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The fate of Landau levels under δ-interactions

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Abstract. We consider the self-adjoint Landau Hamiltonian H_0 in $L^2(\mathbb{R}^2)$ whose spectrum consists of infinitely degenerate eigenvalues Λ_q, $q \in \mathbb{Z}_+$, and the perturbed Landau Hamiltonian $H_\upsilon = H_0 + \upsilon\delta_\Gamma$, where $\Gamma \subset \mathbb{R}^2$ is a regular Jordan $C^{1,1}$-curve and $\upsilon \in L^p(\Gamma;\mathbb{R})$, $p > 1$, has a constant sign. We investigate $\ker(H_\upsilon - \Lambda_q)$, $q \in \mathbb{Z}_+$, and show that generically

$$0 \leq \dim\ker(H_\upsilon - \Lambda_q) - \dim\ker(T_q(\upsilon\delta_\Gamma)) < \infty,$$

where $T_q(\upsilon\delta_\Gamma) = p_q(\upsilon\delta_\Gamma)p_q$, is an operator of Berezin–Toeplitz type, acting in $p_q L^2(\mathbb{R}^2)$, and p_q is the orthogonal projection onto $\ker(H_0 - \Lambda_q)$. If $\upsilon \neq 0$ and $q = 0$, then we prove that $\ker(T_0(\upsilon\delta_\Gamma)) = \{0\}$. If $q \geq 1$ and $\Gamma = \mathcal{C}_r$ is a circle of radius r, then we show that $\dim\ker(T_q(\delta_{\mathcal{C}_r})) \leq q$, and the set of $r \in (0,\infty)$ for which $\dim\ker(T_q(\delta_{\mathcal{C}_r})) \geq 1$ is infinite and discrete.

1. Introduction

The aim of this article is to study the spectral type of the Landau levels of the singularly perturbed Landau Hamiltonian

$$H_\upsilon = (-i\nabla - A)^2 + \upsilon\delta_\Gamma, \tag{1.1}$$

where $A(x) := \frac{b}{2}(-x_2, x_1)$, $x = (x_1, x_2) \in \mathbb{R}^2$, is a magnetic potential which generates a constant scalar magnetic field $b > 0$, and the singular perturbation is supported on a $C^{1,1}$-smooth Jordan curve $\Gamma \subset \mathbb{R}^2$ and has strength $\upsilon \in L^p(\Gamma;\mathbb{R})$, $p > 1$. The expression (1.1) is of formal nature here and the self-adjoint operator H_υ will be defined rigorously via the corresponding quadratic form in Section 2. If the singular perturbation is absent, that is, $\upsilon = 0$ in (1.1), then the operator reduces to the usual self-adjoint Landau Hamiltonian $H_0 = (-i\nabla - A)^2$. It is well known that

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = \bigcup_{q\in\mathbb{Z}_+}\{\Lambda_q\},$$

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where the *Landau levels* $\Lambda_q := b(2q+1)$, $q \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$, are eigenvalues of H_0 of infinite multiplicity. Under our assumption on Γ and υ it turns out that H_υ is a compact perturbation of H_0 in the resolvent sense and hence the essential spectrum remains invariant, that is,

$$\sigma_{\mathrm{ess}}(H_\upsilon) = \sigma_{\mathrm{ess}}(H_0) = \bigcup_{q \in \mathbb{Z}_+} \{\Lambda_q\}.$$

In the spectral gaps $(\Lambda_{q-1}, \Lambda_q)$, where $q \in \mathbb{Z}_+$ and $\Lambda_{-1} := -\infty$, of H_0 there may appear discrete eigenvalues of H_υ which can only accumulate at the Landau levels Λ_q, $q \in \mathbb{Z}_+$. Some results on the asymptotic distribution near any fixed Λ_q of these discrete eigenvalues were obtained in [7]. In particular, it was shown that if either $\upsilon \geq 0$ or $\upsilon \leq 0$ on Γ, $\upsilon \not\equiv 0$, and certain additional regularity assumptions hold, then in a neighborhood of any Λ_q there are infinitely many discrete eigenvalues of H_υ and their accumulation rate to the Landau levels is described in terms of the logarithmic capacity of the interaction support; cf. [10, 20, 35, 37] for similar results on the clustering of eigenvalues of Landau Hamiltonians on unbounded domains with Dirichlet, Neumann, and Robin boundary conditions.

Our main objective in this article is to obtain a deeper understanding of the spectral points Λ_q, $q \in \mathbb{Z}_+$, of the perturbed operator H_υ; in other words, we are interested in the fate of the Landau levels Λ_q under δ-potentials of strength υ. In particular, we would like to know what part of the infinite-dimensional eigenspace $\ker(H_0 - \Lambda_q)$ is transformed into an eigenspace $\ker(H_\upsilon - \Lambda_q)$ under the singular perturbation $\upsilon\delta_\Gamma$.

The analogous problem on the fate of Landau levels under *regular* perturbations of the Landau Hamiltonian H_0 was investigated earlier in [28]. Roughly speaking, it was shown that for any non-negative potential $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$, $V \not\equiv 0$, with $\|V\|_{L^\infty(\mathbb{R}^2)} < 2b$ one has

$$\ker(H_0 \pm V - \Lambda_q) = \{0\}. \tag{1.2}$$

The assumption that V is sign-definite is essential here. In fact, in [28] it was also shown that for every $q \in \mathbb{Z}_+$, there exists a compactly supported $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ with $\|V\|_{L^\infty(\mathbb{R}^2)} < b$ of non-constant sign, such that

$$\dim \ker(H_0 + V - \Lambda_q) = \infty.$$

The key idea in the proof of (1.2) is to show that $\ker(H_0 \pm V - \Lambda_q) \subset \ker(\widehat{T}_q(V))$ and $\ker(\widehat{T}_q(V)) = \{0\}$ if $\|V\|_{L^\infty(\mathbb{R}^2)} < 2b$, where $\widehat{T}_q(V) := p_q V p_q$ is a Berezin–Toeplitz type operator and p_q denotes the orthogonal projection onto the eigenspace $\ker(H_0 - \Lambda_q)$.

In our treatment of the perturbed Landau Hamiltonian with a δ-potential in (1.1) a singular analogue of the Berezin–Toeplitz operator plays a key role; cf. the discussion below (2.11) for more details and references. More precisely, if τ is the restriction operator onto Γ we consider the operator $T_q(\upsilon\delta_\Gamma) := (\tau p_q)^* \upsilon(\tau p_q)$ and in our main results we show that the analysis of $\ker(H_{\pm\upsilon} - \Lambda_q)$ can be reduced to that of $\ker(T_q(\upsilon\delta_\Gamma))$. Namely, under the definiteness assumption $\upsilon \geq 0$, we prove in Theorem 3.1 that

$$\ker(T_q(\upsilon\delta_\Gamma)) \subset \ker(H_{\pm\upsilon} - \Lambda_q), \quad q \in \mathbb{Z}_+,$$

and $0 \leq \dim\ker(H_{\pm\upsilon} - \Lambda_q) - \dim\ker(T_q(\upsilon\delta_\Gamma)) < \infty$ for all $q \in \mathbb{Z}_+$. Furthermore, if $\|\upsilon\|_{L^p(\Gamma)}$ is not too large it turns out that

$$\ker(H_{\pm\upsilon} - \Lambda_q) = \ker(T_q(\upsilon\delta_\Gamma)), \quad q \in \mathbb{Z}_+.$$

As we will see, for $\upsilon \geq 0$ the kernel of $T_q(\upsilon\delta_\Gamma)$ consists of eigenfunctions of H_0 for Λ_q which vanish on the support of υ. Intuitively, it is clear that such functions $u \in \ker(T_q(\upsilon\delta_\Gamma))$ are also eigenfunctions of H_υ, as in this case one formally has $\upsilon\delta_\Gamma u = 0$, i.e., the singular interaction does not have an effect on u, and hence $H_\upsilon u = H_0 u = \Lambda_q u$. This allows one to show with the help of [7, Lemma 3.7] that the kernel of $T_q(\upsilon\delta_\Gamma)$ is finite-dimensional under the assumption that υ is strictly positive. Moreover, this connection provides a direct link to nodal sets for eigenfunctions of H_0 and the non-emptiness of $\ker(H_\upsilon - \Lambda_q)$. The above observation means, in particular, that for all $\upsilon \in L^p(\Gamma)$ one has $\ker(H_{\pm\upsilon} - \Lambda_q) \neq \{0\}$, whenever Γ is contained in a nodal set of an eigenfunction of H_0. This is in strong contrast to the case of regular potentials; cf. (1.2). Additionally, for the first Landau level Λ_0 we find $\ker(T_0(\upsilon\delta_\Gamma)) = \{0\}$ in Theorem 3.6, which leads to

$$\ker(H_\upsilon - \Lambda_0) = \{0\}.$$

At present it is not clear if $\dim\ker(T_q(\upsilon\delta_\Gamma))$ can be further estimated for higher Landau levels and general curves Γ. However, we find it worthwhile to discuss the special case that $\Gamma = \mathcal{C}_r$ is a circle of radius $r \in (0, \infty)$. In this situation we find that

$$\dim\ker(T_q(\delta_{\mathcal{C}_r})) \leq q, \quad q = 1, 2, \ldots,$$

and the radii $r \in (0, \infty)$ for which $\dim\ker(T_q(\delta_{\mathcal{C}_r})) \geq 1$ form an infinite and discrete set; cf. Theorem 3.9. The idea to prove these results is again to study when a circle is a nodal set for an eigenfunction of H_0 and use the fact, that this can be characterized explicitly in terms of zeros of Laguerre polynomials. Translating this observation to the spectral points Λ_q of the perturbed Landau operator (1.1) leads to a precise understanding of the fate of Landau levels under δ-perturbations supported on circles. For example, if $\upsilon \not\equiv 0$ and $\upsilon \geq 0$ on some non-empty open subset of Γ, then Λ_q can only be an eigenvalue of finite multiplicity; cf. Section 3.2 for details.

The article is organized as follows. In the next section we introduce the Landau Hamiltonian perturbed by singular δ-interactions. In Section 3 we formulate our main results. Section 4 contains some auxiliary facts from the spectral theory of the Landau Hamiltonian. Finally, in Section 5 we prove our main theorems.

Note by J. Behrndt, M. Holzmann, and V. Lotoreichik. Our coauthor Georgi Raikov passed away unexpectedly on 9 March 2021, while the work on this manuscript was in its active phase. The topics in the present paper result from various discussions with Georgi dating back to 2018 and the first draft of this paper was written by him. When preparing the final text it was our aim to preserve Georgis original handwriting and genuine style. This paper is a tribute to the memory of Georgi Raikov, an influential mathematician, respected colleague, and good friend. We will miss him.

2. Landau Hamiltonians with δ-interactions supported on curves

Let $b > 0$ be a constant scalar magnetic field. Then

$$A(x) := \frac{b}{2}(-x_2, x_1), \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

is a magnetic potential which generates b, i.e.,

$$b = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}.$$

Denote by

$$\Pi(A) = (\Pi_1(A), \Pi_2(A)) := -i\nabla - A$$

the magnetic gradient. In the following, for $\ell = (\ell_1, \ell_2) \in \mathbb{Z}_+^2$ with $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, the notations $|\ell| := \ell_1 + \ell_2$ and $\Pi(A)^\ell := \Pi_1(A)^{\ell_1}\Pi_2(A)^{\ell_2}$ are used. For an open non-empty set $\Omega \subset \mathbb{R}^2$ and an index $s \in \mathbb{Z}_+$ introduce the magnetic Sobolev spaces

$$\mathrm{H}_A^s(\Omega) := \{u \in \mathcal{D}'(\Omega) \mid \Pi(A)^\ell u \in L^2(\mathbb{R}^2),\ \ell \in \mathbb{Z}_+^2,\ 0 \le |\ell| \le s\}$$

with a norm defined by

$$\|u\|_{\mathrm{H}_A^s(\Omega)}^2 := \sum_{\ell \in \mathbb{Z}_+^2 :\, 0 \le |\ell| \le s} \int_\Omega |\Pi(A)^\ell u|^2 dx.$$

Throughout this paper it is assumed that $\Gamma \subset \mathbb{R}^2$ is a $C^{1,1}$-smooth Jordan curve, i.e., a closed simple curve which is mapped onto the unit circle by a $C^{1,1}$-smooth diffeomorphism. Let $\mathrm{H}^{1/2}(\Gamma)$ be the L^2-based Sobolev space of order $1/2$ on Γ.

The Dirichlet trace operator $\tau : \mathrm{H}_A^1(\mathbb{R}^2) \to \mathrm{H}^{1/2}(\Gamma)$ is the continuous extension of the restriction map

$$\mathrm{H}_A^1(\mathbb{R}^2) \cap C(\mathbb{R}^2) \ni u \mapsto u|_\Gamma.$$

Assume that $\upsilon \in L^p(\Gamma; \mathbb{R})$ with $p > 1$. Denote by H_υ the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the symmetric, densely defined, lower-bounded, and closed quadratic form

$$\int_{\mathbb{R}^2} |\Pi(A)u|^2 dx + \int_\Gamma \upsilon |\tau u|^2 \, ds, \quad u \in \mathrm{H}^1_A(\mathbb{R}^2); \tag{2.1}$$

cf. Appendix A. In particular, for $\upsilon = 0$ one obtains

$$H_0 = \Pi_1(A)^2 + \Pi_2(A)^2 = (-i\nabla - A)^2,$$

which is the *Landau Hamiltonian*, self-adjoint on $\mathrm{H}^2_A(\mathbb{R}^2)$ (see, for example, [18, Appendix A]), and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ (see [30, Theorem 2]). As mentioned in the introduction, one has

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = \bigcup_{q \in \mathbb{Z}_+} \{\Lambda_q\},$$

where $\Lambda_q := b(2q+1)$, $q \in \mathbb{Z}_+$, are the *Landau levels* which are eigenvalues of H_0 of infinite multiplicity (see [4, 17, 29]). In particular,

$$\inf \sigma(H_0) = \Lambda_0 = b > 0.$$

Note that integration by parts allows to find an explicit characterization of H_υ. Denote by Ω_{in} and Ω_{ex} the interior and the exterior of Γ, respectively, and by ν the unit normal vector on Γ pointing outwards of Ω_{in}. Then one can show in the same way as in [7, Section 4] that

$$\begin{aligned}
(H_\upsilon u)_\natural &= (-i\nabla - A)^2 u_\natural \qquad \text{for } \natural = \mathrm{in}, \mathrm{ex} \\
\mathfrak{D}(H_\upsilon) &= \Big\{ u \in \mathrm{H}^1_A(\mathbb{R}^2) \;\Big|\; (-i\nabla - A)^2 u_\natural \in L^2(\Omega_\natural) \text{ for } \natural = \mathrm{in}, \mathrm{ex}, \\
&\qquad\qquad \frac{\partial u_{\mathrm{ex}}}{\partial \nu} - \frac{\partial u_{\mathrm{in}}}{\partial \nu} = \upsilon u \text{ on } \Gamma \Big\}.
\end{aligned} \tag{2.2}$$

In the next lemma it is shown that the difference of the resolvents of H_0 and H_υ is compact, which implies that the essential spectra of H_0 and H_υ coincide. In order to formulate the lemma, define for $\lambda > -b$ the operator

$$G_\upsilon(\lambda) := |\upsilon|^{1/2} \tau (H_0 + \lambda)^{-1/2} \colon L^2(\mathbb{R}^2) \to L^2(\Gamma). \tag{2.3}$$

Lemma 2.1. *Let $\lambda > -b$ and set $J_\upsilon := \operatorname{sign} \upsilon$. Then $G_\upsilon(\lambda)$ is compact and there exists $\lambda_0 > -b$ such that for all $\lambda > \lambda_0$ the resolvent difference of H_0 and H_υ is a compact operator in $L^2(\mathbb{R}^2)$ and admits the factorization*

$$\begin{aligned}
&(H_\upsilon + \lambda)^{-1} - (H_0 + \lambda)^{-1} \\
&\quad = -(H_0 + \lambda)^{-1/2} G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda) (I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda))^{-1} (H_0 + \lambda)^{-1/2}.
\end{aligned}$$

In particular, one has

$$\sigma_{\rm ess}(H_0) = \sigma_{\rm ess}(H_\upsilon) = \bigcup_{q\in\mathbb{Z}_+}\{\Lambda_q\}. \tag{2.4}$$

Proof. Note first that the operator $G_\upsilon(\lambda)$ in (2.3) depends only on $|\upsilon|$ but not on the sign of υ. The assumption $\upsilon \in L^p(\Gamma;\mathbb{R})$ with $p > 1$, and the compactness of the trace $\mathrm{H}^1_A(\mathbb{R}^2) \to L^r(\Gamma)$ for any $r > 1$ (see [33, Section 2.6, Theorem 6.2]), easily imply that $G_\upsilon(\lambda)$ is compact, and

$$\begin{aligned}
\|G_\upsilon(\lambda)\|^2 &= \sup_{0\neq w\in L^2(\mathbb{R}^2)} \frac{\||\upsilon|^{1/2}\tau(H_0+\lambda)^{-1/2}w\|^2_{L^2(\Gamma)}}{\|w\|^2_{L^2(\mathbb{R}^2)}}\\
&= \sup_{0\neq u\in \mathrm{H}^1_A(\mathbb{R}^2)} \frac{\||\upsilon|^{1/2}\tau u\|^2_{L^2(\Gamma)}}{\|(H_0+\lambda)^{1/2}u\|^2_{L^2(\mathbb{R}^2)}}\\
&= \sup_{0\neq u\in \mathrm{H}^1_A(\mathbb{R}^2)} \frac{\int_\Gamma |\upsilon||\tau u|^2 ds}{\int_{\mathbb{R}^2}(|\Pi(A)u|^2+\lambda|u|^2)\,dx},
\end{aligned}$$

so that the Hölder inequality leads to the estimate

$$\|G_\upsilon(\lambda)\|^2 \le C_p(\lambda)\|\upsilon\|_{L^p(\Gamma)} \tag{2.5}$$

with

$$C_p(\lambda) := \sup_{0\neq u\in \mathrm{H}^1_A(\mathbb{R}^2)} \frac{(\int_\Gamma |\tau u|^{2p'} ds)^{1/p'}}{\int_{\mathbb{R}^2}(|\Pi(A)u|^2+\lambda|u|^2)\,dx}, \quad p' := \frac{p}{p-1}, \quad \lambda > -b. \tag{2.6}$$

Set

$$J_\upsilon = \operatorname{sign}\upsilon := \begin{cases} \upsilon|\upsilon|^{-1} & \text{if } \upsilon \neq 0,\\ 0 & \text{if } \upsilon = 0.\end{cases}$$

Let $\lambda > -b$. Then one has for $u \in L^2(\mathbb{R}^2)$, $w := (H_0+\lambda)^{-1/2}u \in \mathrm{H}^1_A(\mathbb{R}^2)$, and the self-adjoint operator $G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda)$

$$\langle (I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda))u, u\rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2}(|\Pi(A)w|^2 + \lambda|w|^2)dx + \int_\Gamma \upsilon|\tau w|^2 ds \tag{2.7}$$

and hence, with (2.1) one concludes that

$$\lambda > -\inf\sigma(H_\upsilon) \tag{2.8}$$

is equivalent to

$$I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda) > 0. \tag{2.9}$$

Assume in the following that $\lambda > -\inf \sigma(H_\upsilon)$ is fixed. Due to the compactness of $G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda)$, it follows from (2.9) that the operator

$$I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda)\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

is boundedly invertible. Thus, we have

$$H_\upsilon + \lambda = M_\upsilon(\lambda)^* M_\upsilon(\lambda) \tag{2.10}$$

where the operator

$$M_\upsilon(\lambda) := (I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda))^{1/2}(H_0 + \lambda)^{1/2}, \quad \mathfrak{D}(M_\upsilon(\lambda)) = \mathrm{H}^1_A(\mathbb{R}^2),$$

is closed in $L^2(\mathbb{R}^2)$, as it is a product of a bijective operator in $L^2(\mathbb{R}^2)$ and the operator $(H_0+\lambda)^{1/2}$, which is bijective from $\mathrm{H}^1_A(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. The representation in (2.10) can be seen with the help of the quadratic form in (2.1) associated with H_υ and a similar calculation as in (2.7). Therefore, the operators $H_\upsilon + \lambda$ and, hence, H_υ are self-adjoint on

$$\mathfrak{D}(H_\upsilon) := \{u \in \mathrm{H}^1_A(\mathbb{R}^2) \mid M_\upsilon(\lambda)u \in \mathfrak{D}(M_\upsilon(\lambda)^*)\}$$

(see [40, Theorem X.25]). In the above construction we have obtained an alternative characterisation of the operator domain of H_υ; cf. (2.2). Moreover,

$$\begin{aligned}
&(H_\upsilon + \lambda)^{-1} - (H_0 + \lambda)^{-1} \\
&\quad = (H_0 + \lambda)^{-1/2}(I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda))^{-1}(H_0 + \lambda)^{-1/2} - (H_0 + \lambda)^{-1} \\
&\quad = -(H_0 + \lambda)^{-1/2} G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda)(I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda))^{-1}(H_0 + \lambda)^{-1/2}.
\end{aligned}$$

Bearing in mind the compactness of the operator $G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda)$, and applying a suitable version of the Weyl theorem on the invariance of the essential spectrum (see, e.g., [9, Chapter 9, Section 1, Theorem 4]), we obtain (2.4). ∎

Consider the operator

$$T_q(\upsilon\delta_\Gamma) := (\tau p_q)^* \upsilon (\tau p_q), \quad q \in \mathbb{Z}_+, \tag{2.11}$$

which can be viewed as a singular analogue of a Berezin–Toeplitz operator. The relation of Landau Hamiltonians coupled with regular potentials V and the Berezin–Toeplitz type operators $\widehat{T}_q(V) = p_q V p_q$ was discovered in [38] and further studied in many publications. Singular Toeplitz operators as in (2.11) play an important role in modern operator theory and are also of independent interest. They were already considered in [2], and in connection with magnetic Laplacians with different types of boundary conditions these types of operators appear in [20, 21, 37]; we also refer the reader to [3, 10, 11, 13, 16, 32, 36, 41, 42] for some other recent related works in this context. Note that the operator $T_q(\upsilon\delta_\Gamma)$ corresponds to the quadratic form

$$t_q(\upsilon\delta_\Gamma)[u] := \int_\Gamma \upsilon(x)|u(x)|^2 ds, \quad u \in p_q L^2(\mathbb{R}^2). \tag{2.12}$$

Lemma 2.2. *The operator $T_q(\upsilon\delta_\Gamma)$, $q \in \mathbb{Z}_+$, is a compact self-adjoint operator in $p_q L^2(\mathbb{R}^2)$. For $\lambda > -b$ and $G_\upsilon(\lambda)$ in (2.3) one has*

$$T_q(\upsilon\delta_\Gamma) = (\Lambda_q + \lambda) p_q G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda) p_q. \tag{2.13}$$

Moreover, if υ has a constant sign, then there exists $\lambda_0 > -b$ such that for all $\lambda > \lambda_0$

$$\ker(p_q Q_\upsilon(\lambda) p_q) = \ker(T_q(\upsilon\delta_\Gamma)), \tag{2.14}$$

where $Q_\upsilon(\lambda) := (H_\upsilon + \lambda)^{-1} - (H_0 + \lambda)^{-1}$.

Proof. Recall that J_υ denotes the sign of υ. A simple calculation involving the form $t_q(\upsilon\delta_\Gamma)$ shows for any $u \in p_q L^2(\mathbb{R}^2)$ and $\lambda > -b$ that

$$\begin{aligned} t_q(\upsilon\delta_\Gamma)[u] &= \int_\Gamma \upsilon(x)|u(x)|^2 ds \\ &= (\Lambda_q + \lambda) \int_\Gamma \upsilon(x)|(\tau(H_0 + \lambda)^{-1/2} u)(x)|^2 ds \\ &= (\Lambda_q + \lambda) \langle J_\upsilon G_\upsilon(\lambda) p_q u, G_\upsilon(\lambda) p_q u \rangle_{L^2(\Gamma)}. \end{aligned}$$

Thus, we get the representation (2.13), which also shows with Lemma 2.1 that $T_q(\upsilon\delta_\Gamma)$ is compact and self-adjoint. Moreover, (2.13) immediately implies

$$\ker(T_q(\upsilon\delta_\Gamma)) = \ker(p_q G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda) p_q). \tag{2.15}$$

It remains to prove equation (2.14). For this, assume that υ has a constant sign, i.e., $\pm\upsilon \geq 0$, and fix $\lambda > -b$ sufficiently large such that $H_\upsilon + \lambda$ is strictly positive. Then the equivalence of (2.8) and (2.9) shows that $I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda)$ is strictly positive and bounded. Thus, we get for $K_\upsilon(\lambda) := G_\upsilon(\lambda)^* G_\upsilon(\lambda) \geq 0$ by Lemma 2.1 that

$$\begin{aligned} &\langle p_q Q_\upsilon(\lambda) p_q u, u \rangle_{L^2(\mathbb{R}^2)} \\ &\quad = -J_\upsilon(\Lambda_q + \lambda)^{-1} \langle K_\upsilon(\lambda)(I + J_\upsilon K_\upsilon(\lambda))^{-1} p_q u, p_q u \rangle_{L^2(\mathbb{R}^2)} \\ &\quad = -J_\upsilon(\Lambda_q + \lambda)^{-1} \langle (I + J_\upsilon K_\upsilon(\lambda))^{-1} (K_\upsilon(\lambda))^{1/2} p_q u, (K_\upsilon(\lambda))^{1/2} p_q u \rangle_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Taking the bijectivity of $I + G_\upsilon(\lambda)^* J_\upsilon G_\upsilon(\lambda)$ and (2.15) into account, this leads to (2.14). ■

Via the form $t_q(\upsilon\delta_\Gamma)$, one also gets another interesting characterization of $\ker(T_q(\upsilon\delta_\Gamma))$ in the case that $\upsilon \geq 0$ almost everywhere on Γ, as then it follows from (2.12) that $T_q(\upsilon\delta_\Gamma)$ is a non-negative operator and that $\ker(T_q(\upsilon\delta_\Gamma)) \neq \{0\}$ if and only if there exists a $u \in p_q L^2(\mathbb{R}^2)$ such that $\upsilon u = 0$ on Γ. This yields

$$\ker(T_q(\upsilon\delta_\Gamma)) = \{u \in p_q L^2(\mathbb{R}^2) \mid u = 0 \text{ on } \operatorname{supp} \upsilon\}, \tag{2.16}$$

where $\operatorname{supp} \upsilon$ denotes the essential support of υ. In other words, this means that $\ker(T_q(\upsilon\delta_\Gamma)) \neq \{0\}$ if and only if the essential support of υ is contained in a nodal set of an eigenfunction of H_0 for Λ_q. Furthermore, the dimension of $\ker(T_q(\upsilon\delta_\Gamma))$ is equal to the number of linearly independent eigenfunctions u of H_0 for Λ_q such that $u = 0$ on $\operatorname{supp} \upsilon$. For studies on nodal sets of eigenfunctions, we refer the reader to, e.g., [24–26, 34]. Clearly, a similar consideration is true if $\upsilon \leq 0$ almost everywhere on Γ.

3. Main results

In this section we formulate our main results on the fate of Landau levels under δ-perturbations supported on curves. The case of general $C^{1,1}$-smooth Jordan curves is treated first and, roughly speaking, we show that the analysis of the eigenspaces $\ker(H_{\pm\upsilon} - \Lambda_q)$ of the perturbed Landau Hamiltonian can be reduced to the analysis of the kernels $\ker(T_q(\upsilon\delta_\Gamma))$ of the Berezin–Toeplitz type operators defined in (2.11). This connection is of independent interest, but also turns out to be useful for a more explicit analysis of the Landau levels. We illustrate this for the special case of δ-perturbations supported on circles.

3.1. Singular interactions supported on $C^{1,1}$-smooth Jordan curves

Throughout this section, let $\Gamma \subset \mathbb{R}^2$ be a $C^{1,1}$-smooth Jordan curve and assume that $\upsilon \in L^p(\Gamma; \mathbb{R})$ for some $p > 1$ is such that $\upsilon \geq 0$ on Γ and $\upsilon \not\equiv 0$. Our first theorem contains two independent statements which concern the operators H_υ and $H_{-\upsilon}$ respectively. The proof of Theorem 3.1 can be found in Section 5.1.

Theorem 3.1. *Let $q \in \mathbb{Z}_+$ and let $T_q(\upsilon\delta_\Gamma)$ be the operator of Berezin–Toeplitz type in* (2.11).

i. *There holds*
$$\ker(T_q(\upsilon\delta_\Gamma)) \subset \ker(H_{\pm\upsilon} - \Lambda_q).$$

ii. *There exist $n_q^\pm \in \mathbb{Z}_+$ depending on υ such that*
$$\dim\ker(H_{\pm\upsilon} - \Lambda_q) \leq \dim\ker(T_q(\upsilon\delta_\Gamma)) + n_q^\pm \tag{3.1}$$
and for $q = 0$ one can choose $n_0^+ = 0$.

iii. *There exist* $\upsilon_q^\pm > 0$ *such that* $\|\upsilon\|_{L^p(\Gamma)} < \upsilon_q^\pm$ *implies*

$$\ker(H_{\pm\upsilon} - \Lambda_q) = \ker(T_q(\upsilon\delta_\Gamma)) \tag{3.2}$$

and for $q = 0$ *one can choose* $\upsilon_0^+ = \infty$*. Moreover, there is a constant* $c > 0$ *independent of* b *and* q *such that*

$$\upsilon_q^+ \geq \frac{2bc}{(\Lambda_q + 1)(\Lambda_{q-1} + 1)} \quad \textit{and} \quad \upsilon_q^- \geq \frac{2bc}{2b + (\Lambda_q + 1)(\Lambda_{q+1} + 1)}. \tag{3.3}$$

Remark 3.2. Writing (3.2), we mean that $u \in \ker(H_{\pm\upsilon} - \Lambda_q)$ implies $u = p_q u$, and

$$u \in \ker(T_q(\upsilon\delta_\Gamma)) \subset p_q L^2(\mathbb{R}^2),$$

and vice versa. A similar remark applies to all further inclusions of the same kind. We also mention that the inequality (3.1) will be obtained by showing that $\ker(H_{\pm\upsilon} - \Lambda_q)$ is the sum of $\ker(T_q(\upsilon\delta_\Gamma))$ and a finite-dimensional space.

Remark 3.3. The definitions of the numbers $\upsilon_q^\pm$ in Theorem 3.1 (iii) are given in (5.36) and (5.45), respectively. Their precise values are not obvious. However, equation (3.2) is also true, if one replaces $\upsilon_q^\pm$ by the lower bounds in (3.3). Since $\Lambda_q = b(2q + 1)$, we see that these lower bounds are decreasing in q; the same is true for $\upsilon_q^\pm$ defined in (5.36) and (5.45).

Remark 3.4. The operator H_υ can be introduced as a self-adjoint extension of the symmetric operator S given by

$$Su = (-i\nabla - A)^2 u, \quad \mathfrak{D}(S) = \{u \in \mathrm{H}_A^2(\mathbb{R}^2) \,|\, u|_\Gamma = 0\}, \tag{3.4}$$

i.e., S is the restriction of H_0 onto functions in $\mathrm{H}_A^2(\mathbb{R}^2)$ that vanish on Γ; cf. [7] for the case $\upsilon \in L^\infty(\Gamma; \mathbb{R})$. In view of (2.16), if $u \in \ker(T_q(\upsilon\delta_\Gamma))$ for $\upsilon > 0$ or $\upsilon < 0$, then $u \in \ker(S - \Lambda_q)$ and thus, as H_υ was defined as an extension of S, $u \in \ker(H_\upsilon - \Lambda_q)$. Thus, the result of Theorem 3.1 (i) can also be interpreted from an extension theoretic point of view.

Remark 3.5. Considering (3.1) the question arises, if $\dim \ker(T_q(\upsilon\delta_\Gamma))$ is finite, as then $\dim \ker(H_{\pm\upsilon} - \Lambda_q) < \infty$. If υ is strictly positive, i.e., if $\upsilon \geq c > 0$ everywhere on Γ, then by [7, Lemma 3.7] and (2.16) one indeed has $\dim \ker(T_q(\upsilon\delta_\Gamma)) < \infty$.

Theorem 3.1 reduces the analysis of $\ker(H_\upsilon - \Lambda_q)$ to that of $\ker(T_q(\upsilon\delta_\Gamma))$. This is why our further results concern $\ker(T_q(\upsilon\delta_\Gamma))$. The situation is particularly simple for the Berezin–Toeplitz operator $T_0(\upsilon\delta_\Gamma)$ as the next theorem shows. Its proof can be found in Section 5.2.

Theorem 3.6. *For $q = 0$ we have*

$$\ker(T_0(\upsilon\delta_\Gamma)) = \{0\}. \tag{3.5}$$

Combining (3.5) and (3.1)–(3.2) with $q = 0$ we obtain the following corollary.

Corollary 3.7. *We have*

$$\ker(H_\upsilon - \Lambda_0) = \{0\} \quad \textit{and} \quad \dim\ker(H_{-\upsilon} - \Lambda_0) < \infty.$$

Moreover, if $\|\upsilon\|_{L^p(\Gamma)}$ is sufficiently small, then $\ker(H_{-\upsilon} - \Lambda_0) = \{0\}$.

The next remark concerns regular perturbations of the Landau Hamiltonian and the fate of the Landau levels as investigated earlier in [28].

Remark 3.8. Assume that $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$ satisfies $V \not\equiv 0$, $V \geq 0$, and

$$\lim_{|x|\to\infty} V(x) = 0.$$

Then, applying the general scheme of the proof of Theorem 3.1 below, one can verify that

$$\dim\ker(H_0 \pm V - \Lambda_q) < \infty \tag{3.6}$$

and

$$\ker(H_0 \pm V - \Lambda_q) = \ker(\widehat{T}_q(V)), \tag{3.7}$$

if $\|V\|_{L^\infty(\mathbb{R}^2)} < V_q^\pm$ for some constants $V_q^\pm > 0$; here $\widehat{T}_q(V) = p_q V p_q$, and p_q is the orthogonal projection onto $\ker(H_0 - \Lambda_q)$.

The main idea of the proof of [28, Theorem 1] is to show that, under the assumptions $\|V\|_{L^\infty(\mathbb{R}^2)} < 2b$ and $V \geq 0$, one has

$$\ker(H_0 \pm V - \Lambda_q) \subset \ker(\widehat{T}_q(V)), \quad q \in \mathbb{Z}_+. \tag{3.8}$$

On the other hand, if additionally $V \not\equiv 0$, then the results of [28, 39] imply

$$\ker(\widehat{T}_q(V)) = \{0\}, \quad q \in \mathbb{Z}_+, \tag{3.9}$$

and (1.2) follows from (3.8) and (3.9).

If one compares (3.6)–(3.7) (following our approach) with (3.8)–(3.9) (following the approach in [28]), then the upper bounds $V_q^\pm$ for $\|V\|_{L^\infty(\mathbb{R}^2)}$ in (3.7) are not as explicit as the upper bound $2b$ in (3.8), but one gets in (3.6) also results for potentials with $\|V\|_{L^\infty(\mathbb{R}^2)} \geq 2b$.

3.2. Singular interactions supported on circles

Now, we illustrate the above results for the special case that $\Gamma = \mathcal{C}_r$ is a circle of radius $r \in (0, \infty)$. In this situation, more explicit results on the structure of $\ker(T_q(\delta_{\mathcal{C}_r}))$ are obtained, as one can use an explicit basis of $\ker(H_0 - \Lambda_q)$ in polar coordinates. Using this and the simple characterization of $\ker(T_q(\delta_{\mathcal{C}_r}))$ from (2.16), it will turn out that $\ker(T_q(\delta_{\mathcal{C}_r})) \neq \{0\}$ is equivalent to the fact that $br^2/2$ is a zero of a suitable Laguerre polynomial. Since (2.16) implies $\ker(T_q(\delta_{\mathcal{C}_r})) = \ker(T_q(\upsilon\delta_{\mathcal{C}_r}))$, if υ is strictly positive or negative everywhere on Γ, this yields further results on $\ker(H_{\pm\upsilon} - \Lambda_q)$. The obtained result on $\ker(T_q(\delta_{\mathcal{C}_r}))$ is the following:

Theorem 3.9. *Let $q \in \mathbb{N}$, assume that $\Gamma = \mathcal{C}_r$ is a circle of radius $r \in (0, \infty)$, and let $T_q(\delta_{\mathcal{C}_r})$ be the corresponding operator of Berezin–Toeplitz type in* (2.11).

i. *For any $r \in (0, \infty)$ we have*

$$\dim \ker(T_q(\delta_{\mathcal{C}_r})) \leq q. \tag{3.10}$$

ii. *The set*

$$\mathcal{D}_q := \{r \in (0, \infty) \mid \dim \ker(T_q(\delta_{\mathcal{C}_r})) \geq 1\}$$

is infinite and discrete.

Theorem 3.9 will be proved in Section 5.3. Since (2.16) implies the relation

$$\ker(T_q(\delta_{\mathcal{C}_r})) = \ker(T_q(\upsilon\delta_{\mathcal{C}_r})),$$

if υ is strictly positive, a combination of (3.1) and (3.10) leads to the following corollary.

Corollary 3.10. *Let $q \in \mathbb{N}$, let $\Gamma = \mathcal{C}_r$ be a circle of radius $r \in (0, \infty)$, and assume that $\upsilon \in L^p(\mathcal{C}_r; \mathbb{R})$ with $p > 1$ satisfies $\upsilon \geq c$ on $\mathcal{C}_r$ with some constant $c > 0$. Then*

$$\dim \ker(H_{\pm\upsilon} - \Lambda_q) < \infty.$$

Remark 3.11. For $q \in \mathbb{N}$ set

$$\mathcal{D}_{q,j} := \{r \in (0, \infty) \mid \dim \ker(T_q(\delta_{\mathcal{C}_r})) = j\}, \quad j = 1, \ldots, q,$$

so that $\mathcal{D}_q = \bigcup_{j=1}^q \mathcal{D}_{q,j}$; note that the union stops at $j = q$ by Theorem 3.9 (i). In the proof of Theorem 3.9, we will describe the dimension of $\ker(T_q(\delta_{\mathcal{C}_r}))$ in terms of the zeros of Laguerre polynomials of q-th degree (see (5.50)–(5.51) below). If $q = 1, 2$, these zeros can be easily calculated, and we obtain explicitly the sets $\mathcal{D}_q$ and their components $\mathcal{D}_{q,j}$, namely

$$\mathcal{D}_1 = \mathcal{D}_{1,1} = \sqrt{(2/b)\mathbb{N}}, \tag{3.11}$$

$$\mathcal{D}_2 = \sqrt{(2/b)((\mathbb{N}+1) - \sqrt{(\mathbb{N}+1)})} \cup \sqrt{(2/b)(\mathbb{N}+\sqrt{\mathbb{N}})}, \tag{3.12}$$

$$\mathcal{D}_{2,2} = \sqrt{(2/b)(\mathbb{N}^2+\mathbb{N})}, \quad \mathcal{D}_{2,1} = \mathcal{D}_2 \setminus \mathcal{D}_{2,2}, \tag{3.13}$$

where $b > 0$ is the magnetic field.

Remark 3.12. Taking the explicit representation (4.6) of the orthonormal basis of $\operatorname{ran}(p_q) = \ker(H_0 - \Lambda_q)$, $q \in \mathbb{N}$, into account, we conclude that Λ_q is still an eigenvalue of the symmetric operator S defined in (3.4) provided that $br^2/2$ is a root of a Laguerre polynomial of q-th degree. Since H_υ is a self-adjoint extension of S for all $\upsilon \in L^p(\Gamma;\mathbb{R})$, Λ_q remains an eigenvalue of H_υ under the same assumption on $r > 0$.

4. Auxiliary results from the spectral theory of H_0

In this section we recall several known facts about H_0 that are necessary for our considerations; the first part follows [37, Section 4.2], while the second part on the basis of $\ker(H_0 - \Lambda_q)$ can be found in [39, Section 3.1], see also [23]. In the following, we describe a suitable spectral representation of H_0. Set

$$\phi(x) := \frac{b|x|^2}{4}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Introduce the magnetic creation operator

$$a^* = \Pi_1(A) - i\Pi_2(A) = -2ie^{\phi}\frac{\partial}{\partial z}e^{-\phi}, \quad z = x_1 + ix_2, \tag{4.1}$$

and the magnetic annihilation operator

$$a = \Pi_1(A) + i\Pi_2(A) = -2ie^{-\phi}\frac{\partial}{\partial \bar{z}}e^{\phi}, \quad \bar{z} = x_1 - ix_2. \tag{4.2}$$

The operators a and a^* are closed on $\mathfrak{D}(a) = \mathfrak{D}(a^*) = \mathrm{H}^1_A(\mathbb{R}^2)$, and are mutually adjoint in $L^2(\mathbb{R}^2)$. Moreover,

$$[a, a^*] = 2b, \tag{4.3}$$

and

$$H_0 = a^*a + b = aa^* - b.$$

Further,

$$\ker(H_0 - \Lambda_q) = (a^*)^q \ker(a), \quad q \in \mathbb{Z}_+. \tag{4.4}$$

By (4.2), we have

$$\ker(a) = \left\{ u \in L^2(\mathbb{R}^2) \;\middle|\; u = e^{-\phi} g,\ \frac{\partial g}{\partial \bar{z}} = 0 \right\}. \tag{4.5}$$

Thus, $e^{\phi}\ \ker(H_0 - \Lambda_0) = e^{\phi}\ \ker(a)$ coincides with the *Fock–Segal–Bargmann space* of entire functions $g \in L^2(\mathbb{R}^2; e^{-2\phi}dx)$ (see, e.g., [23, Section 3.2]). Assume now that

$$u \in \ker(H_0 - \Lambda_q), \quad q \in \mathbb{N}.$$

By (4.4) and (4.1), there exists an entire function $g \in L^2(\mathbb{R}^2; e^{-2\phi}dx)$ such that

$$\begin{aligned}(e^{\phi}u)(x) =: f(x) &= ((e^{\phi}(a^*)^q e^{-\phi})g)(x) = (-2i)^q\Big(\Big(e^{2\phi}\frac{\partial^q}{\partial z^q}\,e^{-2\phi}\Big)g\Big)(x)\\ &= (-2i)^q\sum_{\ell=0}^{q}\binom{q}{\ell}\Big(-\frac{b\bar{z}}{2}\Big)^{\ell} g^{(q-\ell)}(z), \quad x \in \mathbb{R}^2,\ z = x_1 + ix_2.\end{aligned}$$

Evidently, $f \in L^2(\mathbb{R}^2; e^{-2\phi}dx)$ is a polyanalytic function of order $q+1$, i.e., f is a solution of the equation

$$\frac{\partial^{q+1} f}{\partial \bar{z}^{q+1}}(z) = 0, \quad z \in \mathbb{C},$$

(see [1,5,6] and also [42, Section 2.2]). We have

$$\Big\{h \in L^2(\mathbb{R}^2; e^{-2\phi}dx)\ \Big|\ \frac{\partial^{q+1}h}{\partial \bar{z}^{q+1}} = 0\Big\} = \bigoplus_{j=0}^{q} e^{\phi}\ \ker(H_0 - \Lambda_j)$$

and the spaces $e^{\phi}\ \ker(H_0 - \Lambda_j)$, $j = 0, \dots, q$, are called sometimes *true poly-Fock spaces* of order j (see [1,44]).

Next, we introduce an explicit orthonormal basis of every $\ker(H_0 - \Lambda_q)$, $q \in \mathbb{Z}_+$, called sometimes the *angular-momentum basis*. Let at first $q = 0$. Then the functions

$$\tilde{\varphi}_{k,0}(x) = z^k e^{-\phi(x)}, \quad x \in \mathbb{R}^2,\ z = x_1 + ix_2,\ k \in \mathbb{Z}_+,$$

form an orthogonal basis of $\ker(a) = \operatorname{ran}(p_0)$ (see, e.g., [23, Sections 3.1–3.2]). Normalizing, we obtain an orthonormal basis of $\operatorname{ran}(p_0)$, consisting of the functions

$$\varphi_{k,0}(x) := \frac{\tilde{\varphi}_{k,0}(x)}{\|\tilde{\varphi}_{k,0}\|_{L^2(\mathbb{R}^2)}} = \sqrt{\frac{b}{2\pi}}\sqrt{\frac{1}{k!}}\Big(\sqrt{\frac{b}{2}}z\Big)^k e^{-\phi(x)}, \quad x \in \mathbb{R}^2,\ k \in \mathbb{Z}_+.$$

Let now $q \geq 1$. Set

$$\tilde{\varphi}_{k,q} = (a^*)^q\, \varphi_{k,0}, \quad k \in \mathbb{Z}_+.$$

The commutation relation (4.3) easily implies

$$\langle \tilde{\varphi}_{k,q}, \tilde{\varphi}_{\ell,q}\rangle_{L^2(\mathbb{R}^2)} = (2b)^q q!\,\delta_{k\ell}, \quad k,\ell \in \mathbb{Z}_+.$$

Therefore, the functions

$$\varphi_{k,q} := \frac{\tilde{\varphi}_{k,q}}{\|\tilde{\varphi}_{k,q}\|_{L^2(\mathbb{R}^2)}} = \frac{\tilde{\varphi}_{k,q}}{\sqrt{(2b)^q q!}}, \quad k \in \mathbb{Z}_+,$$

form an orthonormal basis of $\operatorname{ran}(p_q)$, $q \in \mathbb{N}$. They admit a more explicit representation (see [39, Section 3.1]), namely

$$\varphi_{k,q}(x) = \frac{1}{i^q}\sqrt{\frac{b}{2\pi}}\sqrt{\frac{q!}{k!}}\left(\sqrt{\frac{b}{2}}z\right)^{k-q} \mathrm{L}_q^{(k-q)}\left(\frac{b|x|^2}{2}\right)e^{-\phi(x)}, \quad x \in \mathbb{R}^2,\ z = x_1 + ix_2,\ k \in \mathbb{Z}_+, \tag{4.6}$$

where

$$\mathrm{L}_q^{(\alpha)}(t) := \frac{t^{-\alpha}e^t}{q!}\frac{d^q}{dt^q}(t^{q+\alpha}e^{-t}), \quad t > 0,\ \alpha \in \mathbb{R},\ q \in \mathbb{Z}_+,$$

are the (generalized) Laguerre polynomials (see [22, eq. 8.970 (1)]). In particular,

$$\mathrm{L}_1^{(\alpha)}(t) = -t + \alpha + 1, \tag{4.7}$$

$$\mathrm{L}_2^{(\alpha)}(t) = \frac{1}{2}(t^2 - 2(\alpha+2)t + (\alpha+2)(\alpha+1)). \tag{4.8}$$

The Laguerre polynomials $\mathrm{L}_q^{(\alpha)}$ with $q \in \mathbb{Z}_+$ and $\alpha > -1$ satisfy

$$\int_0^\infty e^{-t}t^\alpha \mathrm{L}_q^{(\alpha)}(t)\mathrm{L}_p^{(\alpha)}(t)dt = \Gamma(\alpha+1)\binom{q+\alpha}{q}\delta_{qp}, \quad q, p \in \mathbb{Z}_+, \tag{4.9}$$

(see [43, (5.1.1)]).

Finally, for $y \in \mathbb{R}^2$, introduce the *magnetic translations*

$$(\mathcal{T}_y u)(x) := e^{-i\frac{b}{2}(x\wedge y)}u(x-y), \quad x \in \mathbb{R}^2, \tag{4.10}$$

where

$$x \wedge y := x_1y_2 - x_2y_1.$$

Evidently, for each $y \in \mathbb{R}^2$, the operator $\mathcal{T}_y$ is unitary in $L^2(\mathbb{R}^2)$. A direct calculation yields

$$\mathcal{T}_y^* \,\Pi_j(A)\, \mathcal{T}_y = \Pi_j(A), \quad j = 1, 2,$$

and, therefore,

$$\mathcal{T}_y^* \,H_0\, \mathcal{T}_y = H_0, \quad y \in \mathbb{R}^2.$$

Then the spectral theorem implies

$$\mathcal{T}_y^* \,p_q\, \mathcal{T}_y = p_q, \quad y \in \mathbb{R}^2,\ q \in \mathbb{Z}_+. \tag{4.11}$$

5. Proofs of the main results

5.1. Proof of Theorem 3.1

We start with the proof of item (i). Denote by Ω_{in} and Ω_{ex} the interior and the exterior of Γ, respectively, and by ν the unit normal vector on Γ pointing outwards of Ω_{in}. For $w \in L^2(\mathbb{R}^2)$ set

$$w_\natural := w|_{\Omega_\natural}, \quad \natural = \text{in}, \text{ex}.$$

In view of (2.2) we have that $w \in \mathfrak{D}(H_{\pm\upsilon})$ is equivalent to the following conditions:

a. $w \in \mathrm{H}^1_A(\mathbb{R}^2)$;

b. $(-i\nabla - A)^2 w_\natural \in L^2(\Omega_\natural)$, $\natural = \text{in}, \text{ex}$;

c. $(\frac{\partial w_{\text{ex}}}{\partial \nu} - \frac{\partial w_{\text{in}}}{\partial \nu} \mp \upsilon w)_\Gamma = 0$.

Moreover, if $w \in \mathfrak{D}(H_{\pm\upsilon})$, then

$$(H_{\pm\upsilon} w)_\natural = (-i\nabla - A)^2 w_\natural, \qquad \natural = \text{in}, \text{ex}. \tag{5.1}$$

Assume $u \in \ker(T_q(\upsilon\delta_\Gamma))$, $q \in \mathbb{Z}_+$. Since $\upsilon \geq 0$, by (2.16) this is equivalent to

$$u \in \ker(H_0 - \Lambda_q) \subset \mathfrak{D}(H_0) = \mathrm{H}^2_A(\mathbb{R}^2) \tag{5.2}$$

and

$$\upsilon u = 0 \quad \text{on } \Gamma. \tag{5.3}$$

By $u \in \mathrm{H}^2_A(\mathbb{R}^2)$ conditions (a)–(b) are fulfilled and, moreover,

$$\frac{\partial u_{\text{ex}}}{\partial \nu} = \frac{\partial u_{\text{in}}}{\partial \nu} \quad \text{on } \Gamma. \tag{5.4}$$

Combining (5.3) with (5.4) we find that also (c) holds, i.e., $u \in \mathfrak{D}(H_{\pm\upsilon})$. By (5.2) we have

$$H_0 u = (-i\nabla - A)^2 u = \Lambda_q u \quad \text{in } \mathbb{R}^2$$

and, hence,

$$(-i\nabla - A)^2 u_\natural = \Lambda_q u_\natural \quad \text{in } \Omega_\natural,\ \natural = \text{in}, \text{ex}. \tag{5.5}$$

Bearing in mind (5.1), we now find that (5.5) implies $H_{\pm\upsilon} u = \Lambda_q u$, i.e.,

$$u \in \ker(H_{\pm\upsilon} - \Lambda_q).$$

The remaining items (ii) and (iii) will be proved together. To make the proof accessible in an easier way, we have split it into several steps. First, the case $\upsilon \geq 0$ is treated. In Step 1 the claims for the first Landau level Λ_0 are shown. In Steps 2–5 the claims for Λ_q, $q \in \mathbb{N}$, are verified. More precisely, in Step 2 the eigenvalue equation is reduced to equations for operators which are easier accessible for our purposes. Then, in Step 3 a representation of $\ker(T_q(\upsilon\delta_\Gamma))$ involving these new operators is proved. In Step 4 we are putting all this together to verify assertion (ii) for $\upsilon \geq 0$, while in Step 5 the proof of item (iii) in this case is concluded. Finally, in Step 6 the case $\upsilon \leq 0$ is treated.

Let us introduce the notations which will be used throughout the proof. Assume, as usual, that $\upsilon \in L^p(\Gamma;\mathbb{R})$ with $p > 1$, $\upsilon \geq 0$, $\upsilon \neq 0$, and (2.9) holds true. Set

$$Q_\upsilon^+(\lambda) := (H_0+\lambda)^{-1} - (H_\upsilon+\lambda)^{-1},$$
$$Q_\upsilon^-(\lambda) := -(H_0+\lambda)^{-1} + (H_{-\upsilon}+\lambda)^{-1}.$$

By Lemma 2.1, we have $Q_\upsilon^\pm(\lambda) \geq 0$, and the operators $Q_\upsilon^\pm(\lambda)$ are compact in $L^2(\mathbb{R}^2)$. Note that Lemma 2.1 also implies

$$Q_\upsilon^\pm(\lambda) = (H_0+\lambda)^{-1/2} G_\upsilon(\lambda)^* (I \pm G_\upsilon(\lambda)G_\upsilon(\lambda)^*)^{-1} G_\upsilon(\lambda)(H_0+\lambda)^{-1/2}. \quad (5.6)$$

Further, put

$$P_q^+ := \sum_{j=q}^{\infty} p_j \quad \text{and} \quad P_q^- := I - P_q^+,$$

so that $P_0^+ = I$, and $P_0^- = 0$. For $q \geq 1$ the projections $P_q^\pm$ have infinite rank. Finally, set

$$\mu_q(\lambda) := (\Lambda_q + \lambda)^{-1}, \quad q \in \mathbb{Z}_+, \ \lambda > -b.$$

Step 1. We first prove the part of Theorem 3.1 (ii) and (iii) concerning positive perturbations H_υ and start with the case $q = 0$. Assume that

$$u \in \ker(H_\upsilon - \Lambda_0).$$

Then u satisfies

$$(H_\upsilon+\lambda)^{-1}u = \mu_0(\lambda)u$$

or, equivalently,

$$((H_0+\lambda)^{-1} - \mu_0(\lambda))u - Q_\upsilon^+(\lambda)u = 0, \quad \lambda > -b. \quad (5.7)$$

Thus,

$$\langle ((H_0+\lambda)^{-1} - \mu_0(\lambda))u, u\rangle_{L^2(\mathbb{R}^2)} - \langle Q_\upsilon^+(\lambda)u, u\rangle_{L^2(\mathbb{R}^2)} = 0. \quad (5.8)$$

Both terms on the left-hand side of (5.8) are non-positive, and hence they both should vanish. Since $(H_0+\lambda)^{-1} - \mu_0(\lambda)$ is non-positive, the equality

$$\langle ((H_0+\lambda)^{-1} - \mu_0(\lambda))u, u\rangle_{L^2(\mathbb{R}^2)} = 0$$

and the min-max principle imply $u = p_0 u$. Then,

$$\langle Q_v^+(\lambda)u, u\rangle_{L^2(\mathbb{R}^2)} = \langle p_0 Q_v^+(\lambda) p_0 u, u\rangle_{L^2(\mathbb{R}^2)} = 0,$$

where the operator $p_0 Q_v^+(\lambda) p_0$ is self-adjoint and non-negative in the space

$$p_0 L^2(\mathbb{R}^2) = \operatorname{ran}(p_0).$$

Hence, by Lemma 2.2

$$u \in \ker(p_0 Q_v^+(\lambda) p_0) = \ker(T_0(v\delta_\Gamma)). \tag{5.9}$$

Thus, we obtain the inclusion $\subset$ in (3.2) for H_v and $q = 0$. The remaining inclusion $\supset$ in (3.2) is clear by (i). Therefore, item (iii) is shown in the case of positive perturbations and $q = 0$, which also implies assertion (ii) in the same case.

Step 2. Assume now

$$u \in \ker(H_v - \Lambda_q), \quad q \in \mathbb{N}.$$

In this step we reduce the eigenvalue equation for u to equations for operators which are easier accessible for our purposes. Similarly to (5.7) we have

$$((H_0 + \lambda)^{-1} - \mu_q(\lambda))u - Q_v^+(\lambda)u = 0, \quad \lambda > -b. \tag{5.10}$$

Set

$$u^+ := P_q^+ u \quad \text{and} \quad u^- := P_q^- u, \tag{5.11}$$

so that

$$u = u^+ + u^-. \tag{5.12}$$

Since $P_q^\pm$ are functions of H_0, they commute with $(H_0 + \lambda)^{-1}$ and thus, their application to (5.10) implies

$$((H_0 + \lambda)^{-1} - \mu_q(\lambda))u^+ = P_q^+ Q_v^+(\lambda)u^+ + P_q^+ Q_v^+(\lambda)u^-, \tag{5.13}$$

$$((H_0 + \lambda)^{-1} - \mu_q(\lambda))u^- = P_q^- Q_v^+(\lambda)u^+ + P_q^- Q_v^+(\lambda)u^-. \tag{5.14}$$

Let

$$S_q^+(\lambda) := P_q^- Q_v^+(\lambda) P_q^- \tag{5.15}$$

and observe that by Lemma 2.1 the operator $S_q^+(\lambda)$ is compact, self-adjoint, and non-negative in $P_q^- L^2(\mathbb{R}^2)$. Set

$$m_q^+(\lambda) := \inf \sigma\big(((H_0 + \lambda)^{-1} - \mu_q(\lambda))|_{P_q^- L^2(\mathbb{R}^2)}\big) = \frac{2b}{(\Lambda_q + \lambda)(\Lambda_{q-1} + \lambda)}, \tag{5.16}$$

and

$$S_{q,>}^{+}(\lambda) := S_q^{+}(\lambda)\mathbb{1}_{[m_q^{+}(\lambda),\infty)}(S_q^{+}(\lambda)), \quad S_{q,<}^{+}(\lambda) := S_q^{+}(\lambda) - S_{q,>}^{+}(\lambda);$$

here and in the sequel $\mathbb{1}_{\mathcal{J}}(T)$ denotes the spectral projection of the operator $T = T^*$ associated with the Borel set $\mathcal{J} \subset \mathbb{R}$. Note that

$$\operatorname{rank}(S_{q,>}^{+}(\lambda)) < \infty.$$

Moreover, if

$$\|S_q^{+}(\lambda)\| < m_q^{+}(\lambda),$$

then $S_{q,>}^{+}(\lambda) = 0$. Now, (5.14) is equivalent to

$$((H_0+\lambda)^{-1} - \mu_q(\lambda) - S_{q,<}^{+}(\lambda))u^{-} = P_q^{-} Q_v^{+}(\lambda)u^{+} + S_{q,>}^{+}(\lambda)u_{>}^{-}, \tag{5.17}$$

where

$$u_{>}^{-} := P_{q,>}^{-} u \tag{5.18}$$

and

$$P_{q,>}^{-} = P_{q,>}^{-}(\lambda) := \mathbb{1}_{[m_q^{+}(\lambda),\infty)}(S_q^{+}(\lambda))\, P_q^{-}. \tag{5.19}$$

Note that

$$\operatorname{rank}(P_{q,>}^{-}(\lambda)) = \operatorname{rank}(S_{q,>}^{+}(\lambda)) < \infty. \tag{5.20}$$

Since $S_q^{+}(\lambda)$ is compact and $m_q^{+}(\lambda) > 0$, there exists $\varepsilon > 0$ such that

$$\sigma(S_q^{+}(\lambda)) \cap (m_q^{+}(\lambda) - \varepsilon, m_q^{+}(\lambda)) = \emptyset$$

and hence $\|S_{q,<}^{+}(\lambda)\| \le m_q^{+}(\lambda) - \varepsilon < m_q^{+}(\lambda)$. By definition of $m_q^{+}(\lambda)$, we have

$$((H_0+\lambda)^{-1} - \mu_q(\lambda))|_{P_q^{-}L^2(\mathbb{R}^2)} \ge m_q^{+}(\lambda)$$

and thus, the operator

$$((H_0+\lambda)^{-1} - \mu_q(\lambda))|_{P_q^{-}L^2(\mathbb{R}^2)} - S_{q,<}^{+}(\lambda)$$

is positive and boundedly invertible on $P_q^{-}L^2(\mathbb{R}^2)$. Set

$$R_q^{+}(\lambda) := \big(((H_0+\lambda)^{-1} - \mu_q(\lambda))|_{P_q^{-}L^2(\mathbb{R}^2)} - S_{q,<}^{+}(\lambda)\big)^{-1}.$$

Then (5.17) implies

$$u^{-} = R_q^{+}(\lambda) P_q^{-} Q_v^{+}(\lambda)u^{+} + R_q^{+}(\lambda) S_{q,>}^{+}(\lambda)u_{>}^{-}. \tag{5.21}$$

Inserting (5.21) into (5.13) we get

$$P_q^{+} X_q^{+}(\lambda) P_q^{+} u^{+} = -P_q^{+} Y_q^{+}(\lambda) u_{>}^{-}, \tag{5.22}$$

where

$$X_q^+(\lambda) := -(H_0+\lambda)^{-1} + \mu_q(\lambda) + Q_\upsilon^+(\lambda) + Q_\upsilon^+(\lambda) R_q^+(\lambda) P_q^- Q_\upsilon^+(\lambda)$$

and

$$Y_q^+(\lambda) := Q_\upsilon^+(\lambda) R_q^+(\lambda)\, S_{q,>}^+(\lambda).$$

Step 3. Define the operator

$$K_q^+ := P_q^+ X_q^+(\lambda) P_q^+, \tag{5.23}$$

which is self-adjoint and non-negative in $P_q^+ L^2(\mathbb{R}^2)$. In this step we show

$$\ker(K_q^+) = \ker(T_q(\upsilon\delta_\Gamma)). \tag{5.24}$$

In order to check (5.24), assume first that $w \in \ker(T_q(\upsilon\delta_\Gamma))$. Then $w = p_q w$, see Remark 3.2, and

$$\begin{aligned} \langle K_q^+ w, w\rangle_{L^2(\mathbb{R}^2)} &= \langle(-(H_0+\lambda)^{-1} + \mu_q(\lambda)) p_q w, p_q w\rangle_{L^2(\mathbb{R}^2)} \\ &\quad + \langle Q_\upsilon^+(\lambda) p_q w, p_q w\rangle_{L^2(\mathbb{R}^2)} \\ &\quad + \langle Q_\upsilon^+(\lambda) R_q^+(\lambda) P_q^- Q_\upsilon^+(\lambda) p_q w, p_q w\rangle_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{5.25}$$

Using Lemma 2.2 and $\ker(A^*A) = \ker(A)$, one has

$$\ker(T_q(\upsilon\delta_\Gamma)) = \ker(p_q Q_\upsilon^+(\lambda) p_q) = \ker(Q_\upsilon^+(\lambda)^{1/2} p_q). \tag{5.26}$$

Moreover, the definitions of $\mu_q(\lambda)$ and p_q yield

$$((H_0+\lambda)^{-1} - \mu_q(\lambda)) p_q = 0. \tag{5.27}$$

Thus, (5.25)–(5.27) imply

$$\langle K_q^+ w, w\rangle_{L^2(\mathbb{R}^2)} = 0.$$

Since $K_q^+ \geq 0$ we conclude that $w \in \ker(K_q^+)$, i.e.,

$$\ker(T_q(\upsilon\delta_\Gamma)) \subset \ker(K_q^+). \tag{5.28}$$

Let now $w \in \ker(K_q^+)$. Then

$$\begin{aligned} &\langle(-(H_0+\lambda)^{-1} + \mu_q(\lambda)) P_q^+ w, P_q^+ w\rangle_{L^2(\mathbb{R}^2)} + \langle Q_\upsilon^+(\lambda) P_q^+ w, P_q^+ w\rangle_{L^2(\mathbb{R}^2)} \\ &\quad + \langle Q_\upsilon^+(\lambda) R_q^+(\lambda) P_q^- Q_\upsilon^+(\lambda) P_q^+ w, P_q^+ w\rangle_{L^2(\mathbb{R}^2)} = 0. \end{aligned} \tag{5.29}$$

The three terms on the left-hand side of (5.29) are non-negative and therefore all of them vanish. The equality

$$\langle((H_0+\lambda)^{-1} - \mu_q(\lambda)) P_q^+ w, P_q^+ w\rangle_{L^2(\mathbb{R}^2)} = 0$$

implies that $P_q^+ w = p_q w$. Inserting this into

$$\langle Q_\upsilon^+(\lambda) P_q^+ w, P_q^+ w\rangle_{L^2(\mathbb{R}^2)} = 0$$

we obtain

$$\langle Q_\upsilon^+(\lambda) p_q w, p_q w\rangle_{L^2(\mathbb{R}^2)} = 0,$$

i.e., with (5.26)

$$w \in \ker(p_q Q_\upsilon^+(\lambda) p_q) = \ker(T_q(\upsilon\delta_\Gamma)).$$

Therefore,

$$\ker(K_q^+) \subset \ker(T_q(\upsilon\delta_\Gamma)),$$

which combined with (5.28) yields (5.24).

Step 4. Now, we have everything in hands to finish the proof of assertion (ii) for $\upsilon \geq 0$. For this purpose we show that the element $u \in \ker(H_\upsilon - \Lambda_q)$ can be written as $u = u_0^+ + W_q^+ u_>^-$, where $u_0^+ \in \ker(T_q(\upsilon\delta_\Gamma))$, $u_>^- \in \operatorname{ran}(P_{q,>}^-(\lambda))$, and W_q^+ is a suitable operator; this implies (3.1) as $\operatorname{rank}(P_{q,>}^-(\lambda)) < \infty$ by (5.20). Let π_0^+ be the orthogonal projection onto $\ker(K_q^+)$ and $\pi_\perp^+ := P_q^+ - \pi_0^+$. Set

$$u_0^+ := \pi_0^+ u^+ \quad \text{and} \quad u_\perp^+ := \pi_\perp^+ u^+,$$

where u^+ is the function defined in (5.11). Thus,

$$u^+ = u_0^+ + u_\perp^+. \tag{5.30}$$

Denote by $K_{q,\perp}^+$ the operator $\pi_\perp^+ K_q^+ \pi_\perp^+$, which is self-adjoint and positive on the space $\operatorname{ran}(\pi_\perp^+)$. Then, as $u_0^+ \in \ker(K_q^+)$, (5.22) and (5.23) imply

$$K_{q,\perp}^+ u_\perp^+ = -P_q^+ Y_q^+(\lambda) u_>^-. \tag{5.31}$$

Since we started with an arbitrary $u \in \ker(H_\upsilon - \Lambda_q)$ we have by (5.18)

$$u_>^- \in P_{q,>}^- \ker(H_\upsilon - \Lambda_q).$$

Then (5.31) implies

$$P_q^+ Y_q^+(\lambda) P_{q,>}^- \ker(H_\upsilon - \Lambda_q) \subset \operatorname{ran}(K_{q,\perp}^+). \tag{5.32}$$

Recall that $K_{q,\perp}^+$ is positive and hence invertible. Therefore, we can define on $P_{q,>}^- \ker(H_\upsilon - \Lambda_q)$ the operator

$$L_q^+ := -(K_{q,\perp}^+)^{-1} P_q^+ Y_q^+(\lambda).$$

By (5.31) we have

$$u_\perp^+ = L_q^+ u_>^-. \tag{5.33}$$

Putting together the equations (5.12), (5.21), (5.30), and (5.33), we find that for any $u \in \ker(H_\upsilon - \Lambda_q)$ we have

$$u = u_0^+ + W_q^+ u_>^-, \tag{5.34}$$

where

$$u_0^+ \in \ker(K_q^+) = \ker(T_q(\upsilon\delta_\Gamma)), \quad u_>^- \in P_{q,>}^- \ker(H_\upsilon - \Lambda_q),$$

and

$$W_q^+ = R_q^+(\lambda) S_{q,>}^+(\lambda) + (I + R_q^+(\lambda) P_q^- Q_\upsilon^+(\lambda)) L_q^+.$$

Deriving (5.34) we have also taken into account that

$$R_q^+(\lambda) P_q^- Q_\upsilon^+(\lambda) u_0^+ = 0$$

due to $u_0^+ \in \ker(K_q^+)$, (5.24), and (5.26). Now, (5.34) and (5.32) entail (3.1) for the case of H_υ with

$$n_q^+ := \inf_{\lambda \in (-b,\infty)} \operatorname{rank}(P_{q,>}^-(\lambda)), \quad q \in \mathbb{N}.$$

Step 5. Let us prove item (iii) for H_υ and $q \in \mathbb{N}$. Note that we have already shown assertion (i) and hence it suffices to verify the inclusion $\subset$ in (3.2). Since

$$\|(I + G_\upsilon(\lambda)^* G_\upsilon(\lambda))^{-1}\| \leq 1$$

we get by (5.15) and (2.6),

$$\|S_q^+(\lambda)\| \leq \|G_\upsilon(\lambda)\|^2 \leq \|\upsilon\|_{L^p(\Gamma)} C_p(\lambda), \quad \lambda > -b + 1. \tag{5.35}$$

Let

$$\|\upsilon\|_{L^p(\Gamma)} < \upsilon_q^+ := \sup_{\lambda \in (-b+1,\infty)} \frac{m_q^+(\lambda)}{C_p(\lambda)}, \tag{5.36}$$

where $m_q^+(\lambda)$ is the quantity defined in (5.16). Let $R > 0$ be such that Γ is contained in the open ball B_R with radius R. Observe that by the diamagnetic inequality

$$\begin{aligned} C_p(1) &= \sup_{0 \neq u \in \mathrm{H}_A^1(\mathbb{R}^2)} \frac{(\int_\Gamma |\tau u|^{2p'} ds)^{1/p'}}{\int_{\mathbb{R}^2} (|\Pi(A)u|^2 + |u|^2) dx} \\ &\leq \sup_{0 \neq u \in \mathrm{H}_A^1(\mathbb{R}^2)} \frac{(\int_\Gamma |\tau u|^{2p'} ds)^{1/p'}}{\int_{\mathbb{R}^2} (|\nabla |u||^2 + |u|^2) dx} \\ &\leq \sup_{0 \neq u \in \mathrm{H}_A^1(\mathbb{R}^2)} \frac{(\int_\Gamma |\tau u|^{2p'} ds)^{1/p'}}{\int_{B_R} (|\nabla |u||^2 + |u|^2) dx} \\ &= \sup_{0 \neq u \in \mathrm{H}^1(B_R)} \frac{(\int_\Gamma |\tau u|^{2p'} ds)^{1/p'}}{\int_{B_R} (|\nabla |u||^2 + |u|^2) dx} =: c^{-1}, \quad p' = \frac{p}{p-1}, \end{aligned} \tag{5.37}$$

where the constant $c > 0$ does not depend on the magnetic field. Hence, the constant υ_q^+ can be estimated from below as

$$\upsilon_q^+ \geq \frac{m_q^+(1)}{C_p(1)} \geq \frac{2bc}{(\Lambda_q + 1)(\Lambda_{q-1} + 1)},$$

which is the bound in (3.3). Furthermore, (5.35) implies that there exist $\lambda > -b + 1$ such that

$$\|S_q^+(\lambda)\| < m_q^+(\lambda),$$

so that $P_{q,>}^-(\lambda) = 0$ by (5.19). By (5.34), we conclude that if $u \in \ker(H_\upsilon - \Lambda_q)$, then $u \in \ker(T_q(\upsilon\delta_\Gamma))$, as $u_>^- \in \operatorname{ran}(P_{q,>}^-(\lambda)) = \{0\}$. Therefore, together with (i) we conclude that (3.2) holds.

Step 6. Let us now consider $H_{-\upsilon}$, i.e., the Landau Hamiltonian perturbed by a negative δ-potential. The proof of Theorem 3.1 (ii) and (iii) in this case is quite similar to the one concerning H_υ, so that we omit certain details. Assume

$$u \in \ker(H_{-\upsilon} - \Lambda_q), \quad q \in \mathbb{Z}_+.$$

Then, similarly to (5.10), we have

$$((H_0 + \lambda)^{-1} - \mu_q(\lambda))u + Q_\upsilon^-(\lambda)u = 0, \quad \lambda > -\inf \sigma(H_{-\upsilon}). \tag{5.38}$$

Put

$$u^+ := P_{q+1}^+ u \quad \text{and} \quad u^- := P_{q+1}^- u,$$

so that again

$$u = u^+ + u^-.$$

Then (5.38) implies

$$(\mu_q(\lambda) - (H_0 + \lambda)^{-1})u^- = P_{q+1}^- Q_\upsilon^-(\lambda)u^- + P_{q+1}^- Q_\upsilon^-(\lambda)u^+, \tag{5.39}$$

$$(\mu_q(\lambda) - (H_0 + \lambda)^{-1})u^+ = P_{q+1}^+ Q_\upsilon^-(\lambda)u^- + P_{q+1}^+ Q_\upsilon^-(\lambda)u^+. \tag{5.40}$$

Let

$$S_q^-(\lambda) := P_{q+1}^+ Q_\upsilon^-(\lambda) P_{q+1}^+$$

and observe that by Lemma 2.1 the operator $S_q^-(\lambda)$ is compact, self-adjoint, and non-negative in $P_{q+1}^+ L^2(\mathbb{R}^2)$. Set

$$m_q^-(\lambda) := \inf \sigma\big((\mu_q(\lambda) - (H_0 + \lambda)^{-1})|_{P_{q+1}^+ L^2(\mathbb{R}^2)}\big) = \frac{2b}{(\Lambda_q + \lambda)(\Lambda_{q+1} + \lambda)} \tag{5.41}$$

and

$$S_{q,>}^-(\lambda) := S_q^-(\lambda)\mathbb{1}_{[m_q^-(\lambda),\infty)}(S_q^-(\lambda)), \quad S_{q,<}^-(\lambda) := S_q^-(\lambda) - S_{q,>}^-(\lambda).$$

Now, (5.40) is equivalent to

$$\big((\mu_q(\lambda) - (H_0+\lambda)^{-1}) - S_{q,<}^-(\lambda)\big)u^+ = P_{q+1}^+ Q_\upsilon^-(\lambda)u^- + S_{q,>}^-(\lambda)u_>^+, \tag{5.42}$$

where

$$u_>^+ := P_{q+1,>}^+ u$$

and

$$P_{q+1,>}^+ = P_{q+1,>}^+(\lambda) := \mathbb{1}_{[m_q^-(\lambda),\infty)}(S_q^-(\lambda))\, P_{q+1}^+.$$

Note that

$$\operatorname{rank}(P_{q+1,>}^+(\lambda)) = \operatorname{rank}(S_{q,>}^-(\lambda)) < \infty.$$

The operator

$$(\mu_q(\lambda) - (H_0+\lambda)^{-1})|_{P_{q+1}^+ L^2(\mathbb{R}^2)} - S_{q,<}^-(\lambda)$$

is positive and boundedly invertible in $P_{q+1}^+ L^2(\mathbb{R}^2)$. Set

$$R_q^-(\lambda) := \big((\mu_q(\lambda) - (H_0+\lambda)^{-1})|_{P_{q+1}^+ L^2(\mathbb{R}^2)} - S_{q,<}^-(\lambda)\big)^{-1}.$$

Then (5.42) implies

$$u^+ = R_q^-(\lambda)P_{q+1}^+ Q_\upsilon^-(\lambda)u^- + R_q^-(\lambda)S_{q,>}^-(\lambda)u_>^+. \tag{5.43}$$

Inserting (5.43) into (5.39), we obtain

$$P_{q+1}^- X_q^-(\lambda)P_{q+1}^- u^- = -P_{q+1}^- Y_q^-(\lambda)u_>^+,$$

with

$$X_q^-(\lambda) := (H_0+\lambda)^{-1} - \mu_q(\lambda) + Q_\upsilon^-(\lambda) + Q_\upsilon^-(\lambda)R_q^-(\lambda)P_{q+1}^+ Q_\upsilon^-(\lambda)$$

and

$$Y_q^-(\lambda) := Q_\upsilon^-(\lambda)R_q^-(\lambda)\, S_{q,>}^-(\lambda).$$

Then, similarly to (5.34), we find that for any $u \in \ker(H_\upsilon - \Lambda_q)$ we have

$$u = u_0^- + W_q^- u_>^+, \tag{5.44}$$

with

$$u_0^- \in \ker(T_q(\upsilon\delta_\Gamma)), \quad u_>^+ \in P_{q+1,>}^+ \ker(H_{-\upsilon} - \Lambda_q),$$

and an appropriate operator

$$W_q^-\colon P_{q+1,>}^+ \ker(H_{-\upsilon} - \Lambda_q) \to L^2(\mathbb{R}^2).$$

Now, (5.44) entails (3.1) for $H_{-\upsilon}$ with

$$n_q^- := \inf_{\lambda \in (-\inf \sigma(H_\upsilon),\infty)} \operatorname{rank}(P_{q+1,>}^+(\lambda)) < \infty.$$

Finally, we prove (3.2) for $H_{-\upsilon}$. Assume that

$$\|\upsilon\|_{L^p(\Gamma)} < \upsilon_q^- := \sup_{\lambda \in (-b+1,\infty)} \frac{m_q^-(\lambda)}{C_p(\lambda)(1 + m_q^-(\lambda))}, \tag{5.45}$$

where $C_p(\lambda)$ and $m_q^-(\lambda)$ are the quantities defined in (2.6) and (5.41), respectively. Note that using (5.37) we can estimate υ_q^- from below as

$$\upsilon_q^- \geq \frac{m_q^-(1)}{C_p(1)(1 + m_q^-(1))} \geq \frac{2bc}{2b + (\Lambda_q + 1)(\Lambda_{q+1} + 1)}.$$

Furthermore, there exists $\lambda \in (-b + 1, \infty)$ such that

$$\|\upsilon\|_{L^p(\Gamma)}\, C_p(\lambda) < \frac{m_q^-(\lambda)}{1 + m_q^-(\lambda)} < 1.$$

By (2.5)

$$\|G_\upsilon(\lambda)\|^2 \leq C_p(\lambda)\|\upsilon\|_{L^p(\Gamma)} \leq \frac{m_q^-(\lambda)}{1 + m_q^-(\lambda)} < 1. \tag{5.46}$$

On the other hand, since $\|(I - G_\upsilon(\lambda)^* G_\upsilon(\lambda))^{-1}\| \leq (1 - \|G_\upsilon(\lambda)\|^2)^{-1}$, similarly to (5.35) we have

$$\|S_q^-(\lambda)\| \leq \frac{\|G_\upsilon(\lambda)\|^2}{1 - \|G_\upsilon(\lambda)\|^2}. \tag{5.47}$$

Putting together (5.47) and (5.46) we get

$$\|S_q^-(\lambda)\| < m_q^-(\lambda),$$

so that $P_{q+1,>}^+(\lambda) = 0$, and (5.44) implies that for $u \in \ker(H_{-\upsilon} - \Lambda_q)$ we have $u \in \ker(T_q(\upsilon\delta_\Gamma))$, i.e., (3.2) holds for $H_{-\upsilon}$.

5.2. Proof of Theorem 3.6

Assume that

$$u = p_0 u \in \ker(T_0(\upsilon\delta_\Gamma)),$$

which is equivalent to $\upsilon^{1/2}(u|_\Gamma) = 0$ as an element of $L^2(\Gamma)$. Since we assume that $\upsilon \neq 0$ as an element of $L^p(\Gamma; \mathbb{R})$, $p > 1$, we have $u = 0$ on a subset of Γ of positive measure. Since $e^\phi u$ is entire, cf. (4.5), and its zeros are isolated if $u \neq 0$, we easily find that $u = 0$, i.e., (3.5) holds.

Remark 5.1. The above argument is not applicable in the case $q \geq 1$, because there exist polyanalytic functions u which do not vanish identically on $\mathbb{C}$ but vanish on certain regular Jordan curves (see [6, Section 5]).

5.3. Proof of Theorem 3.9

Due to the invariance of p_q under the magnetic translations (see (4.10) and (4.11)) we may assume without loss of generality that $\mathcal{C}_r$ is centered at the origin.

Let $u \in \ker(H_0 - \Lambda_q) = \operatorname{ran}(p_q)$, $q \in \mathbb{N}$. Then we have

$$u(x) = \sum_{k \in \mathbb{Z}_+} c_k \varphi_{k,q}(x), \quad x \in \mathbb{R}^2, \tag{5.48}$$

with $\mathbf{c} := \{c_k\}_{k \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$, where $\{\varphi_{k,q}\}_{k \in \mathbb{Z}_+}$ is the orthonormal basis of the space $\ker(H_0 - \Lambda_q)$ defined in (4.6). Hence, the representation in (5.48) generates a unitary operator $\mathcal{U}_q \colon \ker(H_0 - \Lambda_q) \to \ell^2(\mathbb{Z}_+)$ which maps u to $\mathbf{c}$. On the other hand, we have

$$\langle T_q(\delta_{\mathcal{C}_r})\varphi_{k,q}, \varphi_{\ell,q}\rangle_{L^2(\mathbb{R}^2)} = \lambda_{k,q}(r)\delta_{k\ell},$$

where

$$\begin{aligned} \lambda_{k,q}(r) &:= \langle T_q(\delta_{\mathcal{C}_r})\varphi_{k,q}, \varphi_{k,q}\rangle_{L^2(\mathbb{R}^2)} \\ &= b\frac{q!}{k!}(br^2/2)^{k-q}\, \mathrm{L}_q^{(k-q)}(br^2/2)^2 e^{-br^2/2}, \quad r \in (0, \infty). \end{aligned} \tag{5.49}$$

Then we have

$$\mathcal{U}_q T_q(\delta_{\mathcal{C}_r})\mathcal{U}_q^* = \tau_{q,r},$$

where $\tau_{q,r} \colon \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ is a compact self-adjoint operator defined by

$$(\tau_{q,r}\mathbf{c})_k = \lambda_{k,q}(r)c_k, \quad k \in \mathbb{Z}_+,$$

with $\mathbf{c} = \{c_k\}_{k \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$. In particular, the functions $\varphi_{k,q}$ are eigenfunctions of $T_q(\delta_{\mathcal{C}_r})$ with eigenvalues equal to $\lambda_{k,q}(r)$. For $r \in (0, \infty)$ set

$$m_q(r) := \#\{k \in \mathbb{Z}_+ \mid \mathrm{L}_q^{(k-q)}(br^2/2) = 0\}. \tag{5.50}$$

Then (5.49) implies

$$\dim \ker(T_q(\delta_{\mathcal{C}_r})) = \#\{k \in \mathbb{Z}_+ \mid \lambda_{k,q}(r) = 0\} = m_q(r), \quad r \in (0, \infty), \tag{5.51}$$

and

$$\mathcal{D}_q = \{r \in (0,\infty) \mid \dim\ker(T_q(\delta_{\mathcal{C}_r})) \geq 1\} = \{r \in (0,\infty) \mid m_q(r) \geq 1\}. \quad (5.52)$$

Bearing in mind the expressions for the Laguerre polynomials $\mathrm{L}_q^{(k-q)}$ with $q = 1, 2$, given in (4.7)–(4.8), we find that the zero of $\mathrm{L}_1^{(k-1)}$ is equal to k, while the zeros of $\mathrm{L}_2^{(k-2)}$ are equal to $k \pm \sqrt{k}$, $k \in \mathbb{Z}_+$. Thus, (5.51)–(5.52) easily entail the explicit description of the sets $\mathcal{D}_q$, $q = 1, 2$, and their components $\mathcal{D}_{q,j}$, available in (3.11)–(3.13).

Let us estimate $m_q(r)$ and describe $\mathcal{D}_q$ in the general case. Note that the polynomial $\mathrm{L}_q^{(\alpha)}$ with $\alpha > -1$ has exactly q simple strictly positive zeros (see [43, Theorem 3.3.1] and (4.9)). Denote by $\{\zeta_\ell(\alpha)\}_{\ell=1}^q$, $\alpha \in [0,\infty)$, the set of the zeros of $\mathrm{L}_q^{(\alpha)}$, enumerated in decreasing order. The functions ζ_ℓ, $\ell = 1,\dots,q$, are smooth strictly increasing functions (see [43, Section 6.21 (4)]) which tend to infinity as $\alpha \to \infty$ (see [12]). Thus, we can classify the zeros of $\mathrm{L}_q^{(k-q)}$ with $k \geq q$. In order to handle the polynomials $\mathrm{L}_q^{(k-q)}$ with $0 \leq k < q$ we note that

$$\mathrm{L}_q^{(k-q)}(t) = \frac{k!}{q!}(-t)^{q-k}\mathrm{L}_k^{(q-k)}(t), \quad t \in \mathbb{R}, \quad (5.53)$$

(see [43, (5.2.1)]), so that if $k = 0$ the polynomial $\mathrm{L}_q^{(-q)}(t)$ is proportional to t^q, while if $q \geq 2$ and $1 \leq k < q-1$ the polynomial $\mathrm{L}_q^{(k-q)}$ has k simple positive zeros and a null root of order $q-k$. If $k = 1,\dots,q$, denote by $\{z_{m,k}\}_{m=1}^k$ the set of the positive zeros of $\mathrm{L}_q^{(k-q)}$, enumerated in decreasing order. Note that

$$z_{\ell,q} = \zeta_\ell(0), \quad \ell = 1,\dots,q.$$

Moreover, we have

$$\frac{d}{dt}\mathrm{L}_k^{(q-k)}(t) = -\mathrm{L}_{k-1}^{(q-k+1)}(t), \quad t \in \mathbb{R}, \quad (5.54)$$

(see [43, (5.1.14)]), so that (5.53), (5.54), and Rolle's theorem imply that the zeros $z_{m,k}$ interlace, i.e.,

$$z_{m+1,k} < z_{m,k-1} < z_{m,k}$$

(see [14, 15] for further details). If $q \geq 2$, let us extend the functions ζ_ℓ, $\ell = 1,\dots,q-1$, to the interval $[-q+\ell,\infty)$. To this end, set

$$\zeta_\ell(-n) = z_{\ell,q-n}, \quad n = 1,\dots,q-\ell,$$

and interpolate by linear functions on the intervals $(-n,-n+1)$. Thus, we obtain a family of q increasing Lipschitz functions $\zeta_\ell(\alpha)$, $\ell = 1,\dots,q$, defined on $\alpha \in [-q+\ell,\infty)$, which tend to infinity as $\alpha \to \infty$ and, if $q \geq 2$, we have

$$\zeta_{\ell+1}(\alpha) < \zeta_\ell(\alpha), \quad \alpha \in [-q+\ell,\infty), \quad \ell = 1,\dots,q-1.$$

Set

$$\eta_\ell(\alpha) := \sqrt{2\zeta_\ell(\alpha)/b}, \quad \alpha \in [-q+\ell+1, \infty),\ \ell = 1, \dots, q.$$

Thus, we find that for any $r \in (0, \infty)$ the quantity $m_q(r)$ defined in (5.50) is equal to the number of integers $\ell \in \{1, \dots, q\}$ for which $r \in \operatorname{ran}(\eta_\ell)$ and $\eta_\ell^{-1}(r) \in \mathbb{N} - \{q\}$. Then, evidently, $m_q(r) \leq q$ and combined with (5.51) this implies (3.10).

Finally, the set $\mathcal{D}_q$ is infinite since it contains, for example, all the points $r = \eta_1(k-q)$ with $k \in \mathbb{N}$. On the other hand, $\mathcal{D}_q$ is discrete because it is locally finite.

A. Closedness and semiboundedness of the quadratic form in (2.1)

Recall that we consider for $\upsilon \in L^p(\Gamma; \mathbb{R})$ the following quadratic form:

$$\int_{\mathbb{R}^2} |\Pi(A)u|^2 dx + \int_\Gamma \upsilon |\tau u|^2\, ds, \quad u \in \mathrm{H}^1_A(\mathbb{R}^2). \tag{A.1}$$

The function υ can be decomposed as $\upsilon = \upsilon_1 + \upsilon_2$, where $\upsilon_1 \in L^\infty(\Gamma)$ and where $\|\upsilon_2\|_{L^p(\Gamma)} \leq \delta$ for arbitrarily small $\delta > 0$. First, we get the following elementary estimate:

$$\left| \int_\Gamma \upsilon |\tau u|^2\, ds \right| \leq \int_\Gamma |\upsilon_1| |\tau u|^2\, ds + \int_\Gamma |\upsilon_2| |\tau u|^2\, ds. \tag{A.2}$$

Next, we estimate the two terms on the right-hand side separately. Combining [8, Lemma 2.6], the diamagnetic inequality [31, Theorem 7.21], and that υ_1 is a bounded function, we obtain that for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\int_\Gamma |\upsilon_1| |\tau u|^2\, ds \leq \varepsilon \|\Pi(A)u\|^2_{L^2(\mathbb{R}^2)} + C(\varepsilon) \|u\|^2_{L^2(\mathbb{R}^2)}. \tag{A.3}$$

By [19, Lemma 5.3], the operator of multiplication $\mathcal{M}_{|\upsilon_2|}$ with $|\upsilon_2|$ is bounded from $\mathrm{H}^{1/2}(\Gamma)$ into $\mathrm{H}^{-1/2}(\Gamma)$ and, moreover, its norm between these two spaces is estimated from above by

$$\|\mathcal{M}_{|\upsilon_2|}\|_{\mathrm{H}^{1/2}(\Gamma) \to \mathrm{H}^{-1/2}(\Gamma)} \leq c \|\upsilon_2\|_{L^p(\Gamma)} \leq c\delta,$$

where $c = c(\Gamma, p) > 0$. Using the above estimate of the norm of $\mathcal{M}_{|\upsilon_2|}$ and that the mapping τ is bounded from $\mathrm{H}^1_A(\mathbb{R}^2)$ into $\mathrm{H}^{1/2}(\Gamma)$ we get

$$\begin{aligned} \int_\Gamma |\upsilon_2| |\tau u|^2\, ds &\leq \|\mathcal{M}_{|\upsilon_2|} \tau u\|_{\mathrm{H}^{-1/2}(\Gamma)} \|\tau u\|_{\mathrm{H}^{1/2}(\Gamma)} \\ &\leq c\delta \|\tau\|^2_{\mathrm{H}^1_A(\mathbb{R}^2) \to \mathrm{H}^{1/2}(\Gamma)} \|u\|^2_{\mathrm{H}^1_A(\mathbb{R}^2)}. \end{aligned} \tag{A.4}$$

Combining the estimates (A.2), (A.3), (A.4), and taking into account that the decomposition of υ can be chosen such that the parameter δ is arbitrarily small, we conclude that for any $\varepsilon' > 0$ there exists $C'(\varepsilon') > 0$ so that

$$\left| \int_\Gamma \upsilon |\tau u|^2 \, ds \right| \leq \varepsilon' \|\Pi(A)u\|^2_{L^2(\mathbb{R}^2)} + C'(\varepsilon')\|u\|^2_{L^2(\mathbb{R}^2)}.$$

Hence, it follows from the perturbation result [27, Theorem VI.1.33] that the quadratic form in (A.1) is closed and semibounded.

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