valued operator) in $L^2(\partial\Omega)$ such that $0 \notin \sigma_{\mathrm{ess}}(\Theta)$; see [5, Section 4]. In particular, if $0 \in \rho(\Theta)$, then Θ^{-1} is a bounded, self-adjoint (maximal dissipative, maximal accumulative, respectively) operator. Conversely, for every bounded, self-adjoint (maximal dissipative, maximal accumulative) operator B, the inverse B^{-1} is a self-adjoint (maximal dissipative, maximal accumulative, respectively) relation with $0 \in \rho(B^{-1})$. Hence the restriction $-\Delta^{\Omega}_{B^{-1}}$ of T to the domain*

$$\mathrm{dom}\,(-\Delta^{\Omega}_{B^{-1}}) = \left\{ f \in \mathcal{D}_1(\Omega) \colon \left.\frac{\partial f}{\partial n}\right|_{\partial\Omega} = Bf|_{\partial\Omega} \right\}$$

is a self-adjoint (maximal dissipative, maximal accumulative, respectively) realization of the Laplacian and satisfies

$$(-\Delta^{\Omega}_{B^{-1}} - \lambda)^{-1} - (-\Delta^{\Omega}_N - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega)) \quad \text{for all } p > \frac{n-1}{3}\,.$$

As a special case we can treat (ordinary) Robin boundary conditions

$$\left.\frac{\partial f}{\partial n}\right|_{\partial\Omega} = \beta f|_{\partial\Omega},$$

where the values of $\beta \in L^{\infty}(\partial\Omega)$ are real (have positive/negative imaginary parts, respectively).

Theorem 3.5 does not cover the case of the difference of Dirichlet and Neumann Laplacians since for the Dirichlet Laplacian we have to choose $\Theta = 0$, which does not satisfy $0 \notin \sigma_{\mathrm{ess}}(\Theta)$. However, we obtain the following result, which is due to Birman [6].

Theorem 3.8 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^{∞} boundary $\partial\Omega$. Then*

$$(-\Delta^{\Omega}_D - \lambda)^{-1} - (-\Delta^{\Omega}_N - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega)) \quad \text{for all } p > \frac{n-1}{2} \qquad (3.3)$$

and all $\lambda \in \rho(-\Delta^{\Omega}_D) \cap \rho(-\Delta^{\Omega}_N)$. In particular, for $n = 2$ the resolvent difference is a trace class operator.

Proof. By Proposition 2.5 (ii) we have

$$(-\Delta_D^\Omega - \lambda)^{-1} - (-\Delta_N^\Omega - \lambda)^{-1} = -\gamma(\lambda)M(\lambda)^{-1}\gamma(\overline{\lambda})^*$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The operator $\gamma(\overline{\lambda})^*$ is bounded as an operator from $L^2(\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega)$; see the proof of Theorem 3.5. As an operator from $L^2(\Omega)$ to $H^1(\partial\Omega)$ it is in $\mathcal{S}_p(L^2(\Omega), H^1(\partial\Omega))$ for all $p > 2(n-1)$ according to Lemma 3.4.

By Proposition 2.5 (i), the operator $M(\lambda)^{-1}$ is bounded from $H^1(\partial\Omega)$ to $L^2(\partial\Omega)$ and therefore $M(\lambda)^{-1}\gamma(\overline{\lambda})^* \in \mathcal{S}_p(L^2(\Omega), L^2(\partial\Omega))$ for all $p > 2(n-1)$. As in the proof of Theorem 3.5 we have $\gamma(\lambda) \in \mathcal{S}_p(L^2(\partial\Omega), L^2(\Omega))$ for all $p > \frac{2(n-1)}{3}$. Hence Lemma 3.3 implies that the resolvent difference in (3.3) is in $\mathcal{S}_p(L^2(\Omega))$ for all

$$p > \frac{1}{\frac{1}{2(n-1)} + \frac{3}{2(n-1)}} = \frac{n-1}{2}.$$

The same argument as in the proof of Theorem 3.5 shows that (3.3) holds also for all $\lambda \in \rho(-\Delta_D^\Omega) \cap \rho(-\Delta_N^\Omega)$. □

Remark 3.9 *Comparing the result of Theorem 3.5 with the result of Theorem 3.8 we see that we have $\frac{n-1}{3}$ instead of $\frac{n-1}{2}$. The explanation comes from the fact that $M(\lambda)$ is compact in $L^2(\partial\Omega)$, and hence $M(\lambda)^{-1}$ is unbounded in $L^2(\partial\Omega)$ whereas for Θ as in Theorem 3.5 the operator $(\Theta - M(\lambda))^{-1}$ is bounded in $L^2(\partial\Omega)$.*

Combining Theorems 3.5 and 3.8 we obtain the following corollary.

Corollary 3.10 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$ and let Θ be a self-adjoint (maximal dissipative, maximal accumulative) operator in $L^2(\partial\Omega)$ such that $0 \notin \sigma_{\mathrm{ess}}(\Theta)$. Denote by $-\Delta_\Theta^\Omega$ the generalized Robin Laplacian from Proposition 2.5 (iv). Then*

$$(-\Delta_\Theta^\Omega - \lambda)^{-1} - (-\Delta_D^\Omega - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega)) \quad \text{for all } p > \frac{n-1}{2}$$

and all $\lambda \in \rho(-\Delta_\Theta^\Omega) \cap \rho(-\Delta_D^\Omega)$.

For ordinary boundary triples the resolvent difference belongs to the same Schatten–von Neumann class as the resolvent difference of the operators which parameterize the extensions; see [11, Theorem 2 and Corollary 4]. In the case of quasi boundary triples the situation is different. In the next Theorem we assume that $\Theta_2 - \Theta_1 \in \mathcal{S}_{p_0}(L^2(\partial\Omega))$ for some $p_0 > 0$ and investigate Schatten–von Neumann properties of the resolvent difference of the generalized Robin Laplacians parameterized by Θ_1 and Θ_2.

Theorem 3.11 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Further, let Θ_1 and Θ_2 be bounded self-adjoint, maximal dissipative or maximal accumulative operators in $L^2(\partial\Omega)$ such that $0 \notin \sigma_{\mathrm{ess}}(\Theta_i)$, $i = 1, 2$, and*

$$\Theta_1 - \Theta_2 \in \mathcal{S}_{p_0}(L^2(\partial\Omega))$$

for some $p_0 \in (0, \infty)$. Denote by $-\Delta_{\Theta_i}^\Omega$ the restriction of T as in Proposition 2.5 (iv). Then

$$(-\Delta_{\Theta_1}^\Omega - \lambda)^{-1} - (-\Delta_{\Theta_2}^\Omega - \lambda)^{-1} \in \mathcal{S}_p(L^2(\Omega))$$

$$\text{for all } p > \frac{(n-1)p_0}{n - 1 + 3p_0} \tag{3.4}$$

and all $\lambda \in \rho(-\Delta_{\Theta_1}^\Omega) \cap \rho(-\Delta_{\Theta_2}^\Omega)$.

By Theorem 3.5 and Corollary 3.6 the difference of the resolvents of $-\Delta_{\Theta_1}^\Omega$ and $-\Delta_{\Theta_2}^\Omega$ is a trace class operator for $n = 2$ and $n = 3$ without any further assumptions on $\Theta_1 - \Theta_2$. If, in addition, $\Theta_1 - \Theta_2 \in \mathcal{S}_{p_0}(L^2(\partial\Omega))$ for some $p_0 \in (0, \infty)$, then this also holds for $n = 4$.

Corollary 3.12 *Let the assumptions be as in Theorem 3.11. For $n \in \{2, 3, 4\}$ and all $p_0 \in (0, \infty)$ the resolvent difference in (3.4) is a trace class operator. The same holds for $n > 4$ and $p_0 < \frac{n-1}{n-4}$.*

Proof of Theorem 3.11. Assume first that Θ_2 is self-adjoint and that Θ_1 is self-adjoint (maximal dissipative or maximal accumulative, respectively). According to Proposition 2.5 (iv) we can write

$$(-\Delta_{\Theta_1}^\Omega - \lambda)^{-1} - (-\Delta_{\Theta_2}^\Omega - \lambda)^{-1}$$

$$= \gamma(\lambda)\left[\left(\Theta_1 - M(\lambda)\right)^{-1} - \left(\Theta_2 - M(\lambda)\right)^{-1}\right]\gamma(\bar{\lambda})^*$$

$$= \gamma(\lambda)\left(\Theta_1 - M(\lambda)\right)^{-1}(\Theta_2 - \Theta_1)\left(\Theta_2 - M(\lambda)\right)^{-1}\gamma(\bar{\lambda})^*$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ($\lambda \in \mathbb{C}^-$, $\lambda \in \mathbb{C}^+$, respectively). As in the proof of Theorem 3.5 we have

$$\gamma(\lambda) \in \mathcal{S}_p\left(L^2(\partial\Omega), L^2(\Omega)\right), \quad \gamma(\bar{\lambda})^* \in \mathcal{S}_p\left(L^2(\Omega), L^2(\partial\Omega)\right)$$

$$\text{for all } p > \frac{2(n-1)}{3}.$$

The operators $(\Theta_i - M(\lambda))^{-1}$ are bounded by Proposition 2.5 (iii). Hence, using Lemma 3.3 we obtain that the resolvent difference in (3.4) is in $\mathcal{S}_p(L^2(\Omega))$ for all

$$p > \frac{1}{\frac{3}{2(n-1)} + \frac{1}{p_0} + \frac{3}{2(n-1)}} = \frac{(n-1)p_0}{n - 1 + 3p_0}.$$

The same argument as in the proof of Theorem 3.5 shows that (3.3) holds also for all $\lambda \in \rho(-\Delta^{\Omega}_{\Theta_1}) \cap \rho(-\Delta^{\Omega}_{\Theta_2})$.

In the case that Θ_1 and Θ_2 are both either maximal dissipative or maximal accumulative the above arguments remain valid for $\lambda \in \mathbb{C}^-$ or $\lambda \in \mathbb{C}^+$, respectively, and hence (3.4) holds also in this case.

Let us now consider the case that Θ_1 is maximal dissipative and Θ_2 is maximal accumulative. If Θ_1 is maximal accumulative and Θ_2 is maximal dissipative a similar reasoning applies. As $\Theta_1 - \Theta_2 \in \mathcal{S}_{p_0}(L^2(\partial\Omega))$ we also have

$$\mathrm{Re}\,(\Theta_1 - \Theta_2) \in \mathcal{S}_{p_0}(L^2(\partial\Omega)) \quad \text{and} \quad \mathrm{Im}\,(\Theta_1 - \Theta_2) \in \mathcal{S}_{p_0}(L^2(\partial\Omega)),$$

and since $\mathrm{Im}\,\Theta_1 \geq 0$ and $\mathrm{Im}\,\Theta_2 \leq 0$ we conclude from the inequalities

$$0 \leq \mathrm{Im}\,\Theta_1 \leq \mathrm{Im}\,(\Theta_1 - \Theta_2) \quad \text{and} \quad 0 \leq -\mathrm{Im}\,\Theta_2 \leq \mathrm{Im}\,(\Theta_1 - \Theta_2)$$

that also $\mathrm{Im}\,\Theta_i$, $i = 1, 2$, belong to $\mathcal{S}_{p_0}(L^2(\partial\Omega))$. Therefore

$$\Theta_i - \mathrm{Re}\,\Theta_i \in \mathcal{S}_{p_0}(L^2(\partial\Omega)) \quad \text{and} \quad \sigma_{\mathrm{ess}}(\Theta_i) = \sigma_{\mathrm{ess}}(\mathrm{Re}\,\Theta_i), \qquad i = 1, 2,$$

and by the first part of the proof each of the resolvent differences

$$
\begin{aligned}
&(-\Delta^{\Omega}_{\Theta_1} - \lambda)^{-1} - (-\Delta^{\Omega}_{\mathrm{Re}\,\Theta_1} - \lambda)^{-1}, \\
&(-\Delta^{\Omega}_{\mathrm{Re}\,\Theta_1} - \mu)^{-1} - (-\Delta^{\Omega}_{\mathrm{Re}\,\Theta_2} - \mu)^{-1}, \\
&(-\Delta^{\Omega}_{\mathrm{Re}\,\Theta_2} - \vartheta)^{-1} - (-\Delta^{\Omega}_{\Theta_2} - \vartheta)^{-1}
\end{aligned}
\tag{3.5}
$$

belongs to $\mathcal{S}_p(L^2(\Omega))$, where $p > \frac{(n-1)p_0}{n-1+3p_0}$. Moreover, the resolvents of $-\Delta^{\Omega}_{\Theta_i}$ and $-\Delta^{\Omega}_{\mathrm{Re}\,\Theta_i}$, $i = 1, 2$, are all compact and hence almost all $\lambda \in \mathbb{C}$ belong to the intersection of the resolvent sets of these generalized Robin Laplacians. Then it follows from (3.5) that the difference of the resolvents of $-\Delta^{\Omega}_{\Theta_1}$ and $-\Delta^{\Omega}_{\Theta_2}$ satisfies (3.4). $\qquad\square$

Acknowledgements

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