# Spectral bounds for indefinite singular Sturm-Liouville operators with uniformly locally integrable potentials

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**Abstract**

The non-real spectrum of a singular indefinite Sturm-Liouville operator

$$A = \frac{1}{r}\left(-\frac{d}{dx}\,p\,\frac{d}{dx} + q\right)$$

with a sign changing weight function $r$ consists (under suitable additional assumptions on the real coefficients $1/p, q, r \in L^1_{\rm loc}(\mathbb{R})$) of isolated eigenvalues with finite algebraic multiplicity which are symmetric with respect to the real line. In this paper bounds on the absolute values and the imaginary parts of the non-real eigenvalues of $A$ are proved for uniformly locally integrable potentials $q$ and potentials $q \in L^s(\mathbb{R})$ for some $s \in [1, \infty]$. The bounds depend on the negative part of $q$, on the norm of $1/p$ and in an implicit way on the sign changes and zeros of the weight function.

*Keywords:* Non-real eigenvalue, indefinite Sturm-Liouville operator, Krein space

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## 1. Introduction

In this paper we investigate the non-real spectrum of indefinite singular Sturm-Liouville operators associated to the differential expression

$$\ell = \frac{1}{r}\left(-\frac{d}{dx}\,p\,\frac{d}{dx} + q\right) \tag{1.1}$$

with real coefficients $1/p, q, r \in L^1_{\rm loc}(\mathbb{R})$ satisfying Hypothesis 2.1 below. In particular we suppose that $p$ is positive and $r$ does not vanish almost everywhere on $\mathbb{R}$. We emphasize that the weight function $r$ is assumed to have different signs, more precisely, it is allowed that $r$ has finitely or even infinitely many sign changes within a compact interval. For this reason $\ell$ is called indefinite and the associated Sturm-Liouville operators may exhibit non-real spectrum, see, e.g. [7, 10, 15, 16, 18, 19, 21, 25]. For $1/p$ we assume that it is contained in $L^\eta(\mathbb{R})$ for some $\eta \in [1, \infty]$. For the potential $q$ in (1.1) we only assume uniform local integrability so that, in particular, potentials in $L^s(\mathbb{R})$ for any $s \in [1, \infty]$ are allowed. The assumptions in Hypothesis 2.1 naturally generalize the case $p = 1$, $r = \operatorname{sgn}$ and $q \in L^1(\mathbb{R})$ or $q \in L^\infty(\mathbb{R})$ studied in [4, 5].

The differential operators associated to $\ell$ act in the weighted $L^2$-space $L^2_r(\mathbb{R})$ of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that $f^2 r \in L^1(\mathbb{R})$. Equipped with the usual scalar product

$$(f, g)_r := \int_{\mathbb{R}} f(t)\overline{g(t)}|r(t)|\,\mathrm{d}t, \qquad f, g \in L^2_r(\mathbb{R}), \tag{1.2}$$

Hypothesis 2.1 (b) ensures that $L^2_r(\mathbb{R})$ is a Hilbert space. We are interested in the non-real spectrum of the maximal operator

$$A = \frac{1}{r}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x} + q\right), \qquad \operatorname{dom}(A) = \mathcal{D}_{\max},$$

associated to $\ell$ in $L^2_r(\mathbb{R})$ with

$$\mathcal{D}_{\max} = \left\{ f \in L^2_r(\mathbb{R}) : f, pf' \in \mathcal{AC}(\mathbb{R}),\ \ell f \in L^2_r(\mathbb{R}) \right\},$$

where $\mathcal{AC}(\mathbb{R})$ denotes the space of absolutely continuous functions. Since the weight function $r$ changes its sign the operator $A$ is not symmetric nor self-adjoint with respect to the Hilbert space inner product (1.2) but becomes self-adjoint with respect to the indefinite inner product

$$[f, g]_r := \int_{\mathbb{R}} f(t)\overline{g(t)}r(t)\,\mathrm{d}t, \quad f, g \in L^2_r(\mathbb{R}).$$

Hence, the non-real spectrum of $A$ is symmetric with respect to the real axis, and from Hypothesis 2.1 and perturbation methods we conclude that the non-real spectrum consists of isolated eigenvalues with finite algebraic multiplicity. The properties of $A$ are collected in the following theorem; a self-contained proof using standard techniques in Sturm-Liouville theory is presented in Appendix A.

**Theorem 1.1.** *Suppose Hypothesis* 2.1 *holds. Then the operator $A$ is self-adjoint with respect to the indefinite inner product $[\cdot,\cdot]_r$, the non-real spectrum of $A$ is symmetric with respect to the real axis and consists of isolated eigenvalues with finite algebraic multiplicity.*

The main objective of the present paper is to prove bounds on the absolute values and the imaginary parts of the non-real eigenvalues of $A$. For the case of regular indefinite Sturm-Liouville operators related estimates were obtained in [1, 8, 9, 14, 17, 20]. The more difficult case of singular indefinite Sturm-Liouville operators was so far only studied in very special situations; cf. [4, 5] for $p = 1$, $r = \operatorname{sgn}$, and $q \in L^s(\mathbb{R})$ for $s = 1$ or $s = \infty$ (see also [2, 6]). In contrast to the abovementioned contributions here we impose only rather weak assumptions in Hypothesis 2.1 on the coefficients in (1.1), in particular, we treat weight functions $r$ with finitely or infinitely many sign changes within a compact interval, functions $1/p \in L^\eta(\mathbb{R})$ for $\eta \in [1,\infty]$ and uniformly locally integrable potentials $q$ or $q \in L^s(\mathbb{R})$ for $s \in [1,\infty]$.

All main results are collected in Section 2; their proofs are postponed to Section 4. Theorem 2.2 treats the case $1/p \in L^\infty(\mathbb{R})$ and $q$ is supposed to be uniformly locally integrable; regarding the assumptions on $q$ this is the most general result. Our estimates depend on the norm of the negative part $q_-$ of $q$, the $L^\infty$-norm of $1/p$ and in an implicit form on the sign changes and zeros of the weight function $r$. Under the slightly stronger assumption $q_- \in L^s(\mathbb{R})$ for some $s \in [1,\infty]$ by simultaneously allowing $1/p$ to be in $L^\eta(\mathbb{R})$ for some $\eta \in [1,\infty]$ we obtain similar estimates (except the case $\eta = s = 1$) in terms of the $L^s$-norm of $q_-$ and the $L^\eta$-norm of $1/p$ in Theorem 2.4. In the case $\eta = s = 1$ we find a sufficient condition for the absence of non-real spectrum in Theorem 2.5. The estimates in Theorem 2.2 and in Theorem 2.4 are in some sense implicit as they are expressed in terms of an auxiliary function $g \in \mathcal{AC}(\mathbb{R})$ which neutralizes the behaviour of the weight function $r$. The construction of such a function $g$ is the topic of Section 3. A similar technique was used in [1] for regular indefinite Sturm-Liouville problems.

In the special case that the weight function $r$ has only finitely many sign changes we obtain explicit estimates in Theorem 2.6 which again depend on the norms of $q_-$ and $1/p$, and on the sign changes and zeros of the weight function $r$. These estimates become very simple for $r = \operatorname{sgn}$ and $p = 1$ in Corollary 2.7. If, e.g. $q$ is uniformly locally integrable with $q_- \in L^s(\mathbb{R})$ for $s \in [1,\infty)$ then every non-real eigenvalue $\lambda$ of $A$ satisfies

$$|\operatorname{Im}\lambda| \leq 2^{\frac{2s+1}{2s-1}} \cdot 3 \cdot \sqrt{3}\|q_-\|_s^{\frac{2s}{2s-1}}$$

and

$$|\lambda| \leq \left(2^{\frac{2s+1}{2s-1}} \cdot 3 \cdot \sqrt{3} + 2^{\frac{3-2s}{2s-1}} \cdot 9\right)\|q_-\|_s^{\frac{2s}{2s-1}}.$$

It is remarkable that for $s = 1$ these bounds reduce to those obtained in [5, Theorem 1.3] recently. If $q_- \in L^\infty(\mathbb{R})$ then

$$|\operatorname{Im}\lambda| \leq 6\cdot\sqrt{3}\|q_-\|_\infty \quad \text{and} \quad |\lambda| \leq \left(6\cdot\sqrt{3}+\frac{9}{2}\right)\|q_-\|_\infty. \tag{1.3}$$

We emphasize that, in contrast to the bounds obtained in [4, 5], it is only needed that the negative part $q_-$ is contained in $L^s(\mathbb{R})$ for $s=1$ or $s=\infty$ while $q$ is assumed to be uniformly locally integrable. Moreover, if the negative part $q_-$ is small compared to $q$ the bounds in (1.3) may be stronger than those obtained in [4], e.g. if $\|q_-\|_\infty < 3^{-\frac{3}{2}}\|q\|_\infty$ then the bounds in (1.3) are better than the ones in [4].

## 2. Main results

In this section we state all main results of the paper without proofs. The proofs are presented in Section 4. Our standing assumptions on the functions $r$, $p$, and $q$ are collected in Hypothesis 2.1 below. For this recall that the normed space of uniformly locally integrable functions $L^1_{\mathrm{u}}(\mathbb{R})$ is defined as

$$L^1_{\mathrm{u}}(\mathbb{R}) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \sup_{n\in\mathbb{Z}} \int_n^{n+1} |f(t)|\,\mathrm{d}t < \infty \right\}, \quad \|f\|_{\mathrm{u}} := \sup_{n\in\mathbb{Z}} \int_n^{n+1} |f(t)|\,\mathrm{d}t.$$

Observe, that $L^s(\mathbb{R})$, $s \in [1,\infty]$, is contained in $L^1_{\mathrm{u}}(\mathbb{R})$.

**Hypothesis 2.1.** *The real coefficients $p, q, r$ satisfy the following:*

(a) $q \in L^1_{\mathrm{u}}(\mathbb{R})$*;*

(b) $r \in L^1_{\mathrm{loc}}(\mathbb{R})$ *such that* $r(x) \neq 0$ *for a.a.* $x \in \mathbb{R}$*, and the sets*

$$\Delta_+^r := \{x \in \mathbb{R} : r(x) > 0\} \quad \textit{and} \quad \Delta_-^r := \{x \in \mathbb{R} : r(x) < 0\}$$

*have positive Lebesgue measure;*

(c) *there exist* $a, b \in \mathbb{R}$*,* $a < b$*, such that*

$$C_r := \operatorname*{ess\,inf}_{x\in\mathbb{R}\setminus[a,b]} |r(x)| > 0$$

*and $r$ has constant sign a.e. in $(-\infty, a)$ and constant sign a.e. in $(b,\infty)$;*

(d) $0 < p(x) < \infty$ *for a.a.* $x \in \mathbb{R}$ *and* $1/p \in L^\eta(\mathbb{R})$ *for some* $\eta \in [1,\infty]$*.*

Beside these conditions we require the existence of a locally absolutely continuous real function $g$ which neutralizes the behaviour of the weight $r$ in the sense that $rg \geq \gamma$ holds true for some positive $\gamma$ on a sufficiently large subset of the real line. To make this more precise we use the notation $\{rg < 0\}$ for the set $\{x \in \mathbb{R} : r(x)g(x) < 0\}$; the sets $\{|g| < 1\}$ and $\{|r| < \gamma\}$ are defined analogously. Then the characterisation for a suitable pair $g$ and $\gamma$ is that

$$\omega_{\gamma,g} := \left( \mu\Big( \{|r| < \gamma\} \cup \{|g| < 1\} \cup \{rg < 0\} \Big) + \frac{1}{\gamma} \int_{\{rg<0\}} |r(t)| \,\mathrm{d}t \right)$$

is below a certain bound, where $\mu$ denotes Lebesgue measure. Actually, this is no restriction on the conditions in Hypothesis 2.1 since $\gamma$ and $g$ with the abovementioned properties always exist; cf. Remark 2.3 and Theorem 3.4.

The following theorem is the first main result. It provides bounds on the non-real eigenvalues of $A$ under the general assumption $q \in L^1_{\mathrm{u}}(\mathbb{R})$. Here the weight function $r$ may have infinitely many sign changes within the compact interval $[a, b]$. We decompose the potential $q = q_+ - q_-$ into its positive part $q_+(x) = \max\{0, q(x)\}$ and its negative part $q_-(x) = \max\{0, -q(x)\}$, $x \in \mathbb{R}$. Note that the bounds below do not depend on the positive part $q_+$.

**Theorem 2.2.** *Assume that Hypothesis* 2.1 *holds with* $1/p \in L^\infty(\mathbb{R})$ *and define*

$$\alpha := 2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty \|q_-\|_{\mathrm{u}}^2.$$

*Choose* $\gamma > 0$ *and* $g \in \mathcal{AC}(\mathbb{R})$ *real such that* $\|g\|_\infty = 1$*,* $g'$ *has compact support,* $\sqrt{p}g' \in L^2(\mathbb{R})$ *and*

$$\omega_{\gamma,g} (4\|1/p\|_\infty \alpha)^{1/2} < 1 \tag{2.1}$$

*holds. Then every non-real eigenvalue* $\lambda$ *of* $A$ *satisfies*

$$|\operatorname{Im} \lambda| \leq \frac{\sqrt{2}\|1/p\|_\infty^{\frac{1}{4}} \|\sqrt{p}g'\|_2 \alpha^{\frac{3}{4}}}{\gamma\big(1 - \omega_{\gamma,g}\,(4\|1/p\|_\infty \alpha)^{\frac{1}{2}}\big)} \quad \textit{and} \quad |\lambda| \leq \frac{\sqrt{2}\|1/p\|_\infty^{\frac{1}{4}} \|\sqrt{p}g'\|_2 \alpha^{\frac{3}{4}} + 3\alpha}{\gamma\big(1 - \omega_{\gamma,g}\,(4\|1/p\|_\infty \alpha)^{\frac{1}{2}}\big)}.$$

*Remark* 2.3. The estimates in Theorem 2.2 (and also in Theorem 2.4 below) depend on the choice of the constant $\gamma$ and the function $g$. In Theorem 3.4 for every $\beta > 0$ a constant $\gamma > 0$ and a real function $g \in \mathcal{AC}(\mathbb{R})$ with $\|g\|_\infty = 1$ is constructed such that $g'$ has compact support, $\sqrt{p}g' \in L^2(\mathbb{R})$ and

$$\omega_{\gamma,g}\beta < 1.$$

Hence, for given $r, p, q$ satisfying Hypothesis 2.1, there always exist $\gamma$ and $g$ such that (2.1) holds. The same holds true for the corresponding conditions on $\omega_{\gamma,g}$ in Theorem 2.4.

If, in addition to Hypothesis 2.1, the negative part $q_-$ of the potential belongs to $L^s(\mathbb{R})$ for some $s \in [1, \infty]$ we obtain the following estimates.

**Theorem 2.4.** *Assume that Hypothesis* 2.1 *holds with* $1/p \in L^\eta(\mathbb{R})$ *for some* $\eta \in [1, \infty]$ *and let* $q_- \in L^s(\mathbb{R})$ *for some* $s \in [1, \infty]$*. Let* $\eta + s > 2$ *and define*

$$\beta = \begin{cases} \left( \left(\frac{2\eta - 1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s \right)^{\frac{\eta s}{2\eta s - \eta - s}} & \textit{if } \eta, s \in [1, \infty), \\ (4\|1/p\|_\infty \|q_-\|_s)^{\frac{s}{2s-1}} & \textit{if } \eta = \infty, s \in [1, \infty), \\ (4\|1/p\|_\infty \|q_-\|_\infty)^{\frac{1}{2}} & \textit{if } \eta = \infty, s = \infty. \end{cases} \tag{2.2}$$

*Choose $\gamma > 0$ and $g \in \mathcal{AC}(\mathbb{R})$ real such that $\|g\|_\infty = 1$, $g'$ has compact support, $\sqrt{p}g' \in L^2(\mathbb{R})$ and $\omega_{\gamma,g}\beta < 1$ holds. Then every non-real eigenvalue $\lambda$ of $A$ satisfies the following bounds.*

(i) *If $\eta \in [1,\infty)$ and $s \in [1,\infty)$ with $2 < \eta + s$ or $\eta = \infty$ and $s \in [1,\infty)$ then*

$$|\operatorname{Im}\lambda| \leq \frac{\|q_-\|_s^{\frac{1}{2}}\beta^{\frac{s+1}{2s}}\|\sqrt{p}g'\|_2}{\gamma(1-\omega_{\gamma,g}\beta)} \quad \text{and} \quad |\lambda| \leq \frac{\|q_-\|_s^{\frac{1}{2}}\beta^{\frac{s+1}{2s}}\|\sqrt{p}g'\|_2 + 3\|q_-\|_s\beta^{\frac{1}{s}}}{\gamma(1-\omega_{\gamma,g}\beta)}.$$

(ii) *If $\eta = s = \infty$ then*

$$|\operatorname{Im}\lambda| \leq \frac{\big(\|q_-\|_\infty\beta\big)^{\frac{1}{2}}\|\sqrt{p}g'\|_2}{\gamma(1-\omega_{\gamma,g}\beta)} \quad \text{and} \quad |\lambda| \leq \frac{\big(\|q_-\|_\infty\beta\big)^{\frac{1}{2}}\|\sqrt{p}g'\|_2 + 3\|q_-\|_\infty}{\gamma(1-\omega_{\gamma,g}\beta)}.$$

The case $\eta = s = 1$ is excluded in Theorem 2.4. In this situation, which is slightly different, we can give a sufficient criterion for the non-real spectrum of $A$ to be empty.

**Theorem 2.5.** *Assume that Hypothesis 2.1 holds with $1/p \in L^1(\mathbb{R})$ and $q_- \in L^1(\mathbb{R})$. If, in addition, $\|1/p\|_1\|q_-\|_1 < 1$ then the spectrum of $A$ is real.*

In the above results the weight function $r$ is allowed to have infinitely many sign changes. If we restrict to the case of finitely many sign changes or of one sign change we obtain simpler estimates. In particular, in the estimates the function $g$ does not appear anymore and the parameter $\gamma$ is given explicitly. To formulate the results we first discuss the notion of sign changes or, more precisely, *turning points of $r$*. In [10] turning points of $r$ are defined as the elements of $\overline{\Delta_+^r} \cap \overline{\Delta_-^r}$. Since this definition depends on the representative of $r$ in the equivalence class in $L^1_{\mathrm{loc}}(\mathbb{R})$ we use a slightly different approach. Here turning points of $r$ are elements in

$$\mathcal{T}_r := \big\{x \in \mathbb{R} : \mu(\Delta_+^r \cap I) > 0, \mu(\Delta_-^r \cap I) > 0 \text{ for all open intervals } I \text{ with } x \in I\big\}. \tag{2.3}$$

The set $\mathcal{T}_r$ is a closed subset of $\overline{\Delta_+^r} \cap \overline{\Delta_-^r}$. Under Hypothesis 2.1 the set $\mathcal{T}_r$ is bounded and, thus, compact. Hypothesis 2.1 (b) also ensures that $\mathcal{T}_r \neq \emptyset$. Furthermore, the set $\mathcal{T}_r$ does not depend on the representative of the equivalence class of $r$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Besides $\mathcal{T}_r$, the set of points where $r$ is close to zero also plays an important role. More precisely, define

$$\mathcal{Z}_r := \left\{x \in \mathbb{R} : \operatorname*{ess\,inf}_{y \in I} |r(y)| = 0 \text{ for all open intervals } I \text{ with } x \in I\right\}, \tag{2.4}$$

which is again independent of the representative of $r$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Note that $\mathcal{Z}_r$ and $\{r = 0\}$ in general do not coincide.

**Theorem 2.6.** *Assume that Hypothesis* 2.1 *holds with* $1/p \in L^\eta(\mathbb{R})$ *for some* $\eta \in [1,\infty]$ *and that the set* $\mathcal{T}_r \cup \mathcal{Z}_r$ *contains* $n < \infty$ *elements. For*

$$0 < \delta < \frac{1}{2} \min \left\{ |x - x'| : x, x' \in \mathcal{T}_r,\ x \neq x' \right\} \tag{2.5}$$

*define* $\Omega := \bigcup_{x \in \mathcal{Z}_r} (x - \delta, x + \delta)$, $\gamma := \operatorname{ess\,inf}_{x \in \mathbb{R} \setminus \Omega} |r(x)|$ *and*

$$P := \left( \sum_{x \in \mathcal{T}_r} \left( \int_{x-\delta}^{x} \frac{1}{p(t)}\, \mathrm{d}t \right)^{-1} + \sum_{x \in \mathcal{T}_r} \left( \int_{x}^{x+\delta} \frac{1}{p(t)}\, \mathrm{d}t \right)^{-1} \right)^{\frac{1}{2}}.$$

*Then the following estimates hold for every non-real eigenvalue* $\lambda$ *of* $A$.

(i) *If* $\eta = \infty$ *and* $2\delta n (4\|1/p\|_\infty \alpha)^{1/2} < 1$ *for* $\alpha := 2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty \|q_-\|_{\mathrm{u}}^2$ *then*

$$|\operatorname{Im} \lambda| \leq \frac{\sqrt{2} \|1/p\|_\infty^{\frac{1}{4}} \alpha^{\frac{3}{4}} P}{\gamma \big(1 - 2\delta n \,(4\|1/p\|_\infty \alpha)^{\frac{1}{2}}\big)} \quad \text{and} \quad |\lambda| \leq \frac{\sqrt{2} \|1/p\|_\infty^{\frac{1}{4}} \alpha^{\frac{3}{4}} P + 3\alpha}{\gamma \big(1 - 2\delta n \,(4\|1/p\|_\infty \alpha)^{\frac{1}{2}}\big)}.$$

*Suppose* $q_- \in L^s(\mathbb{R})$ *for some* $s \in [1,\infty]$, *where* $\eta + s > 2$, *and define* $\beta$ *as in* (2.2).

(ii) *If* $\eta \in [1,\infty)$ *and* $s \in [1,\infty)$ *with* $2 < \eta + s$ *or* $\eta = \infty$ *and* $s \in [1,\infty)$, *and* $2\delta n \beta < 1$ *then*

$$|\operatorname{Im} \lambda| \leq \frac{\|q_-\|_s^{\frac{1}{2}} \beta^{\frac{s+1}{2s}} P}{\gamma (1 - 2\delta n \beta)} \quad \text{and} \quad |\lambda| \leq \frac{\|q_-\|_s^{\frac{1}{2}} \beta^{\frac{s+1}{2s}} P + 3\|q_-\|_s \beta^{\frac{1}{s}}}{\gamma (1 - 2\delta n \beta)}.$$

(iii) *If* $s = \eta = \infty$ *and* $2\delta n \beta < 1$ *then*

$$|\operatorname{Im} \lambda| \leq \frac{\big(\|q_-\|_\infty \beta\big)^{\frac{1}{2}} P}{\gamma (1 - 2\delta n \beta)} \quad \text{and} \quad |\lambda| \leq \frac{\big(\|q_-\|_\infty \beta\big)^{\frac{1}{2}} P + 3\|q_-\|_\infty}{\gamma (1 - 2\delta n \beta)}.$$

If $\mathcal{T}_r$ is a singleton then the set on the right hand side of (2.5) is empty and, hence, $\delta$ in (2.5) can be choosen arbitrarily large. This is the case in the next corollary, where we apply the previous theorem to $r = \operatorname{sgn}$ and $p = 1$. The bounds in item (ii) for the special case $s = 1$ coincide with those obtained in [5].

**Corollary 2.7.** *Let* $p = 1$, $r = \operatorname{sgn}$ *and* $q \in L^1_{\mathrm{u}}(\mathbb{R})$. *Then the following estimates hold for every non-real eigenvalue* $\lambda$ *of* $A$.

(i) *One has*

$$|\operatorname{Im} \lambda| \leq 12 \cdot \sqrt{3} \left( \|q_-\|_{\mathrm{u}} + 2\|q_-\|_{\mathrm{u}}^2 \right)$$

*and*

$$|\lambda| \leq (12 \cdot \sqrt{3} + 9) \left( \|q_-\|_{\mathrm{u}} + 2\|q_-\|_{\mathrm{u}}^2 \right).$$

(ii) *If* $q_- \in L^s(\mathbb{R})$ *for* $s \in [1,\infty)$ *then*

$$|\operatorname{Im}\lambda| \leq 2^{\frac{2s+1}{2s-1}} \cdot 3 \cdot \sqrt{3}\|q_-\|_s^{\frac{2s}{2s-1}}$$

*and*

$$|\lambda| \leq \big(2^{\frac{2s+1}{2s-1}} \cdot 3 \cdot \sqrt{3} + 2^{\frac{3-2s}{2s-1}} \cdot 9\big)\|q_-\|_s^{\frac{2s}{2s-1}}.$$

*In particular, if* $s = 1$ *then*

$$|\operatorname{Im}\lambda| \leq 24 \cdot \sqrt{3}\|q_-\|_1^2 \quad \textit{and} \quad |\lambda| \leq \big(24 \cdot \sqrt{3} + 18\big)\|q_-\|_1^2.$$

(iii) *If* $q_- \in L^\infty(\mathbb{R})$ *then*

$$|\operatorname{Im}\lambda| \leq 6 \cdot \sqrt{3}\|q_-\|_\infty \quad \textit{and} \quad |\lambda| \leq \left(6 \cdot \sqrt{3} + \frac{9}{2}\right)\|q_-\|_\infty.$$

## 3. The parameter $\gamma$ and the function $g$

In this section we discuss the choice of the parameters $\gamma$ and $g$ in Theorems 2.2 and 2.4. In particular, it will turn out in Theorem 3.4 that there always exist a function $g$ and a constant $\gamma$ as required in the assumptions of the Theorem 2.2 and Theorem 2.4.

In the next lemma it is shown that in the case of a finite set $\mathcal{T}_r$ one can choose a representative $r \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $\mathcal{T}_r = \overline{\Delta^r_+} \cap \overline{\Delta^r_-}$.

**Lemma 3.1.** *Let* $r$ *be as in Hypothesis* 2.1 *and assume that* $\mathcal{T}_r$ *is finite. Then there exists a function* $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ *with* $w = r$ *a.e. such that the disjoint sets*

$$\Delta^w_+ := \{x \in \mathbb{R} : w(x) > 0\} \quad \textit{and} \quad \Delta^w_- := \{x \in \mathbb{R} : w(x) < 0\}$$

*are finite unions of disjoint open intervals and for the boundaries* $\partial\Delta^w_\pm$ *of* $\Delta^w_\pm$ *we have*

$$\partial\Delta^w_+ = \partial\Delta^w_- = \overline{\Delta^w_+} \cap \overline{\Delta^w_-} = \{w = 0\} = \mathcal{T}_r. \tag{3.1}$$

*Proof.* Let $\mathcal{F}_\pm$ be the family of all open intervals $I$ such that $\mu(I \cap \Delta^r_\pm) > 0$ and $\mu(I \cap \Delta^r_\mp) = 0$, and consider the open sets

$$\Upsilon_+ = \bigcup_{I \in \mathcal{F}_+} I \quad \text{and} \quad \Upsilon_- = \bigcup_{I \in \mathcal{F}_-} I. \tag{3.2}$$

For the union in (3.2) it suffices to consider open intervals $I$ with rational endpoints, and hence $\Upsilon_+$ and $\Upsilon_-$ can be viewed as unions of countable many open intervals. Together with the $\sigma$-subadditivity of the Lebesgue measure this implies

$$\mu(\Upsilon_+ \cap \Delta^r_-) = 0 \quad \text{and} \quad \mu(\Upsilon_- \cap \Delta^r_+) = 0. \tag{3.3}$$

Moreover, the sets $\Upsilon_+$, $\Upsilon_-$ and $\mathcal{T}_r$ are disjoint by definition. Consider an arbitrary $x \in \mathbb{R}$ and an open interval $I$ containing $x$. The real line is the disjoint union of $\Delta_+^r$, $\Delta_-^r$ and $\{r = 0\}$ where the latter has Lebesgue measure zero. At least one of the sets $I \cap \Delta_+^r$ and $I \cap \Delta_-^r$ has positive Lebesgue measure and $x$ is contained either in $\Upsilon_+$, $\Upsilon_-$ or $\mathcal{T}_r$. This shows

$$\mathbb{R} = \Upsilon_+ \cup \Upsilon_- \cup \mathcal{T}_r, \tag{3.4}$$

where the union is disjoint.

Another helpful observation is the following. Consider an open nonempty interval $I$ with $I \cap \mathcal{T}_r = \emptyset$. The interval $I$ is a connected set. On the other hand since $I \cap \mathcal{T}_r = \emptyset$ it can be represented as the disjoint union of the open sets $I \cap \Upsilon_+$ and $I \cap \Upsilon_-$. Thus, either $I \cap \Upsilon_+ = \emptyset$ and $I \subset \Upsilon_-$ or $I \cap \Upsilon_- = \emptyset$ and $I \subset \Upsilon_+$. Since $\Upsilon_+$ and $\Upsilon_-$ are open and disjoint we have

$$\partial\Upsilon_+ \cap \Upsilon_+ = \emptyset, \quad \partial\Upsilon_- \cap \Upsilon_- = \emptyset, \quad \partial\Upsilon_+ \cap \Upsilon_- = \emptyset, \quad \partial\Upsilon_- \cap \Upsilon_+ = \emptyset. \tag{3.5}$$

Here, (3.4) implies $\partial\Upsilon_+ \subset \mathcal{T}_r$ and $\partial\Upsilon_- \subset \mathcal{T}_r$. Consider $x \in \mathcal{T}_r$. Since $\mathcal{T}_r$ is finite there exists an nonempty open interval $(a, b)$ with $(a, b) \cap \mathcal{T}_r = \{x\}$. Then $(a, x) \cap \mathcal{T}_r$ is empty and, by the consideration above, either $(a, x) \subset \Upsilon_+$ or $(a, x) \subset \Upsilon_-$. Hence, by (3.3), $(x, b) \subset \Upsilon_-$ or $(x, b) \subset \Upsilon_+$ respectively. This shows $x \in \partial\Upsilon_+ \cap \partial\Upsilon_-$ and, therefore, $\mathcal{T}_r \subset \partial\Upsilon_+ \cap \partial\Upsilon_-$. From (3.5) we obtain $\partial\Upsilon_+ = \partial\Upsilon_- = \mathcal{T}_r$ and by (3.5) also $\overline{\Upsilon_+} \cap \overline{\Upsilon_-} = \mathcal{T}_r$. As $\mathcal{T}_r$ is finite the sets $\Upsilon_+$ and $\Upsilon_-$ consist of finitely many disjoint open intervals.

We define

$$w(x) = \begin{cases} 1 & \text{if } x \in \Upsilon_+ \setminus \Delta_+^r, \\ -1 & \text{if } x \in \Upsilon_- \setminus \Delta_-^r, \\ r(x) & \text{if } x \in (\Upsilon_+ \cap \Delta_+^r) \cup (\Upsilon_- \cap \Delta_-^r), \\ 0 & \text{if } x \in \mathcal{T}_r. \end{cases}$$

Then $\Delta_+^w = \{w > 0\} = \Upsilon_+$ and $\Delta_-^w = \{w < 0\} = \Upsilon_-$ consist of finitely many open disjoint intervals and we have $\mathcal{T}_r = \{w = 0\}$. Since $\mathcal{T}_r$ has Lebesgue measure zero as well as $\Upsilon_+ \setminus \Delta_+^r = \Upsilon_+ \cap (\Delta_-^r \cup \{r = 0\})$ and $\Upsilon_- \setminus \Delta_-^r = \Upsilon_- \cap (\Delta_+^r \cup \{r = 0\})$, see (3.3), we have $w = r$ a.e. Finally, the properties in (3.1) hold by construction of the sets $\Delta_\pm^w$. $\square$

**Lemma 3.2.** *Let $r$ be as in Hypothesis 2.1, assume that $\mathcal{T}_r$ is finite, and let*

$$0 < \delta < \frac{1}{2} \min\left\{ |x - x'| : x, x' \in \mathcal{T}_r,\ x \neq x' \right\}.$$

*Then there exists $g \in \mathcal{AC}(\mathbb{R})$ real with $rg > 0$ a.e. and $\|g\|_\infty = 1$ such that $g'$ has compact support and $\sqrt{p}g' \in L^2(\mathbb{R})$, where*

$$\|\sqrt{p}g'\|_2^2 = \sum_{x \in \mathcal{T}_r} \left( \int_{x-\delta}^{x} \frac{1}{p(t)}\,\mathrm{d}t \right)^{-1} + \sum_{x \in \mathcal{T}_r} \left( \int_{x}^{x+\delta} \frac{1}{p(t)}\,\mathrm{d}t \right)^{-1},$$

*and*

$$\{|g| < 1\} = \bigcup_{x \in \mathcal{T}_r} (x - \delta, x + \delta). \tag{3.6}$$

*Proof.* By Lemma 3.1 we can assume without loss of generality that the sets $\{r > 0\}$ and $\{r < 0\}$ consist of finitely many disjoint open intervals, where the boundaries of $\{r > 0\}$ and $\{r < 0\}$ equal $\{r = 0\} = \mathcal{T}_r$. Then the function $x \mapsto \operatorname{sgn} r(x)$ is piecewise constant with finitely many discontinuities in $\mathcal{T}_r$. Let

$$g(x) := \operatorname{sgn}(r(x)) \begin{cases} \left(\int_y^x \frac{1}{p(t)}\,\mathrm{d}t\right)\left(\int_y^{y+\delta} \frac{1}{p(t)}\,\mathrm{d}t\right)^{-1} & \text{if } x \in [y, y+\delta),\ y \in \mathcal{T}_r, \\ \left(\int_x^y \frac{1}{p(t)}\,\mathrm{d}t\right)\left(\int_{y-\delta}^{y} \frac{1}{p(t)}\,\mathrm{d}t\right)^{-1} & \text{if } x \in (y-\delta, y),\ y \in \mathcal{T}_r, \\ 1 & \text{otherwise.} \end{cases}$$

Then $g \in \mathcal{AC}(\mathbb{R})$, $\|g\|_\infty = 1$, and (3.6) holds. Further,

$$\begin{aligned} \|\sqrt{p}g'\|_2^2 &= \sum_{y \in \mathcal{T}_r} \left( \int_{y-\delta}^{y} p(t)|g'(t)|^2\,\mathrm{d}t + \int_y^{y+\delta} p(t)|g'(t)|^2\,\mathrm{d}t \right) \\ &= \sum_{y \in \mathcal{T}_r} \left( \int_{y-\delta}^{y} \frac{1}{p(t)}\,\mathrm{d}t \right)^{-1} + \sum_{y \in \mathcal{T}_r} \left( \int_y^{y+\delta} \frac{1}{p(t)}\,\mathrm{d}t \right)^{-1} < \infty \end{aligned}$$

since $1/p \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $1/p > 0$ a.e. Since $\mathcal{T}_r$ is finite the function $g$ is constant near $\infty$ and $-\infty$. Hence, $g'$ has compact support. Moreover, since $\{g > 0\} = \{r > 0\}$ and $\{g < 0\} = \{r < 0\}$ the product $rg$ is positive a.e. $\square$

**Lemma 3.3.** *Let $r$ be as in Hypothesis* 2.1 *and let $\mathcal{Z}_r$ be as in* (2.4)*. For every $\delta > 0$ and $\Omega = \bigcup_{x \in \mathcal{Z}_r} (x - \delta, x + \delta)$ we have*

$$\operatorname{ess\,inf}_{x \in \mathbb{R} \setminus \Omega} |r(x)| > 0.$$

*Proof.* Let $[a, b] \subset \mathbb{R}$ and $C_r = \operatorname{ess\,inf}_{x \in \mathbb{R} \setminus [a,b]} |r(x)| > 0$ be as in Hypothesis 2.1 and consider the open set $\Omega = \bigcup_{x \in \mathcal{Z}_r} (x - \delta, x + \delta)$. By the definition of $\mathcal{Z}_r$ in (2.4) there exists for every $x \notin \mathcal{Z}_r$ an open interval $I_x$ containing $x$ such that $c_x := \operatorname{ess\,inf}_{y \in I_x} |r(y)| > 0$. Since $[a, b] \setminus \Omega$ is compact and

$$\big([a, b] \setminus \Omega\big) \subset \big(\mathbb{R} \setminus \mathcal{Z}_r\big) \subset \bigcup_{x \notin \mathcal{Z}_r} I_x$$

we find $x_1, \ldots, x_m \notin \mathcal{Z}_r$, $m \in \mathbb{N}$, such that $[a, b] \setminus \Omega \subset \bigcup_{k=1}^m I_{x_k}$. Thus, since $(\mathbb{R} \setminus \Omega) \subset (\mathbb{R} \setminus [a, b]) \cup ([a, b] \setminus \Omega)$ we have

$$\operatorname*{ess\,inf}_{x \in \mathbb{R} \setminus \Omega} |r(x)| \geq \min\{C_r, c_{x_1}, \ldots, c_{x_m}\} > 0.$$

$\square$

Finally we show that there always exist a function $g$ and a constant $\gamma$ as required in the assumptions of Theorem 2.2 and Theorem 2.4.

**Theorem 3.4.** *Assume that Hypothesis* 2.1 *holds. Then for every $\beta > 0$ there exist $\gamma > 0$ and $g \in \mathcal{AC}(\mathbb{R})$ real with $\|g\|_\infty = 1$ such that $g'$ has compact support, $\sqrt{p}g' \in L^2(\mathbb{R})$ and $\omega_{\gamma,g}\beta < 1$ holds with*

$$\omega_{\gamma,g} := \left(\mu\Big(\{|r| < \gamma\} \cup \{|g| < 1\} \cup \{rg < 0\}\Big) + \frac{1}{\gamma}\int_{\{rg<0\}} |r(t)|\,\mathrm{d}t\right).$$

*Proof.* Fix $\beta > 0$ and consider the interval $[a, b] \subset \mathbb{R}$ from Hypothesis 2.1. Since $\lim_{n\to\infty} \mu(\{|r| < \frac{1}{n}\}) = \mu(\{r = 0\}) = 0$ there exists $\gamma > 0$ such that

$$\mu(\{|r| < \gamma\}) \leq \frac{1}{4\beta}. \tag{3.7}$$

Consider the compact set $\mathcal{T}_r \subset [a, b]$, see (2.3), and let $a_0 := \min \mathcal{T}_r$ and $b_0 := \max \mathcal{T}_r$. The set $\varOmega_+ := \varDelta_+^r \cap [a_0, b_0]$ has finite Lebesgue measure. By [22, Part One, Chapter 3, Proposition 15] for $\varepsilon > 0$ there is a finite union $\Omega_\varepsilon$ of bounded open intervals such that $\mu(\varOmega_+\Delta\varOmega_\varepsilon) < \varepsilon$, where $\Delta$ denotes the symmetric difference of two sets. Hence the characteristic functions $\mathbb{1}_{\varOmega_+\Delta\varOmega_\varepsilon}$ tend to zero in measure for $\varepsilon \to 0$ and we can choose a sequence $\varOmega_m$ such that $\mu(\varOmega_+\Delta\varOmega_m) \to 0$ and $\mathbb{1}_{\varOmega_+\Delta\varOmega_m}$ converge a.e. to zero for $m \to \infty$. Dominated convergence then implies

$$\lim_{m\to\infty} \int_{\varOmega_+\Delta\varOmega_m} |r(t)|\,\mathrm{d}t = 0.$$

Hence, there exist $N \in \mathbb{N}$ and intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_N, b_N)$ such that for $\varOmega := \bigcup_{k=1}^N (a_k, b_k)$

$$\mu\left(\varOmega_+\Delta\varOmega\right) \leq \frac{1}{4\beta} \quad \text{and} \quad \int_{\varOmega_+\Delta\varOmega} |r(t)|\,\mathrm{d}t \leq \frac{\gamma}{4\beta}. \tag{3.8}$$

We have $\varOmega_+ \subset [a_0, b_0]$ but it may happen that $\varOmega \not\subseteq [a_0, b_0]$. In the latter case we replace $\varOmega$ by $\varOmega \cap (a_0, b_0)$. It is clear that for this modified set $\varOmega$ (3.8) still holds. Therefore, without loss of generality, we may assume that the intervals $(a_k, b_k)$, $k = 1, \ldots, N$, are disjoint and ordered in the way that $b_k < a_{k+1}$, $k = 1, \ldots, N-1$ with $(a_k, b_k) \subset [a_0, b_0]$. In particular, $\varOmega \subset [a_0, b_0]$. Define

$$\tilde{r}(x) = \begin{cases} 1 & \text{if } x \in \varOmega, \\ -1 & \text{if } x \in [a_0, b_0] \setminus \varOmega, \\ \operatorname{sgn}(r(x)) & \text{if } x \in (-\infty, a_0) \cup (b_0, \infty). \end{cases}$$

Since $\mathcal{T}_r \subset [a_0, b_0]$ and the signs of $r$ and $\tilde{r}$ coincide outside of $[a_0, b_0]$ we have $\mathcal{T}_{\tilde{r}} \subset [a_0, b_0]$. More precisely, $\mathcal{T}_{\tilde{r}} \subset \{a_0, b_0, a_1, b_1, \ldots, a_N, b_N\}$. Further, from $\varOmega \subset [a_0, b_0]$ and $\{r\tilde{r} < 0\} \subset [a_0, b_0]$ we obtain

$$\{r\tilde{r} < 0\} = \big((\varDelta_+^r \cap [a_0, b_0]) \setminus \varOmega\big) \cup \big(\varDelta_-^r \cap [a_0, b_0] \cap \varOmega\big) \subset \varOmega_+\Delta\varOmega, \tag{3.9}$$

where it was used that $\Omega_+ \setminus \Omega = (\Delta_+^r \cap [a_0, b_0]) \setminus \Omega$ and

$$\Omega \setminus \Omega_+ = \Omega \setminus \Delta_+^r \supset \Omega \setminus (\Delta_+^r \cup \{r = 0\}) = \Omega \cap \Delta_-^r = \Delta_-^r \cap [a_0, b_0] \cap \Omega.$$

Hence, (3.8) together with (3.9) implies

$$\mu\big(\{r\tilde{r} < 0\}\big) \le \frac{1}{4\beta} \quad \text{and} \quad \int_{\{r\tilde{r}<0\}} |r(t)|\,\mathrm{d}t \le \frac{\gamma}{4\beta}. \tag{3.10}$$

Observe, that $\mathcal{T}_{\tilde{r}}$ consists of at most $2(N+1)$ elements. Choose $\delta > 0$ such that

$$\delta < \frac{1}{2}\min\Big\{|x - x'| : x, x' \in \mathcal{T}_{\tilde{r}},\, x \ne x'\Big\}, \quad 4\delta(N+1) < \frac{1}{4\beta}.$$

Lemma 3.2 provides a real function $g \in \mathcal{AC}(\mathbb{R})$, such that $\tilde{r}g > 0$ a.e., $\|g\|_\infty = 1$, $\mu(\{|g| < 1\}) = 2\delta \cdot 2(N+1) < \frac{1}{4\beta}$. Further $g'$ has compact support with $\sqrt{p}g' \in L^2(\mathbb{R})$. Since $\tilde{r}g > 0$ a.e. (3.10) implies

$$\mu\left(\{rg < 0\}\right) \le \frac{1}{4\beta} \quad \text{and} \quad \int_{\{rg<0\}} |r(t)|\,\mathrm{d}t \le \frac{\gamma}{4\beta}.$$

Together with (3.7) this yields $\omega_{\gamma,g} < \frac{1}{\beta}$. □

## 4. Proof of Theorem 2.2, Theorem 2.4, Theorem 2.5, and Theorem 2.6

In this section we prove the theorems in Section 2 on the absolute values and imaginary parts of the non-real eigenvalues of $A$. We first collect some useful estimates for functions contained in $\operatorname{dom}(A) = \mathcal{D}_{\max}$ in the next lemma. Observe also that $\sqrt{p}f' \in L^2(\mathbb{R})$ by Lemma A.2 (i).

**Lemma 4.1.** *Suppose Hypothesis* 2.1 *holds with* $1/p \in L^\eta(\mathbb{R})$ *for* $\eta \in [1, \infty]$ *and let* $f \in \mathcal{D}_{\max}$.

(i) *If* $\eta \in [1, \infty)$ *then*

$$\|f\|_\infty \le \left(\frac{2\eta - 1}{\eta}\sqrt{\|1/p\|_\eta}\|\sqrt{p}f'\|_2\right)^{\frac{\eta}{2\eta-1}} \|f\|_2^{\frac{\eta-1}{2\eta-1}}. \tag{4.1}$$

(ii) *If* $\eta = \infty$ *then*

$$\|f\|_\infty \le \left(2\sqrt{\|1/p\|_\infty}\|\sqrt{p}f'\|_2\|f\|_2\right)^{\frac{1}{2}}. \tag{4.2}$$

*Proof.* Let $f \in \mathcal{D}_{\max}$ and $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $y_n \to -\infty$ and $f(y_n) \to 0$ as $n \to \infty$; cf. Lemma A.2 (ii). First, consider the case $\eta \in [1, \infty)$. Define $\theta := \frac{2\eta-1}{\eta}$. For arbitrary $x \in \mathbb{R}$ we obtain

$$|f(x)|^\theta \le |f(y_n)|^\theta + \theta \int_{y_n}^x |f(t)|^{\theta-1}|f'(t)|\,\mathrm{d}t$$

and, thus,

$$\|f\|_\infty^\theta \le \theta \int_{\mathbb{R}} |f(t)|^{\theta-1}|f'(t)|\,\mathrm{d}t. \tag{4.3}$$

The integral in (4.3) can be further estimated by means of the Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}} |f(t)|^{\theta-1}|f'(t)|\,\mathrm{d}t &\le \|\sqrt{p}f'\|_2 \left(\int_{\mathbb{R}} \frac{|f(t)|^{2(\theta-1)}}{p(t)}\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\le \|\sqrt{p}f'\|_2 \sqrt{\|1/p\|_\eta} \left(\int_{\mathbb{R}} |f(t)|^{\frac{2(\theta-1)\eta}{\eta-1}}\,\mathrm{d}t\right)^{\frac{\eta-1}{2\eta}} \\ &\le \|\sqrt{p}f'\|_2 \sqrt{\|1/p\|_\eta} \|f\|_2^{\frac{\eta-1}{\eta}}.\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) leads to (4.1).

If $s = \infty$ for arbitrary $x \in \mathbb{R}$ we obtain

$$|f(x)|^2 = |f(y_n)|^2 + 2\int_{y_n}^x \overline{f(t)}f'(t)\,\mathrm{d}t$$

and, therefore,

$$\|f\|_\infty^2 \le 2\left(\int_{\mathbb{R}} p(t)|f'(t)|^2\,\mathrm{d}t \int_{\mathbb{R}} \frac{|f(t)|^2}{p(t)}\,\mathrm{d}t\right)^{\frac{1}{2}} \le 2\sqrt{\|1/p\|_\infty}\|\sqrt{p}f'\|_2\|f\|_2.$$

This shows (4.2). □

In the following for an eigenfunction $f$ of $A$ we consider

$$U(x) := \int_x^\infty r(t)|f(t)|^2\,\mathrm{d}t \quad \text{and} \quad V(x) := \int_x^\infty \Big(p(t)|f'(t)|^2 + q(t)|f(t)|^2\Big)\,\mathrm{d}t \tag{4.5}$$

for $x \in \mathbb{R}$. Recall that $\sqrt{p}f' \in L^2(\mathbb{R})$ and $qf^2 \in L^1(\mathbb{R})$ by Lemma A.2 (i). Hence, both functions $U$ and $V$ are well-defined on $\mathbb{R}$, real and absolutely continuous.

**Lemma 4.2.** *Suppose Hypothesis* 2.1 *holds. Let $f$ be an eigenfunction of $A$ corresponding to a non-real eigenvalue $\lambda$. Then $\lambda U(x) = (pf')(x)\overline{f(x)} + V(x)$ for all $x \in \mathbb{R}$ and*

$$\lim_{|x|\to\infty} U(x) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} V(x) = 0 \tag{4.6}$$

*hold. In particular,*

$$\|\sqrt{p}f'\|_2^2 \le \|q_- f^2\|_1. \tag{4.7}$$

*Proof.* We multiply the identity $Af = \lambda f$ by $r\overline{f}$ and integrate by parts. This together with Lemma A.2 (iii) yields

$$\lambda U(x) = \int_x^\infty -(pf')'(t)\overline{f(t)}\,\mathrm{d}t + \int_x^\infty q(t)|f(t)|^2\,\mathrm{d}t = (pf')(x)\overline{f(x)} + V(x)$$

for all $x \in \mathbb{R}$. Again from Lemma A.2 (iii) it follows that

$$\begin{aligned}\lambda \lim_{x\to-\infty} U(x) &= \lambda \int_{\mathbb{R}} r(t)|f(t)|^2\,\mathrm{d}t = \int_{\mathbb{R}} \big(p(t)|f'(t)|^2 + q(t)|f(t)|^2\big)\mathrm{d}t \\ &= \lim_{x\to-\infty} V(x)\end{aligned}$$

and by comparing the imaginary parts we obtain (4.6) since $\lambda \in \mathbb{C}\setminus\mathbb{R}$. For the estimate (4.7) note that $\lim_{x\to-\infty} V(x) = 0$ implies

$$\|\sqrt{p}f'\|_2^2 = \int_{\mathbb{R}} p(t)|f'(t)|^2\,\mathrm{d}t = -\int_{\mathbb{R}} q(t)|f(t)|^2\,\mathrm{d}t \le \int_{\mathbb{R}} q_-(t)|f(t)|^2\,\mathrm{d}t = \|q_- f^2\|_1.$$

□

**Lemma 4.3.** *Suppose Hypothesis* 2.1 *holds. Let $f$ be an eigenfunction of $A$ corresponding to a non-real eigenvalue $\lambda$ and assume that there exist $\alpha > 0$, $\beta > 0$ (not depending on $f$ and $\lambda$) such that*

$$\|q_- f^2\|_1 \le \alpha\|f\|_2^2 \quad \text{and} \quad \|f\|_\infty^2 \le \beta\|f\|_2^2. \tag{4.8}$$

*Furthermore, choose $\gamma > 0$ and $g \in \mathcal{AC}(\mathbb{R})$ real with $\|g\|_\infty = 1$ in such a way that $g'$ has compact support, $\sqrt{p}g' \in L^2(\mathbb{R})$ and*

$$\omega_{\gamma,g} = \left(\mu\Big(\{|r| < \gamma\} \cup \{|g| < 1\} \cup \{rg < 0\}\Big) + \frac{1}{\gamma}\int_{\{rg<0\}} |r(t)|\,\mathrm{d}t\right)$$

*satisfies $\omega_{\gamma,g}\beta < 1$. Then*

$$|\operatorname{Im}\lambda| \le \frac{\sqrt{\alpha\beta}\|\sqrt{p}g'\|_2}{\gamma\,(1-\omega_{\gamma,g}\beta)} \quad \text{and} \quad |\lambda| \le \frac{\sqrt{\alpha\beta}\|\sqrt{p}g'\|_2 + 3\alpha}{\gamma\,(1-\omega_{\gamma,g}\beta)}. \tag{4.9}$$

We mention that constants $\gamma$ and functions $g$ with the properties mentioned in Lemma 4.3 always exist; cf. Theorem 3.4.

*Proof.* Let $U$ and $V$ be as in (4.5). As $\lim_{x\to-\infty} V(x) = 0$ by Lemma 4.2 we have

$$\begin{aligned}\|\sqrt{p}f'\|_2^2 &= -\int_{\mathbb{R}} q(t)|f(t)|^2\,\mathrm{d}t = -\int_{\mathbb{R}} (q_+(t) - q_-(t))|f(t)|^2\,\mathrm{d}t \\ &\le \|q_- f^2\|_1 \le \alpha\|f\|_2^2.\end{aligned} \tag{4.10}$$

As $\|\sqrt{p}f'\|_2^2 \geq 0$ we conclude from (4.10) $\|q_+ f^2\|_1 \leq \|q_- f^2\|_1$ and, thus,

$$\int_{\mathbb{R}} |q(t)||f(t)|^2 \,\mathrm{d}t = \int_{\mathbb{R}} (q_+(t) + q_-(t))|f(t)|^2 \,\mathrm{d}t \leq 2\int_{\mathbb{R}} q_-(t)|f(t)|^2 \,\mathrm{d}t \leq 2\alpha\|f\|_2^2. \tag{4.11}$$

As a consequence of Lemma 4.2 the identity

$$\lambda \int_{\mathbb{R}} g'(x)U(x)\mathrm{d}x = \int_{\mathbb{R}} g'(x)(pf')(x)\overline{f(x)}\,\mathrm{d}x + \int_{\mathbb{R}} g'(x)V(x)\,\mathrm{d}x \tag{4.12}$$

holds, where the compact support of $g'$ guarantees the existence of the integrals. We estimate the first integral on the right hand side of (4.12) by

$$\left|\int_{\mathbb{R}} g'(x)\overline{f(x)}(pf')(x)\,\mathrm{d}x\right| \leq \|f\|_\infty \|\sqrt{p}g'\|_2 \|\sqrt{p}f'\|_2 \leq \sqrt{\alpha\beta}\|\sqrt{p}g'\|_2\|f\|_2^2, \tag{4.13}$$

where we have used (4.8) and (4.10). For the second term in (4.12) integration by parts together with $\lim_{|x|\to\infty} V(x) = 0$ and the inequalities (4.11), (4.10) yields

$$\begin{aligned}\left|\int_{\mathbb{R}} g'(x)V(x)\,\mathrm{d}x\right| &= \left|-\int_{\mathbb{R}} g(x)V'(x)\,\mathrm{d}x\right| \\ &\leq \|g\|_\infty \int_{\mathbb{R}} \left(p(x)|f'(x)|^2 + |q(x)||f(x)|^2\right)\,\mathrm{d}x \leq 3\alpha\|f\|_2^2.\end{aligned} \tag{4.14}$$

We want to find a lower bound for the left hand side in (4.12). The notation $\Gamma := \{|r| < \gamma\} \cup \{|g| < 1\}$ will be useful here. From integration by parts and $\lim_{|x|\to\infty} U(x) = 0$ we obtain

$$\begin{aligned}\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x &= -\int_{\mathbb{R}} g(x)U'(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)r(x)|f(x)|^2\,\mathrm{d}x \\ &= \int_{\{rg<0\}} g(x)r(x)|f(x)|^2\,\mathrm{d}x + \int_{\{rg<0\}^c} g(x)r(x)|f(x)|^2\,\mathrm{d}x.\end{aligned} \tag{4.15}$$

For the first term on the right hand side we have with (4.8)

$$\begin{aligned}\int_{\{rg<0\}} g(x)r(x)|f(x)|^2\,\mathrm{d}x &\geq -\|g\|_\infty \int_{\{rg<0\}} |r(x)||f(x)|^2\,\mathrm{d}x \\ &\geq -\|f\|_\infty^2 \int_{\{rg<0\}} |r(x)|\,\mathrm{d}x \\ &\geq -\beta\|f\|_2^2 \int_{\{rg<0\}} |r(x)|\,\mathrm{d}x.\end{aligned}$$

As $g(x)r(x) \geq \gamma$ for all $x \in \{rg < 0\}^{\mathrm{c}} \cap \Gamma^{\mathrm{c}}$ we obtain

$$
\begin{aligned}
\int_{\{rg<0\}^{\mathrm{c}}} g(x)r(x)|f(x)|^2 \,\mathrm{d}x &\geq \int_{\{rg<0\}^{\mathrm{c}}\cap\Gamma^{\mathrm{c}}} g(x)r(x)|f(x)|^2 \,\mathrm{d}x \\
&\geq \gamma \int_{\{rg<0\}^{\mathrm{c}}\cap\Gamma^{\mathrm{c}}} |f(x)|^2 \,\mathrm{d}x \\
&= \gamma \left( \|f\|_2^2 - \int_{\{rg<0\}\cup\Gamma} |f(x)|^2 \,\mathrm{d}x \right) \qquad (4.16)\\
&\geq \gamma\big(\|f\|_2^2 - \mu(\{rg < 0\} \cup \Gamma)\|f\|_\infty^2\big) \\
&\geq \gamma\big(1 - \mu(\{rg < 0\} \cup \Gamma)\beta\big)\|f\|_2^2,
\end{aligned}
$$

where we used again (4.8). From (4.15)–(4.16) it follows

$$
\begin{aligned}
\int_{\mathbb{R}} g'(x)U(x) \,\mathrm{d}x &\geq \left( \gamma\big(1 - \mu(\{rg < 0\} \cup \Gamma)\beta\big) - \beta \int_{\{rg<0\}} |r(x)| \,\mathrm{d}x \right) \|f\|_2^2 \\
&= \gamma(1 - \omega_{\gamma,g}\beta)\|f\|_2^2 > 0.
\end{aligned} \qquad (4.17)
$$

We compare the imaginary parts in (4.12) and apply (4.13), (4.17), and Lemma 4.2. Consequently,

$$
\begin{aligned}
|\operatorname{Im} \lambda|\gamma\,(1 - \omega_{\gamma,g}\beta)\,\|f\|_2^2 &\leq \left| \operatorname{Im} \left( \lambda \int_{\mathbb{R}} g'(x)U(x)\mathrm{d}x \right) \right| \\
&= \left| \operatorname{Im} \left( \int_{\mathbb{R}} g'(x)(pf')(x)\overline{f(x)} \,\mathrm{d}x \right) \right| \\
&\leq \sqrt{\alpha\beta}\|\sqrt{p}g'\|_2\|f\|_2^2,
\end{aligned}
$$

which proves the first estimate in (4.9). We compare both sides in (4.12) with respect to the absolute value. Then by (4.13), (4.14), (4.17), and Lemma 4.2 we obtain

$$
\begin{aligned}
|\lambda|\gamma\,(1 - \omega_{\gamma,g}\beta)\,\|f\|_2^2 &\leq \left| \left( \lambda \int_{\mathbb{R}} g'(x)U(x)\mathrm{d}x \right) \right| \\
&= \left| \int_{\mathbb{R}} g'(x) \left( (pf')(x)\overline{f(x)} + V(x) \right) \,\mathrm{d}x \right| \\
&\leq \left( \sqrt{\alpha\beta}\|\sqrt{p}g'\|_2 + 3\alpha \right) \|f\|_2^2,
\end{aligned}
$$

which shows the second inequality in (4.9). □

*Proof of Theorem* 2.2. Without restriction we assume $\|q_-\|_{\mathrm{u}} > 0$. Let $\lambda$ be a non-real eigenvalue of $A$ with a corresponding eigenfunction $f$. Since $1/p \in$

$L^\infty(\mathbb{R})$ and

$$\|f'\|_2^2 \le \|1/p\|_\infty \|\sqrt{p}f'\|_2^2 \tag{4.18}$$

we have $f, f' \in L^2(\mathbb{R})$ by Lemma A.2 (i). Thus, for all $\varepsilon > 0$ and every $n \in \mathbb{N}$

$$\sup_{x\in[n,n+1]} |f(x)|^2 \le \varepsilon \int_n^{n+1} |f'(t)|^2\,\mathrm{d}t + \left(1+\frac{1}{\varepsilon}\right)\int_n^{n+1} |f(t)|^2\,\mathrm{d}t \tag{4.19}$$

holds, see, e.g. [23, Lemma 9.32]. Set

$$\alpha := 2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty \|q_-\|_{\mathrm{u}}^2 \quad \text{and} \quad \beta := (4\|1/p\|_\infty \alpha)^{\frac{1}{2}}$$

and let $\varepsilon = (2\|q_-\|_{\mathrm{u}}\|1/p\|_\infty)^{-1} > 0$. With (4.19) and (4.18) we estimate

$$\begin{aligned}
\int_{\mathbb{R}} q_-(t)|f(t)|^2\,\mathrm{d}t &\le \|q_-\|_{\mathrm{u}} \sum_{n\in\mathbb{Z}} \sup_{x\in[n,n+1]} |f(x)|^2 \\
&\le \|q_-\|_{\mathrm{u}} \left(\varepsilon \|f'\|_2^2 + \left(1+\frac{1}{\varepsilon}\right)\|f\|_2^2\right) \\
&\le \|q_-\|_{\mathrm{u}} \left(\varepsilon \|1/p\|_\infty \|\sqrt{p}f'\|_2^2 + \left(1+\frac{1}{\varepsilon}\right)\|f\|_2^2\right) \\
&= \frac{1}{2}\|\sqrt{p}f'\|_2^2 + \left(\|q_-\|_{\mathrm{u}} + 2\|1/p\|_\infty \|q_-\|_{\mathrm{u}}^2\right)\|f\|_2^2 \\
&= \frac{1}{2}\|\sqrt{p}f'\|_2^2 + \frac{\alpha}{2}\|f\|_2^2.
\end{aligned} \tag{4.20}$$

Together with (4.7) we obtain

$$\|\sqrt{p}f'\|_2^2 = 2\|\sqrt{p}f'\|_2^2 - \|\sqrt{p}f'\|_2^2 \le 2\|q_- f^2\|_1 - \|\sqrt{p}f'\|_2^2 \le \alpha\|f\|_2^2$$

and with (4.2) from Lemma 4.1 and (4.20) we find

$$\|f\|_\infty^2 \le 2\sqrt{\|1/p\|_\infty \alpha}\|f\|_2^2 = \beta\|f\|_2^2 \quad \text{and} \quad \|q_- f^2\|_1 \le \alpha\|f\|_2^2.$$

With the choice of $\alpha$ and $\beta$ we have

$$\sqrt{\alpha\beta} = \sqrt{2}\|1/p\|_\infty^{1/4}\alpha^{3/4}$$

and an application of Lemma 4.3 finishes the proof. □

*Proof of Theorem* 2.4. Suppose that Hypothesis 2.1 holds and let $\lambda$ be a nonreal eigenvalue of $A$ corresponding to the eigenfunction $f$. (i) We first consider the case $s, \eta \in [1,\infty)$ where $s+\eta > 2$. Then

$$2\eta s - \eta - s = \eta(s-1) + s(\eta-1) \ge s - 1 + \eta - 1 > 0.$$

Choose

$$\beta = \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s\right)^{\frac{\eta s}{2\eta s-\eta-s}} \qquad \text{and} \quad \alpha = \|q_-\|_s \beta^{\frac{1}{s}}.$$

Hölder's inequality yields

$$\begin{aligned}
\|q_- f^2\|_1 &\le \|f\|_\infty^{\frac{2}{s}} \int_{\mathbb{R}} |q_-(t)||f(t)|^{\frac{2(s-1)}{s}} \,\mathrm{d}t \\
&\le \|f\|_\infty^{\frac{2}{s}} \left(\int_{\mathbb{R}} |q_-(t)|^s \,\mathrm{d}t\right)^{\frac{1}{s}} \left(\int_{\mathbb{R}} |f(t)|^2 \,\mathrm{d}t\right)^{\frac{s-1}{s}} \qquad (4.21) \\
&= \|q_-\|_s \|f\|_\infty^{\frac{2}{s}} \|f\|_2^{\frac{2(s-1)}{s}}.
\end{aligned}$$

Thus, together with Lemma 4.1 and (4.7) we obtain

$$\begin{aligned}
\|f\|_\infty^2 = \left(\frac{\|f\|_\infty^{\frac{2(2\eta-1)}{\eta}}}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{\eta s}{2\eta s-\eta-s}} &\le \left(\frac{\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|\sqrt{p} f'\|_2^2 \|f\|_2^{\frac{2(\eta-1)}{\eta}}}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{\eta s}{2\eta s-\eta-s}} \\
&\le \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s\right)^{\frac{\eta s}{2\eta s-\eta-s}} \|f\|_2^2 = \beta \|f\|_2^2.
\end{aligned}$$

Using this estimate in (4.21) yields

$$\|q_- f^2\|_1 \le \|q_-\|_s \beta^{\frac{1}{s}} \|f\|_2^2 = \alpha \|f\|_2^2.$$

Hence (4.8) is valid. By the choice of $\alpha$ and $\beta$ we have $\sqrt{\alpha\beta} = \|q_-\|_s^{\frac{1}{2}} \beta^{\frac{s+1}{2s}}$ and Lemma 4.3 implies the bounds for $\lambda$.

Now consider the case $\eta = \infty$ and $s \in [1, \infty)$. Choose

$$\beta = (4\|1/p\|_\infty \|q_-\|_s)^{\frac{s}{2s-1}} \quad \text{and} \quad \alpha = \|q_-\|_s \beta^{\frac{1}{s}}.$$

The same estimate as in (4.21) applies here. Thus, Lemma 4.1 together with (4.7) and (4.21) imply

$$\begin{aligned}
\|f\|_\infty^2 = \left(\frac{\|f\|_\infty^4}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{s}{2s-1}} &\le \left(\frac{4\|1/p\|_\infty \|\sqrt{p} f'\|_2^2 \|f\|_2^2}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{s}{2s-1}} \\
&\le (4\|1/p\|_\infty \|q_-\|_s)^{\frac{s}{2s-1}} \|f\|_2^2 = \beta \|f\|_2^2.
\end{aligned}$$

Combining this with the estimate in (4.21) yields

$$\|q_- f^2\|_1 \le \|q_-\|_s \beta^{\frac{1}{s}} \|f\|_2^2 = \alpha \|f\|_2^2.$$

Hence (4.8) is valid and $\sqrt{\alpha\beta} = \|q_-\|_s^{\frac{1}{2}} \beta^{\frac{s+1}{2s}}$. Lemma 4.3 implies the bounds for $\lambda$ and the assertion (i) is shown.

(ii) Assume $\eta = \infty$ and $s = \infty$. Choose

$$\alpha = \|q_-\|_\infty \quad \text{and} \quad \beta = 2\sqrt{\|1/p\|_\infty \|q_-\|_\infty},$$

so that

$$\sqrt{\alpha\beta} = \sqrt{2}\|1/p\|_\infty^{\frac{1}{4}} \|q_-\|_\infty^{\frac{3}{4}}.$$

We show that (4.8) holds. Observe, that

$$\|q_- f^2\|_1 \leq \|q_-\|_\infty \|f\|_2^2 = \alpha \|f\|_2^2. \tag{4.22}$$

Lemma 4.1 in combination with (4.7) and (4.22) leads to

$$\|f\|_\infty^2 \leq 2\sqrt{\|1/p\|_\infty} \|\sqrt{p} f'\|_2 \|f\|_2 \leq 2\sqrt{\|1/p\|_\infty \|q_-\|_\infty} \|f\|_2^2 = \beta \|f\|_2^2.$$

Hence (4.8) is valid and Lemma 4.3 implies (ii). □

*Proof of Theorem* 2.5. Assume that $\eta = s = 1$ and $\|1/p\|_1 \|q_-\|_1 < 1$. Let $f$ be an eigenfunction corresponding to a non-real eigenvalue $\lambda$ of $A$. Then Lemma 4.1 (i) and (4.7) yield

$$\|f\|_\infty^2 \leq \|1/p\|_1 \|\sqrt{p} f'\|_2^2 \leq \|1/p\|_1 \|q_- f^2\|_1 \leq \|1/p\|_1 \|q_-\|_1 \|f\|_\infty^2 < \|f\|_\infty^2;$$

a contradiction. □

*Proof of Theorem* 2.6. By Lemma 3.2 for every $\delta$ satisfying (2.5) there exists a real function $g \in \mathcal{AC}(\mathbb{R})$ with $rg > 0$ a.e., $\|g\|_\infty = 1$ such that the support of $g'$ is compact and $\|\sqrt{p} g'\|_2 = P$, where

$$\{|g| < 1\} = \bigcup_{x \in \mathcal{T}_r} (x - \delta, x + \delta).$$

Lemma 3.3 ensures that $\gamma$ is positive and $\{|r| < \gamma\} \subset \Omega$. As $rg > 0$ a.e. we have

$$\omega_{\gamma,g} = \mu\big(\{|r| < \gamma\} \cup \{|g| < 1\}\big) \leq \sum_{x \in \mathcal{T}_r \cup \mathcal{Z}_r} 2\delta = 2\delta n.$$

Choosing $\delta$ sufficiently small as stated in (i), (ii) and (iii), respectively, together with Theorem 2.2 and Theorem 2.4 completes the proof. □

*Proof of Corollary* 2.7. We apply Theorem 2.6. Since $\mathcal{T}_r = \{0\}$ and $\mathcal{Z}_r = \emptyset$ we have $n = 1$, $\gamma = 1$ and $P = \sqrt{2/\delta}$. Without restriction we assume $q_- \neq 0$. The estimates then follow with the choice $\delta = \frac{1}{12}(2\|q_-\|_{\mathrm{u}} + 4\|q_-\|_{\mathrm{u}}^2)^{-\frac{1}{2}}$ in (i), $\delta = \frac{1}{6}(4\|q_-\|_s)^{-\frac{s}{2s-1}}$ in (ii), and $\delta = \frac{1}{12}\|q_-\|_\infty^{-\frac{1}{2}}$ in (iii). □

## A. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Here the main objective is to prove self-adjointness of the corresponding definite Sturm-Liouville operator associated to the definite differential expression

$$\tau = \frac{1}{|r|}\left(\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x} + q\right)$$

on $\mathbb{R}$. Let $T$ be the maximal operator in $L^2_r(\mathbb{R})$ associated to $\tau$,

$$Tf = \tau f, \quad \mathrm{dom}(T) = \mathcal{D}_{\max},$$

where

$$\mathcal{D}_{\max} = \left\{f \in L^2_r(\mathbb{R}) : f, pf' \in \mathcal{AC}(\mathbb{R}), \ell f \in L^2_r(\mathbb{R})\right\}.$$

In the next two lemmas we collect properties of the $T$ and the maximal domain $\mathcal{D}_{\max}$ applying standard techniques in Sturm-Liouville theory, see e.g. in [12, 13] and [24, Appendix to section 6].

**Lemma A.1.** *Suppose Hypothesis* 2.1 *holds. For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $\xi \in \mathbb{R}$*

$$\sup_{x\in[\xi,\xi+1]} |f(x)|^2 \le C_\varepsilon \int_\xi^{\xi+1} |f(t)|^2|r(t)|\,\mathrm{d}t + \varepsilon \int_\xi^{\xi+1} p(t)|f'(t)|^2\,\mathrm{d}t \tag{A.1}$$

*for every $f \in \mathcal{D}_{\max}$.*

*Proof.* Let $\varepsilon > 0$. Since $1/p \in L^\eta(\mathbb{R})$ for some $1 \le \eta \le \infty$ there exists $\delta > 0$ such that

$$\int_{x-\delta}^{x+\delta} \frac{1}{p(t)}\,\mathrm{d}t < \frac{\varepsilon}{2}$$

for all $x \in \mathbb{R}$; this can be seen with the help of the Hölder inequality. We can also assume that $\delta < \frac{1}{2}$. Using conditions (b) and (c) in Hypothesis 2.1 it follows that there exists $c > 0$ such that

$$\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} |r(t)|\,\mathrm{d}t > c$$

for all $x \in \mathbb{R}$. For $f \in \mathcal{D}_{\max}$ and $x, y \in \mathbb{R}$

$$\begin{aligned} |f(x)|^2 &\le 2|f(y)|^2 + 2|f(x) - f(y)|^2 = 2|f(y)|^2 + 2\left|\int_y^x f'(t)\,\mathrm{d}t\right|^2 \\ &\le 2|f(y)|^2 + 2\int_y^x p(t)|f'(t)|^2\,\mathrm{d}t \int_y^x \frac{1}{p(t)}\,\mathrm{d}t. \end{aligned} \tag{A.2}$$

We multiply (A.2) with $|r(y)|$ and integrate over $I(x,\xi) := [\xi,\xi+1]\cap[x-\delta,x+\delta]$ for arbitrary $\xi \in \mathbb{R}$ and $x \in [\xi,\xi+1]$. The length of each interval $I(x,\xi)$ is $\geq \delta$. Then

$$\begin{aligned}
|f(x)|^2 \int_{I(x,\xi)} |r(y)|\,\mathrm{d}y \leq{}& 2\int_{I(x,\xi)} |f(y)|^2|r(y)|\,\mathrm{d}y \\
&+ 2\int_{I(x,\xi)} \left(\int_y^x p(t)|f'(t)|^2\,\mathrm{d}t \int_y^x \frac{1}{p(t)}\,\mathrm{d}t\right)|r(y)|\,\mathrm{d}y \\
\leq{}& 2\int_\xi^{\xi+1} |f(y)|^2|r(y)|\,\mathrm{d}y \\
&+ 2\int_{I(x,\xi)} |r(y)|\,\mathrm{d}y \int_{I(x,\xi)} \frac{1}{p(y)}\,\mathrm{d}y \int_\xi^{\xi+1} p(y)|f'(y)|^2\,\mathrm{d}y.
\end{aligned}$$

We divide this by $\int_{I(x,\xi)} |r(y)|\,\mathrm{d}y$ and define $C_\varepsilon := 2c^{-1}$. This proves (A.1). $\square$

Some parts in item (i) and the assertion of item (iii) in the next lemma were proven under slightly different assumptions in [12] and [13].

**Lemma A.2.** *Under Hypothesis* 2.1 *all* $f,g \in \mathcal{D}_{\max}$ *satisfy*

(i) $f, \sqrt{p}f' \in L^2(\mathbb{R})$ *and* $qf^2 \in L^1(\mathbb{R})$,

(ii) *there exist sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *with* $\lim_{n\to\infty} x_n = \infty$ *and* $\lim_{n\to\infty} y_n = -\infty$ *such that* $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = 0$,

(iii) $\lim_{x\to\pm\infty}(pf')(x)\overline{g(x)} = 0$.

*Moreover, the operator* $T$ *is self-adjoint in* $L^2_r(\mathbb{R})$ *with respect to the scalar product* $(\cdot,\cdot)_r$ *and semibounded from below.*

*Proof.* Let $f,g \in \mathcal{D}_{\max}$. Then integration by parts yields

$$\begin{aligned}
\int_y^x (Tf)(t)\overline{g(t)}|r(t)|\,\mathrm{d}t = \int_y^x \left(p(t)f'(t)\overline{g'(t)} + q(t)f(t)\overline{g(t)}\right)\mathrm{d}t \\
+ (pf')(y)\overline{g(y)} - (pf')(x)\overline{g(x)}
\end{aligned} \tag{A.3}$$

for all $y < x$. We show item (i). By Hypothesis 2.1 (c) there exists $C_r > 0$ with $|r(x)| \geq C_r$ for a.a. $x$ outside of a compact interval $[a,b]$ and we obtain

$$\int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t \leq (b-a)\sup_{x\in[a,b]}|f(x)|^2 + \frac{1}{C_r}\int_{\mathbb{R}\setminus[a,b]} |f(t)|^2|r(t)|\,\mathrm{d}t < \infty,$$

where the continuity of $f$ implies the boundedness on $[a,b]$. This shows $f \in L^2(\mathbb{R})$.

From now on we assume that $f$ is a real function; this is no restriction. Let $y < x$ such that $1 \le x - y$. For $n \in \mathbb{N}$ with $1 \le n \le x - y < n + 1$ we have

$$\begin{aligned}\int_y^x |q(t)||f(t)|^2 &\le \int_{x-n}^x |q(t)||f(t)|^2 \,\mathrm{d}t + \int_y^{y+n} |q(t)||f(t)|^2 \,\mathrm{d}t \\ &= \sum_{k=1}^n \left( \int_{x-k}^{x-k+1} |q(t)||f(t)|^2 \,\mathrm{d}t + \int_{y+k-1}^{y+k} |q(t)||f(t)|^2 \,\mathrm{d}t \right) \\ &\le 2\|q\|_{\mathrm{u}} \sum_{k=1}^n \sup_{t\in[x-k,x-k+1]} |f(t)|^2 + 2\|q\|_{\mathrm{u}} \sum_{k=1}^n \sup_{t\in[y+k-1,y+k]} |f(t)|^2 \,\mathrm{d}t.\end{aligned}$$

Fix $\varepsilon > 0$ such that $4\|q_-\|_{\mathrm{u}}\varepsilon \le \frac{1}{2}$ and let $C_\varepsilon$ as in Lemma A.1. Then

$$\begin{aligned}\int_y^x |q(t)||f(t)|^2 \le 2\|q\|_{\mathrm{u}} &\left( \varepsilon \int_{x-n}^x p(t)|f'(t)|^2 \,\mathrm{d}t + C_\varepsilon \int_{x-n}^x |f(t)|^2 |r(t)| \,\mathrm{d}t \right) \\ &+ 2\|q\|_{\mathrm{u}} \left( \varepsilon \int_y^{y+n} p(t)|f'(t)|^2 \,\mathrm{d}t + C_\varepsilon \int_y^{y+n} |f(t)|^2 |r(t)| \,\mathrm{d}t \right).\end{aligned}$$

Thus, for all $y < x$ with $1 \le x - y$ we obtain

$$\int_y^x |q(t)||f(t)|^2 \le 4\|q\|_{\mathrm{u}} \left( \varepsilon \int_y^x p(t)|f'(t)|^2 \,\mathrm{d}t + C_\varepsilon \int_y^x |f(t)|^2 |r(t)| \,\mathrm{d}t \right). \quad \text{(A.4)}$$

Let $\lambda = -4\|q\|_{\mathrm{u}} C_\varepsilon$. Then by (A.3) and (A.4) we obtain

$$\begin{aligned}\int_y^x \big((T-\lambda)f\big)(t)\overline{f(t)}|r(t)| \,\mathrm{d}t &\ge \int_y^x \Big( p(t)|f'(t)|^2 - \big(|q(t)| + \lambda|r(t)|\big)|f(t)|^2 \Big) \,\mathrm{d}t \\ &\quad + (pf')(y)f(y) - (pf')(x)f(x) \\ &\ge \frac{1}{2} \int_y^x p(t)|f'(t)|^2 \,\mathrm{d}t + (pf')(y)f(y) - (pf')(x)f(x)\end{aligned} \quad \text{(A.5)}$$

for all $y < x$ with $1 \le x - y$. Assume that $\sqrt{p}f'$ is not square integrable over $(0, \infty)$ and fix $y = 0$. Since $f \in \mathcal{D}_{\max}$ the left hand side in (A.5) is bounded for all $x > 0$. The integral on the right hand side is nonnegative for all $x > 0$ and tends monotonically to $\infty$ as $x \to \infty$. Thus, there exists $b > 0$ such that $(pf')(x)f(x)$ is positive for all $x \ge b$. Due to Hypothesis 2.1 the function $|r|$ is bounded from below on $[b, \infty)$ by some $C_r > 0$ (one possibly needs to increase $b$) and we obtain

$$\begin{aligned}\int_b^\infty |f(t)|^2 |r(t)| \,\mathrm{d}t &= \int_b^\infty \left( |f(b)|^2 + 2\int_b^t \frac{(pf')(s)f(s)}{p(s)} \,\mathrm{d}s \right) |r(t)| \,\mathrm{d}t \\ &\ge C_r \int_b^\infty |f(b)|^2 \,\mathrm{d}t = \infty,\end{aligned}$$

which contradicts $f \in L^2_r(\mathbb{R})$. This shows that $\sqrt{p}f'$ is square integrable over $(0,\infty)$. In the same manner one obtains the integrability over $(-\infty,0)$. From (A.4) together with $\sqrt{p}f' \in L^2(\mathbb{R})$ and $f \in L^2_r(\mathbb{R})$ one obtains $qf^2 \in L^1(\mathbb{R})$, which finishes the proof of (i). Moreover, the continuity of $f$ and $f \in L^2(\mathbb{R})$ imply (ii). In fact, by the mean value theorem we find a strictly increasing sequence $(x_n)_{n\in\mathbb{N}}$ with

$$\int_0^\infty |f(t)|^2\,\mathrm{d}t = \sum_{n\in\mathbb{N}} \int_n^{n+1} |f(t)|^2\,\mathrm{d}t = \sum_{n\in\mathbb{N}} |f(x_n)|^2.$$

Thus, $f(x_n) \to 0$ as $n \to \infty$. The sequence $(y_n)_{n\in\mathbb{N}}$ can be constructed in the same way. This proves (ii).

We show that $\lim_{x\to\infty}(pf')(x)\overline{g(x)} = 0$ for all $f, g \in \mathcal{D}_{\max}$, where it is again sufficient to consider only real functions. Let $f, g \in \mathcal{D}_{\max}$ be real. Due to (A.3) the limits $\lim_{x\to\pm\infty}(pf'g)(x)$ exist and are finite. Assume

$$\lim_{x\to\infty} p(x)|f(x)g'(x)| =: \alpha > 0.$$

Then there exists $b > 0$ such that $|f(x)| > 0$ and

$$p(x)|g'(x)| \geq \frac{\alpha}{2|f(x)|}$$

for $x \in [b,\infty)$. Multiplication with $|f'(x)|$ and integration leads to

$$\int_b^x p(t)|f'(t)g'(t)|\,\mathrm{d}t \geq \frac{\alpha}{2}\int_b^x \left|\frac{f'(t)}{f(t)}\right|\,\mathrm{d}t \geq \frac{\alpha}{2}\left|\int_b^x \frac{f'(t)}{f(t)}\,\mathrm{d}t\right| = \frac{\alpha}{2}\left|\ln\left|\frac{f(x)}{f(b)}\right|\right|. \tag{A.6}$$

Let $x$ in (A.6) run through the sequence $(x_n)_{n\in\mathbb{N}}$ from (ii). One obtains that the right hand side grows to $\infty$ while the left hand side is still bounded since (i) holds. This is a contradiction and hence the assumption $\alpha > 0$ was false; thus $\lim_{x\to\infty}(pf')(x)g(x) = 0$. The analog result for $x \to -\infty$ follows in the same way. This shows (iii).

From (iii) we obtain

$$\lim_{x\to\infty}\left(\overline{(pg')(x)}f(x) - (pf')(x)\overline{g(x)}\right) - \lim_{y\to-\infty}\left(\overline{(pg')(y)}f(y) - (pf')(y)\overline{g(y)}\right) = 0$$

for all $f, g \in \mathcal{D}_{\max}$. This implies the self-adjointness of $T$, see e.g. [11, Theorem 5.1]. From (A.3), (A.4) and (iii) we obtain for $\varepsilon > 0$ with $4\|q_-\|_{\mathrm{u}}\varepsilon \leq \frac{1}{2}$

$$\begin{aligned}(Tf,f)_r &= \int_{\mathbb{R}} \left(p(t)|f'(t)|^2 + q(t)|f(t)|^2\,\mathrm{d}t\right) \geq \|\sqrt{p}f'\|_2^2 - \|qf^2\|_1 \\ &\geq \frac{1}{2}\|\sqrt{p}f'\|_2^2 - 4\|q_-\|_{\mathrm{u}} C_\varepsilon (f,f)_r \geq -4\|q_-\|_{\mathrm{u}} C_\varepsilon (f,f)_r\end{aligned}$$

for all $f \in \mathcal{D}_{\max}$, which shows that $T$ is semibounded. □

*Proof of Theorem* 1.1. The map

$$(Jf)(x) := \operatorname{sgn}(r(x))f(x), \quad x \in \mathbb{R}, \quad f \in L^2_r(\mathbb{R}),$$

is a fundamental symmetry of the Krein space $(L^2_r(\mathbb{R}), [\cdot,\cdot]_r)$ such that $[\cdot,\cdot]_r = (J\cdot,\cdot)_r$. Therefore, as $T$ is self-adjoint in the Hilbert space $L^2_r(\mathbb{R})$ it is clear that

$$A = JT$$

with $\operatorname{dom}(A) = \mathcal{D}_{\max}$ is self-adjoint in the Krein space $(L^2_r(\mathbb{R}), [\cdot,\cdot]_r)$. It remains to show the assertions on the non-real spectrum of $A$. If the signs of $r$ near $\infty$ and $-\infty$ differ then the claim follows by [3, Theorem 4.2]. Otherwise, one obtains the proposed spectral properties of $A$ in a similar way as in [3, Theorem 4.2] by applying [3, Corollary 3.9]. $\square$

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