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The non-real spectrum of a singular indefinite Sturm–Liouville operator with regular left endpoint

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We provide bounds on the non-real spectra of indefinite Sturm–Liouville differential operators of the form $(Af)(x) = \operatorname{sgn}(x)(-f''(x) + q(x)f(x))$ on the interval $[a, \infty)$, $-\infty < a < 0$, with real potential $q \in L^1(a, \infty)$. The bounds depend only on the L^1-norm of the negative part of q and the boundary condition at the regular endpoint a.

1 Introduction and main result

We consider the Sturm–Liouville differential operator

$$(A_\alpha f)(x) = \operatorname{sgn}(x)\big(-f''(x) + q(x)f(x)\big), \quad D(A_\alpha) = \left\{ f \in L^2(a,\infty) \;\middle|\; \begin{array}{l} f, f' \in AC[a,\infty), \\ -f'' + qf \in L^2(a,\infty), \\ \cos(\alpha)f(a) = \sin(\alpha)f'(a) \end{array} \right\}, \quad (1)$$

in $L^2(a,\infty)$ for $\alpha \in [0,\pi)$, where $q \in L^1(a,\infty)$ is a real function and $-\infty < a < 0$. Here, $AC[a,\infty)$ denotes the space of functions which are absolutely continuous on every compact subset of $[a,\infty)$. As the weight sgn changes the sign the operator A_α is neither symmetric nor self-adjoint in $L^2(a,\infty)$ with respect to the usual scalar product $(\cdot,\cdot)$. Hence, A_α may have non-real spectrum. But, equipped with the inner product $[\cdot,\cdot]$,

$$[f,g] = \int_a^\infty f(x)\overline{g(x)} \operatorname{sgn}(x)\,\mathrm{d}x, \quad f, g \in L^2(a,\infty),$$

$L^2(a,\infty)$ is a Krein space with the fundamental symmetry $J : L^2(a,\infty) \to L^2(a,\infty)$, $(Jf)(x) = \operatorname{sgn}(x)f(x)$, where A_α is self-adjoint with respect to $[\cdot,\cdot]$; for the basic notions in Krein spaces we refer to [1] and [7]. Indeed, while the finite endpoint a is regular the integrability of q implies the limit point case at the singular endpoint ∞, cf. [13, Lemma 9.37]. Hence, the definite Sturm–Liouville operator JA_α on $D(JA_\alpha) = D(A_\alpha)$ is self-adjoint in the Hilbert space $L^2(a,\infty)$ and due to $(\cdot,\cdot) = [J\cdot,\cdot]$ the self-adjointness of A_α with respect to $[\cdot,\cdot]$ follows. By [2, Corollary 3.9] the operator A_α has nonempty resolvent set and its essential spectrum coincides with the essential spectrum of JA_α, where $\sigma_{\mathrm{ess}}(JA_\alpha) = [0,\infty)$, see Theorem 9.38 and the note below in [13].

Recently, bounds for the non-real spectra of indefinite Sturm–Liouville operators were developed in [3, 8–12] for operators with two regular endpoints and in [4–6] for operators with two singular endpoints. The result in Theorem 1.1 addresses singular operators with one regular and one singular endpoint. The proof is based on techniques developed in [5, 6]. In the following let $q = q_+ - q_-$, where $q_+(x) = \max\{q(x), 0\}$ and $q_-(x) = \max\{-q(x), 0\}$.

Theorem 1.1 *The operator A_α, $\alpha \in [0,\pi)$, in* (1) *is self-adjoint with respect to the inner product $[\cdot,\cdot]$. Let $c_\alpha = 0$ if $\alpha = 0$ and $c_\alpha = \cot(\alpha)$ if $\alpha \in (0,\pi)$. The essential spectrum of A_α equals $[0,\infty)$ and the non-real spectrum of A_α is purely discrete. Every non-real eigenvalue λ of A_α satisfies*

$$|\operatorname{Im}\lambda| \le 24\sqrt{3}\big(\|q_-\|_1 + |c_\alpha|\big)^2 \quad \textit{and} \quad |\lambda| \le \big(24\sqrt{3} + 18\big)\big(\|q_-\|_1 + |c_\alpha|\big)^2 + 6|c_\alpha|\big(\|q_-\|_1 + |c_\alpha|\big). \quad (2)$$

Proof. Consider an eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of A_α and a corresponding eigenfunction f with $\|f\|_2 = 1$. Let

$$U(x) = \int_x^\infty |f|^2 \operatorname{sgn}, \quad V(x) = \int_x^\infty \big(|f'|^2 + q|f|^2\big), \quad (3)$$

for $x \in [a,\infty)$. One can show that f satisfies $\lim_{x\to\infty} f'(x)\overline{f(x)} = 0$, $f' \in L^2(a,\infty)$ and $q|f^2| \in L^1(a,\infty)$, cf. [6, Appendix A]. Hence, the functions V and U given by (3) are well-defined on $[a,\infty)$ with values in $\mathbb{R}$. Moreover, $\lim_{x\to\infty} U(x) = 0$ and $\lim_{x\to\infty} V(x) = 0$. Integration by parts together with the eigenvalue equation $\lambda f = A_\alpha f$ yields

$$\lambda U(x) = \int_x^\infty (A_\alpha f)\overline{f} \operatorname{sgn} = V(x) + f'(x)\overline{f(x)}. \quad (4)$$

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As $f \in D(A_\alpha)$ we have $f'(a)\overline{f(a)} = c_\alpha |f(a)|^2 \in \mathbb{R}$. Evaluating (4) at $x = a$ and comparing the imaginary parts we obtain $U(a) = 0$ and $V(a) = -c_\alpha |f(a)|^2$. Hence,

$$0 \leq \|f'\|_2^2 = -\int_a^\infty \big(q_+ - q_-\big)|f|^2 - c_\alpha |f(a)|^2 \leq \big(\|q_-\|_1 + |c_\alpha|\big)\|f\|_\infty^2. \tag{5}$$

Furthermore, this implies

$$\int_a^\infty q_+ |f|^2 \leq \big(\|q_-\|_1 + |c_\alpha|\big)\|f\|_\infty^2, \quad \|qf^2\|_1 \leq 2\big(\|q_-\|_1 + |c_\alpha|\big)\|f\|_\infty^2. \tag{6}$$

Here, the norm $\|f\|_\infty$ can be estimated as follows. Since $f \in L^2(\mathbb{R})$ is continuous there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in (a, ∞) with $y_n \to \infty$ and $f(y_n) \to 0$ as $n \to 0$. Thus,

$$|f(x)|^2 = |f(y_n)|^2 + 2\,\mathrm{Re}\int_{y_n}^x f'\overline{f}, \quad \|f\|_\infty^2 \leq 2\|f'\|_2,$$

where we used the Cauchy–Schwarz inequality and $\|f\|_2 = 1$ in the last estimate. This together with (5) leads to

$$\|f'\|_2^2 \leq 4\big(\|q_-\|_1 + |c_\alpha|\big)^2 \quad \text{and} \quad \|f\|_\infty^2 \leq 4\big(\|q_-\|_1 + |c_\alpha|\big). \tag{7}$$

Observe, that it is no restriction to consider $\|q_-\|_1 + |c_\alpha| > 0$ since otherwise f is constantly zero. We define an absolutely continuous function g by

$$g(x) = \begin{cases} \frac{x}{\delta} & \text{if } x \in (-\delta, \delta), \\ \mathrm{sgn}(x) & \text{if } x \in [a, -\delta] \cup [\delta, \infty), \end{cases} \qquad \text{where } \delta = \frac{1}{24\big(\|q_-\|_1 + |c_\alpha|\big)}.$$

Here, the interval $[a, -\delta]$ is considered to be empty if $-\delta < a$. We have $\|g\|_\infty = 1$ and $\|g'\|_2 = \sqrt{2/\delta}$. Then

$$\begin{aligned} \int_a^\infty g'U &= \int_a^\infty g|f|^2\,\mathrm{sgn} \geq \int_{(a,\infty)\setminus(-\delta,\delta)} |f|^2 = 1 - \int_{-\delta}^{\delta} |f|^2 \\ &\geq 1 - 2\delta\|f\|_\infty^2 \geq 1 - 8\delta\big(\|q_-\|_1 + |c_\alpha|\big) \geq \frac{2}{3}. \end{aligned} \tag{8}$$

Further, we obtain with (6) and (7)

$$\begin{aligned} \left|\int_a^\infty g'V\right| &= \left|\int_a^\infty g\big(|f'|^2 + q|f|^2\big) - g(a)V(a)\right| \leq \|f'\|_2^2 + \|qf^2\|_1 + |c_\alpha|\|f\|_\infty^2 \\ &\leq 12\big(\|q_-\|_1 + |c_\alpha|\big)^2 + 4|c_\alpha|\big(\|q_-\|_1 + |c_\alpha|\big) \end{aligned} \tag{9}$$

and with (7)

$$\left|\int_a^\infty g'\overline{f}f'\right| \leq \|f\|_\infty \|f'\|_2 \|g'\|_2 \leq 4\sqrt{2/\delta}\big(\|q_-\|_1 + |c_\alpha|\big)^{\frac{3}{2}} \leq 16\sqrt{3}\big(\|q_-\|_1 + |c_\alpha|\big)^2. \tag{10}$$

By (4) we have

$$\lambda \int_a^\infty g'U = \int_a^\infty g'\big(V + f'\overline{f}\big). \tag{11}$$

A comparison of the imaginary parts and the absolute values in (11) together with the estimates (8)–(10) shows (2). □

References

[1] T. Ya. Azizov and I. S. Iokhvidov, Linear Operators in Space with an Indefinite Metric (John Wiley & Sons Ltd., Chichester, 1989).
[2] J. Behrndt and F. Philipp, J. Differ. Equations **248**, 2015–2037 (2010).
[3] J. Behrndt, S. Chen, F. Philipp, and J. Qi, Proc. R. Soc. Edinb., Sect. A, Math. **144**, 1113–1126 (2014).
[4] J. Behrndt, F. Philipp, and C. Trunk, Math. Ann. **357**, 185–213 (2013).
[5] J. Behrndt, S. Schmitz, and C. Trunk, Proc. Amer. Math. Soc. **146**, 3935–3942 (2018).
[6] J. Behrndt, S. Schmitz, and C. Trunk, J. Differ. Equations **267**, 468–493 (2019).
[7] J. Bognar, Indefinite Inner Product Spaces (Springer, Berlin, 1974).
[8] S. Chen and J. Qi, J. Spectr. Theory **4**, 53–63 (2014).
[9] S. Chen, J. Qi, and B. Xie, Proc. Amer. Math. Soc. **144**, 547–559 (2016).
[10] X. Guo, H. Sun, and B. Xie, Electron. J. Qual. Theory Differ. Equ. **2017**, 1–14 (2017).
[11] M. Kikonko and A. B. Mingarelli, J. Differ. Equations **261**, 6221–6232 (2016).
[12] J. Qi and B. Xie, J. Differ. Equations **255**, 2291–2301 (2013).
[13] G. Teschl, Mathematical Methods in Quantum Mechanics (Amer. Math. Soc., Providence, RI, 2009).