Singular Indefinite Sturm-Liouville Operators with a Spectral Gap

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Singular Sturm-Liouville operators with the indefinite weight $\operatorname{sgn}(\cdot)$ and a symmetric potential which has a positive limit at ∞ have a gap in the essential spectrum. Under an additional condition it is shown that in this gap are no eigenvalues.

1 Introduction and main result

In this note we consider the maximal differential operator in $L^2(\mathbb{R})$ associated to the indefinite Sturm-Liouville differential expression

$$\tau = \operatorname{sgn}(\cdot)\left(-\frac{d^2}{dx^2} + q\right), \tag{1}$$

where $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ is real-valued, symmetric with respect to 0, i.e., $q(x) = q(-x)$, $x \in \mathbb{R}$, and $\lim_{x\to\pm\infty} q(x) = q_\infty$ exists and is positive, $q_\infty > 0$. Note that the differential expression τ is not formally symmetric with respect to the scalar product in $L^2(\mathbb{R})$. The maximal differential operator A associated to τ is defined as

$$(Af)(x) = \operatorname{sgn}(x)\big(-f''(x) + q(x)f(x)\big), \quad x \in \mathbb{R}, \qquad f \in \operatorname{dom} A = \mathcal{D},$$

where $\mathcal{D}$ denotes the usual maximal domain given by $\mathcal{D} = \{f \in L^2(\mathbb{R}) : f, f' \text{ absolutely continuous}, \tau f \in L^2(\mathbb{R})\}$.

Spectral properties of indefinite Sturm-Liouville operators play an important role in various applications and have attracted a lot of attention in the recent past, we refer the reader to [2, 4–6, 8, 10] for more details and further references. The following theorem summarizes some facts on the spectrum $\sigma(A)$ and the essential spectrum $\sigma_{\mathrm{ess}}(A)$ of A. A proof can be found in, e.g., [1, 4, 7]. We emphasize that the assumption $\lim_{x\to\infty} q(x) = q_\infty > 0$ is essential for this statement.

Theorem 1.1 *$\sigma(A) \cap (\mathbb{C} \setminus \mathbb{R})$ consists of at most finitely many pairs $\{\mu, \bar{\mu}\}$ of eigenvalues and $\sigma_{\mathrm{ess}}(A) = \mathbb{R} \setminus (-q_\infty, q_\infty)$.*

The main objective of this note is to study the spectrum of A in the gap $(-q_\infty, q_\infty)$ of the essential spectrum. For this it is convenient to introduce the maximal operator B associated to the definite Sturm-Liouville expression $\ell = -d^2/dx^2 + q$,

$$(Bf)(x) = -f''(x) + q(x)f(x), \quad x \in \mathbb{R}, \qquad f \in \operatorname{dom} B = \mathcal{D}.$$

It is well known that under the above assumptions on q the differential expression ℓ is in the limit point case at both endpoints $\pm\infty$ and therefore B is a selfadjoint operator in the Hilbert space $L^2(\mathbb{R})$, see, e.g., [3, 9, 10]. Furthermore, B is semibounded from below and the essential spectrum $\sigma_{\mathrm{ess}}(B)$ is the whole interval $[q_\infty, \infty)$. The next well known statement is a refinement of Theorem 1.1. The set of eigenvalues of B is denoted by $\sigma_p(B)$.

Theorem 1.2 *If $\sigma_p(B) \cap (-\infty, 0) = \emptyset$, then $\sigma(A) \subset \mathbb{R}$ and $\sigma_{\mathrm{ess}}(A) = \mathbb{R} \setminus (-q_\infty, q_\infty)$.*

The following theorem is the main result of this note. Under slightly stronger assumptions on $\sigma_p(B)$ we get a precise description of the spectrum of A.

Theorem 1.3 *If $\sigma_p(B) \cap (-\infty, q_\infty) = \emptyset$, then $\sigma_p(A) \cap (-q_\infty, q_\infty) = \emptyset$ and $\sigma(A) = \sigma_{\mathrm{ess}}(A) = \mathbb{R} \setminus (-q_\infty, q_\infty)$.*

2 Proof of Theorem 1.3

The statements in Theorem 1.3 follow from Theorem 1.2 if we show that A has no eigenvalues in the interval $(-q_\infty, q_\infty)$. The proof of this is based on elementary facts on solutions of linear ordinary differential equations, see, e.g. [3, 9]. Furthermore, the observations in Lemma 2.1 and Lemma 2.2 below are essential ingredients in the proof of Theorem 1.3.

We define $\mathcal{D}_+$ and $\mathcal{D}_-$ in the same way as $\mathcal{D}$, where $\mathbb{R}$ is replaced by $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, and τ is replaced by the restrictions $\tau_+ = -d^2/dx^2 + q$ and $\tau_- = d^2/dx^2 - q$ of τ onto $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. Since $\ell = -d^2/dx^2 + q$ is in the limit point case and $\sigma_{\mathrm{ess}}(B) = [q_\infty, \infty)$ it follows that for each $\lambda \in \mathbb{C} \setminus [q_\infty, \infty)$ there exists (up to a constant multiple) exactly one solution $g_\lambda \in \mathcal{D}_+$ of $\tau_+ u = \lambda u$; cf. [9, Satz 13.22]. The same is true for each $\lambda \in \mathbb{C} \setminus (-\infty, -q_\infty]$ and the solutions $h_\lambda \in \mathcal{D}_-$ of $\tau_- v = \lambda v$.

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Lemma 2.1 *Let $\lambda \in (-q_\infty, q_\infty)$ and let $g_\lambda \in \mathcal{D}_+$ and $h_\lambda \in \mathcal{D}_-$ be nontrivial solutions of $\tau_+ u = \lambda u$ and $\tau_- v = \lambda v$, respectively. Then each of the numbers $g_\lambda(0)$, $g'_\lambda(0)$, $h_\lambda(0)$, and $h'_\lambda(0)$ is nonzero.*

P r o o f. Suppose $g_\lambda \in \mathcal{D}_+$ is a nontrivial solution of $\tau_+ u = \lambda u$ such that $g_\lambda(0) = 0$. Then the function

$$f_\lambda(x) = \begin{cases} g_\lambda(x), & x \in \mathbb{R}^+, \\ -g_\lambda(-x), & x \in \mathbb{R}^-, \end{cases} \tag{2}$$

and its derivative are continuous at 0 and hence $f_\lambda \in \mathcal{D}$. Furthermore, the equation $-f''_\lambda + qf_\lambda = \lambda f_\lambda$ holds and hence $f_\lambda \in \ker(B - \lambda)$, $f_\lambda \neq 0$; a contradiction to the assumption $\sigma_p(B) \cap (-\infty, q_\infty) = \emptyset$. The same argument with $-g_\lambda(-x)$, $x \in \mathbb{R}^-$, in (2) replaced by $g_\lambda(-x)$, $x \in \mathbb{R}^-$, shows $g'_\lambda(0) \neq 0$. The claim for $h_\lambda \in \mathcal{D}_-$ can be proved analogously, but follows also by observing that the function $\mathbb{R}^+ \ni x \mapsto h_\lambda(-x)$ in $\mathcal{D}_+$ is a solution of $\tau_+ u = -\lambda u$ and $-\lambda \in (-q_\infty, q_\infty)$. □

Lemma 2.2 *For $\lambda \in (-q_\infty, q_\infty)$ the following assertions are equivalent:*

(i) *λ is an eigenvalue of A;*

(ii) *there exist nontrivial solutions $g_\lambda \in \mathcal{D}_+$ and $h_\lambda \in \mathcal{D}_-$ of $\tau_+ u = \lambda u$ and $\tau_- v = \lambda v$, respectively, such that*

$$\frac{g'_\lambda(0)}{g_\lambda(0)} - \frac{h'_\lambda(0)}{h_\lambda(0)} = 0. \tag{3}$$

P r o o f. (i) ⇒ (ii) Let $f_\lambda \in \ker(A - \lambda)$, $f_\lambda \neq 0$, be an eigenfunction corresponding to $\lambda \in (-q_\infty, q_\infty)$. Then the restrictions $g_\lambda = f_\lambda|_{\mathbb{R}^+} \in \mathcal{D}_+$ and $h_\lambda = f_\lambda|_{\mathbb{R}_-} \in \mathcal{D}_-$ are nontrivial solutions of the equations $\tau_+ u = \lambda u$ and $\tau_- v = \lambda v$, respectively. Furthermore, since $f_\lambda \in \operatorname{dom} A$ it is clear that $h_\lambda(0) = g_\lambda(0)$ and $h'_\lambda(0) = g'_\lambda(0)$ holds. By Lemma 2.1 we also have $0 \neq g_\lambda(0) = h_\lambda(0)$. This implies (ii).

(ii) ⇒ (i) If $g_\lambda \in \mathcal{D}_+$ and $h_\lambda \in \mathcal{D}_-$ are nontrivial solutions of $\tau_+ u = \lambda u$ and $\tau_- v = \lambda v$, respectively, then by Lemma 2.1 $g_\lambda(0) \neq 0$ and $h_\lambda(0) \neq 0$. Since both terms in (3) do not depend on the particular choice of $g_\lambda \in \mathcal{D}_+$ and $h_\lambda \in \mathcal{D}_-$ it is no restriction to assume that $g_\lambda(0) = h_\lambda(0)$ holds. Then (3) implies $g'_\lambda(0) = h'_\lambda(0)$ and therefore the function

$$f_\lambda(x) = \begin{cases} g_\lambda(x), & x \in \mathbb{R}^+, \\ h_\lambda(x), & x \in \mathbb{R}^-, \end{cases}$$

belongs to $\mathcal{D}$ and is a nontrivial solution of $\tau w = \lambda w$, i.e., $f_\lambda \in \ker(A - \lambda)$ and λ is an eigenvalue of A. □

Remark 2.3 The statements in Lemma 2.1 and Lemma 2.2 hold also for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and Lemma 2.1 is also valid for g_λ (h_λ) if $\lambda \in (-\infty, -q_\infty]$ ($\lambda \in [q_\infty, \infty)$, respectively).

Proof of Theorem 1.3. Let $\lambda \in (-q_\infty, q_\infty)$ and let $g_\lambda \in \mathcal{D}_+$ be a nontrivial solution of $\tau_+ u = \lambda u$. We consider the function m defined by

$$(-q_\infty, q_\infty) \ni \lambda \mapsto m(\lambda) = \frac{g'_\lambda(0)}{g_\lambda(0)}. \tag{4}$$

We mention that the function m is (a restriction) of the usual Titchmarsh-Weyl m-function associated to τ_+; cf. [3]. It follows from $\lambda \in \mathbb{R}$ that the values of m are real and by Lemma 2.1 the function m has no poles or zeros in $(-q_\infty, q_\infty)$. Since $\lambda \mapsto g_\lambda(0)$ and $\lambda \mapsto g'_\lambda(0)$ are continuous also m is continuous. Therefore m does not change its sign in $(-q_\infty, q_\infty)$.

Let $\lambda \in (-q_\infty, q_\infty)$ and let $h_\lambda \in \mathcal{D}_-$ be a nontrivial solution of $\tau_- v = \lambda v$. Then the function $g_{-\lambda}(x) := h_\lambda(-x)$, $x \in \mathbb{R}^-$, in $\mathcal{D}_+$ is a nontrivial solution of $\tau_+ u = -\lambda u$ and we conclude

$$m(-\lambda) = \frac{g'_{-\lambda}(0)}{g_{-\lambda}(0)} = -\frac{h'_\lambda(0)}{h_\lambda(0)}, \qquad \lambda \in (-q_\infty, q_\infty). \tag{5}$$

By (4) and (5) the left hand side of (3) coincides with $m(\lambda) + m(-\lambda)$, and as m does not change its sign in $(-q_\infty, q_\infty)$ the function $\lambda \mapsto m(\lambda) + m(-\lambda)$ has no zeros in $(-q_\infty, q_\infty)$. Now Lemma 2.2 implies $\sigma_p(A) \cap (-q_\infty, q_\infty) = \emptyset$. □

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