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State-based nested iteration solution of optimal control problems with PDE constraints*

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Abstract

We consider an abstract framework for the numerical solution of optimal control problems (OCPs) subject to partial differential equations (PDEs). Examples include not only the distributed control of elliptic PDEs such as the Poisson equation discussed in this paper in detail but also parabolic and hyperbolic equations. The approach covers the standard L^2 setting as well as the more recent energy regularization, also including state and control constraints. We discretize OCPs subject to parabolic or hyperbolic PDEs by means of space-time finite elements similar as in the elliptic case. We discuss regularization and finite element error estimates, and derive an optimal relation between the regularization parameter and the finite element mesh size in order to balance the accuracy, and the energy costs for the corresponding control. Finally, we also discuss the efficient solution of the resulting systems of algebraic equations, and their use in a state-based nested iteration procedure that allows us to compute finite element approximations to the state and the control in asymptotically optimal complexity. The numerical results illustrate the theoretical findings quantitatively.

Keywords: PDE constrained optimal control problems, finite element method, error estimates, solvers, nested iteration

2010 MSC: 49J20, 49M05, 35J05, 65M60, 65N22, 65F10

1 Introduction, motivation, and preliminaries

Since Lions' pioneering monograph [45] on the optimal control of systems described by partial differential equations (PDEs) of elliptic, parabolic, or hyperbolic types, the investigation of optimal control problems (OCPs) for PDEs and their numerical solution have developed into a well-established research field in Applied Mathematics with many applications in different areas in science and engineering. Since then the development of the mathematical analysis on OCPs for PDEs is documented by a huge number of publications. We here only refer the reader to the books [5, 12, 32, 64], the collections [28, 29, 44], and the survey paper [4]. Tracking-type

*This paper is dedicated to our friend and colleague Arnd Rösch in occasion of his 60th birthday.

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OCPs for PDEs can be posed as follows: Find the optimal control $u_\varrho \in U$ and the corresponding state $y_\varrho \in Y$ minimizing the cost functional

$$\mathcal{J}(y_\varrho, u_\varrho) \,=\, \frac{1}{2}\,\|y_\varrho - \overline{y}\|^2_{H_Y} + \frac{1}{2}\,\varrho\,\|u_\varrho\|^2_U \tag{1.1}$$

subject to (s.t.) the state equation

$$By_\varrho = u_\varrho \quad \text{in } \, U, \tag{1.2}$$

where we are thinking about elliptic (e.g., Poission's equation: $B = -\Delta$), parabolic (e.g., heat equation: $B = \partial_t - \Delta_x$), and hyperbolic (e.g., wave equation: $B = \partial_{tt} - \Delta_x$) PDEs or systems of such PDEs with appropriate boundary and initial conditions. Here $\overline{y} \in H_Y$ denotes the given desired state (target) that we want to track as close as possible, and $\varrho > 0$ is a suitable chosen regularization parameter that also defines the energy cost $\|u_\varrho\|^2_U$ of the control $u_\varrho \in U$ appearing as right-hand side of the state equation (1.2). In this paper, the spaces Y, U, and H_Y are Hilbert spaces with $Y \subset H_Y \subset Y^*$ being a Gelfand triple, and $B : Y \to U$ is always assumed to be an isomorphism. Of course, one can consider also nonlinear PDEs or systems of PDEs represented then by a nonlinear operator B acting between Banach spaces in general; see, e.g., [64]. The practical realization of the control u_ϱ sometimes requires additional, so-called box-constraints imposed on the control, i.e., we look for some optimal control $u_\varrho \in U_{\text{ad}} = K_c \subset U$ in a non-empty, closed, and convex subset K_c of U. Similarly, we can also request box-constraints imposed on the state, i.e. we look for $y_\varrho \in Y_{\text{ad}} = K_s \subset Y$ in a non-empty, closed, and convex subset K_s of Y. These are the main assumptions ensuring existence and uniqueness of an optimal solution $(y_\varrho, u_\varrho) \in Y \times U$ of the optimal control problem (1.1)-(1.2) or the coresponding box-constrained problems where U or Y is replaced by U_{ad} or Y_{ad}; see, e.g., [12, 45, 64].

One of the most simple examples of such kind of tracking-type OCPs is the distributed optimal control of the Poission equation with L^2 regularization: Find $u_\varrho \in U = L^2(\Omega)$ and $y_\varrho \in Y = H^1_0(\Omega)$ minimizing the cost functional

$$\mathcal{J}(y_\varrho, u_\varrho) = \frac{1}{2}\,\|y_\varrho - \overline{y}\|^2_{H_Y = L^2(\Omega)} + \frac{1}{2}\,\varrho\,\|u_\varrho\|^2_{U = L^2(\Omega)} \tag{1.3}$$

s.t. the Dirichlet boundary value problem for the Poisson equation

$$-\Delta y_\varrho = u_\varrho \quad \text{in } \Omega, \quad y_\varrho = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with boundary $\partial\Omega$, $d = 1, 2, 3$. It is well known that the solution of the OCP (1.3)–(1.4) is characterized by the gradient equation

$$p_\varrho + \varrho\, u_\varrho = 0 \quad \text{in } \Omega, \tag{1.5}$$

where p_ϱ solves the adjoint Dirichlet boundary value problem

$$-\Delta p_\varrho = y_\varrho - \overline{y} \quad \text{in } \Omega, \quad p_\varrho = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

When inserting $u_\varrho = -\Delta y_\varrho$ into the gradient equation (1.5), we get $p_\varrho = \varrho\,\Delta y_\varrho$. Hence, we have to solve the BiLaplace equation

$$\varrho\,\Delta^2 y_\varrho + y_\varrho = \overline{y} \quad \text{in } \Omega, \quad y_\varrho = \Delta y_\varrho = 0 \quad \text{on } \partial\Omega. \tag{1.7}$$

Since the operator $B = -\Delta$ is an isomorphism from $H^1_0(\Omega, \Delta) = \{y \in H^1_0(\Omega) : \Delta u \in L^2(\Omega)\}$ onto $U = L^2(\Omega)$, the natural choice for the state space would be $Y = H^1_0(\Omega, \Delta)$ rather than $Y = H^1_0(\Omega)$ in the case of L^2 regularization. On the other hand, if we choose $Y = H^1_0(\Omega)$ as state space, then we can permit controls from $U = H^{-1}(\Omega)$, and now $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism as well. Therefore, instead of using the L^2 regularization $\|u_\varrho\|^2_{L^2(\Omega)}$ in (1.3), we may also consider the energy regularization $\|u_\varrho\|^2_{H^{-1}(\Omega)}$ to minimize

$$\mathcal{J}(y_\varrho, u_\varrho) = \frac{1}{2} \int_\Omega [y_\varrho(x) - \overline{y}(x)]^2 \, dx + \frac{1}{2} \, \varrho \, \|u_\varrho\|^2_{H^{-1}(\Omega)} \tag{1.8}$$

subject to the Poisson equation (1.4). By duality we have $\|u_\varrho\|_{H^{-1}(\Omega)} = \|\nabla y_\varrho\|_{L^2(\Omega)}$, and hence we can write (1.8) as the reduced cost functional

$$\widetilde{\mathcal{J}}(y_\varrho) = \frac{1}{2} \int_\Omega [y_\varrho(x) - \overline{y}(x)]^2 \, dx + \frac{1}{2} \, \varrho \int_\Omega |\nabla y_\varrho(x)|^2 \, dx, \tag{1.9}$$

whose minimizer is given as the unique solution of the gradient equation

$$-\varrho \, \Delta y_\varrho + y_\varrho = \overline{y} \quad \text{in } \Omega, \quad y_\varrho = 0 \quad \text{on } \partial\Omega. \tag{1.10}$$

To underline the differences in considering the optimal control problems (1.3) and (1.8) subject to (1.4), i.e., when measuring the control u_ϱ either in $L^2(\Omega)$ or in $H^{-1}(\Omega)$, let us consider three simple examples. Therefore, we will study three different target functions $\overline{y}$ on the one-dimensional ($d = 1$) domain $\Omega = (0,1)$, namely the regular target

$$\overline{y}_1(x) = 4x(1-x) \quad \text{for } x \in (0,1) \tag{1.11}$$

with $\overline{y}_1 \in H_0^1(\Omega) \cap H^2(0,1)$, the piecewise linear target

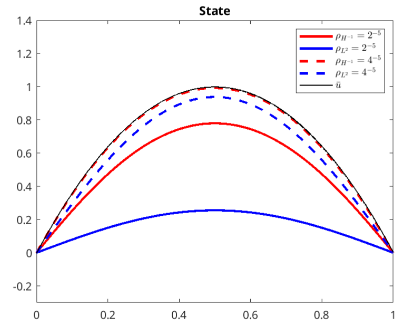
$$\overline{y}_2(x) = \begin{cases} 1, & x = 0.5, \\ 0, & x \in (0, 0.25) \cup (0.75, 1), \\ \text{piecewise linear}, & \text{else}, \end{cases} \tag{1.12}$$

belonging to $H_0^1(\Omega) \cap H^s(\Omega)$ for all $s < 1.5$, and the discontinuous target

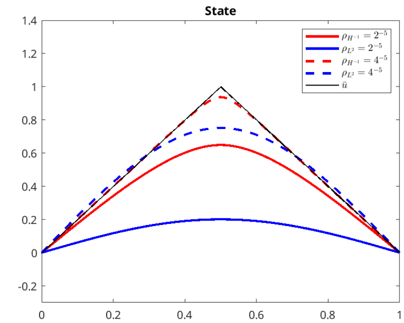
$$\overline{y}_3(x) = \begin{cases} 1, & x \in (0.25, 0.75), \\ 0, & \text{else}, \end{cases} \tag{1.13}$$

with $\overline{y}_3 \in H^s(\Omega)$, $s < 0.5$. In all of these cases, we can solve both the gradient equation (1.10) and the BiLaplace equation (1.7) analytically in order to compute the state functions $y_{i,\varrho}$, $i = 1,2,3$ for different values of the relaxation or cost parameter ϱ, see Fig. 1. There we also plot the errors $\|y_{i,\varrho} - \overline{y}_i\|_{L^2(\Omega)}$ for both regularization norms as a function of the regularization parameter ϱ. We observe that in all cases the error $\|y_{i,\varrho} - \overline{y}_i\|_{L^2(\Omega)}$ is smaller when using energy regularization in $H^{-1}(\Omega)$ instead of using the regularization in $L^2(\Omega)$. In fact, the errors coincide when considering $\varrho_{L^2} = \varrho^2_{H^{-1}}$. Note that the related regularization error estimates were already shown in [51].

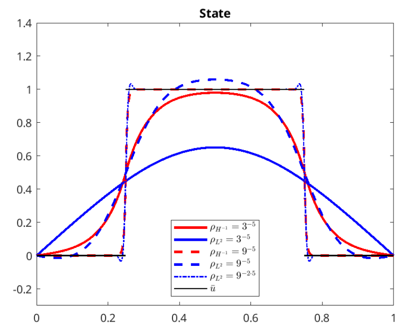
When the state $y_{i,\varrho}$ is known we can compute the control $u_{i,\varrho}(x) = -y''_{i,\varrho}(x)$ as well as the costs $\|u_{i,\varrho}\|_{L^2(\Omega)}$ and $\|u_{i,\varrho}\|_{H^{-1}(\Omega)} = \|y'_{i,\varrho}\|_{L^2(\Omega)}$ for both regularization norms. We observe that in the case of the smooth target $\overline{y}_1$ all costs remain bounded for $\varrho \to 0$, while for the discontinuous target $\overline{y}_3$ all costs tend to infinity as $\varrho \to 0$. The situation is different for the piecewise linear target $\overline{y}_2$ where the costs remain bounded in the case of energy regularization, but tend to infinity as $\varrho \to 0$ when using L^2 regularization. As discussed later in this paper, these observations are in complete agreement with our theoretical results. Moreover, they show that our theoretical results are sharp.



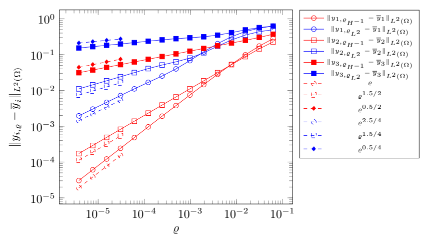
(a) Smooth target
$\overline{y}_1 \in H_0^1(0,1) \cap H^2(0,1)$.



(b) Piecewise linear target
$\overline{y}_2 \in H_0^1(0,1) \cap H^s(0,1)$, $s < 1.5$.

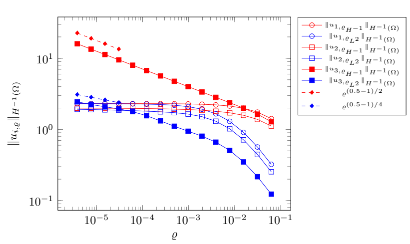


(c) Discontinuous target
$\overline{y}_3 \in H^s(0,1)$, $s < 0.5$.

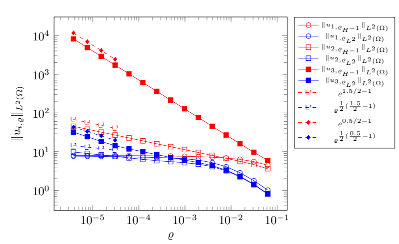


(d) Errors $\|y_{i,\varrho} - \overline{y}_i\|_{L^2(\Omega)}$

Figure 1: Targets $\overline{y}_i$, state solutions $y_{i,\varrho_{L^2}}$ and $y_{i,\varrho_{H^{-1}}}$, and errors $\|y_{i,\varrho} - \overline{y}_i\|_{L^2(\Omega)}$ for different choices of regularization parameters and regularization norms.



(a) Cost $\|u_{i,\varrho}\|_{H^{-1}(\Omega)}$



(b) Cost $\|u_{i,\varrho}\|_{L^2(\Omega)}$

Figure 2: Error and different types of regularization with $\varrho = \varrho_{H^{-1}} = \varrho_{L^2}$.

The finite element discretization of OPCs such as (1.1)–(1.2) in general, or (1.3)–(1.4) and (1.8)–(1.4) in particular, is usually investigated for fixed $\varrho > 0$, and discretization error estimates are provided for the finite element errors $\|y_\varrho - y_{\varrho h}\|$ and $\|u_\varrho - u_{\varrho h}\|$ in different norms; see, e.g., [41, 42, 49, 55, 56] However, as shown above, the distance of the computed finite element state $y_{\varrho h}$ from the desired state $\overline{y}$ is basically given by the distance $\|y_\varrho - \overline{y}\|$ that is fixed for fixed ϱ. Similarly, the computed costs $\|u_{\varrho h}\|$ approximate the cost $\|u_\varrho\|$ that is also fixed for fixed ϱ. If we want to improve the distance $\|y_{\varrho h} - \overline{y}\|$, then we have to diminish ϱ that leads to higher cost $\|u_{\varrho h}\|$. On the other side, if we want to reduce the cost, then we have to enlarge ϱ that yields a larger distance $\|y_{\varrho h} - \overline{y}\|$. A careful analysis of the discretization error $\|y_{\varrho h} - \overline{y}\|$ in terms of ϱ and h shows that we have to relate the regularization parameter ϱ to the mesh-size h in order to get an asymptotically optimal convergence of $y_{\varrho h}$ to $\overline{y}$ in balance with the energy cost for the control. It is shown in [37, 39] that we should choose $\varrho = h^2$ and $\varrho = h^4$ for OCPs (1.3)–(1.4) with L^2 regularization and (1.8)–(1.4) with H^{-1} regularization, respectively. The same choices as in elliptic OPCs hold for parabolic OPCs [43] and hyperbolic OPCs [47, 40] when using space-time finite element discretizations. If we permit functions as regularization, then ϱ can be adapted to the local mesh-size that can heavily vary over the computational domain in the case of an adaptive mesh refinement; see [38]. In this connection, we also refer to the very recent paper [54] where both the regularization parameter and the mesh-size are dynamically adjusted locally on the basis of a posteriori error estimates. The incorporation of box constraints imposed on the state or the control finally leads to state-based variational inequalities of first kind which can be again discretised by (space-time) finite elements; see [15] for the elliptic OPC (1.8)–(1.4) with state or control constraints. As in the unconstrained case, the choice $\varrho = h^2$ again gives asymptotically optimal convergence. For constant $\varrho > 0$, we refer to the recent papers [7, 19, 23] and the references therein.

An approximate solution to OCPs such as (1.1)–(1.2) can be found via the finite element discretization of the optimality system consisting of the state equation, the adjoint equation, and the gradient equation for defining the optimal state y_ϱ, the optimal adjoint (co-state) p_ϱ, and the optimal control u_ϱ; cf. (1.4)–(1.6) and (1.5) for the simple elliptic OCP (1.8)–(1.4) with L^2 regularization. The finite element discretization of the OCP (1.1)–(1.2) finally leads to a linear symmetric, but indefinite linear system of algebraic equations (saddle point system) for defining finite element vectors $\mathbf{y}_{\varrho h}, \mathbf{p}_{\varrho h}, \mathbf{u}_{\varrho h}$ related to the finite element approximations $y_{\varrho h}, p_{\varrho h}, u_{\varrho h}$ to $y_\varrho, p_\varrho, u_\varrho$ via the finite element isomorphism. This 3×3 block system can be reduced to the 2×2 block system

$$\begin{bmatrix} \varrho^{-1}\mathbf{A}_h & \mathbf{B}_h \\ \mathbf{B}_h^\top & -\mathbf{M}_h \end{bmatrix} \begin{bmatrix} \mathbf{p}_{\varrho h} \\ \mathbf{y}_{\varrho h} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_h \\ -\overline{\mathbf{y}}_h \end{bmatrix} \tag{1.14}$$

by eliminating the control $\mathbf{u}_{\varrho h}$. Here, the matrix $\mathbf{B}_h$ arises from the finite element discretization of B, $\mathbf{M}_h$ denotes the mass matrix, and $\mathbf{A}_h$ represents the regularisation term. Since we have to vary ϱ in order to adapt the accuracy of the approximation of the target $\overline{y}$ and the energy cost for the control u_ϱ to the practical requirements and to our budget, we would like to have not only a solver for the symmetric and indefinite system (1.14) or for the original 3×3 block system that runs in asymptotically optimal complexity in terms of h but also a solver that is robust in ϱ. Such kind of robust iterative solvers were proposed and analysed in, e.g., [57, 58, 68]. Eliminating the adjoint $\mathbf{p}_{\varrho h}$ from (1.14), we arrive at the symmetric and positive definite (spd), state-based Schur-complement system

$$\mathbf{S}_{\varrho h}\mathbf{y}_{\varrho h} = \overline{\mathbf{y}}_h, \tag{1.15}$$

where $\mathbf{S}_{\varrho h} = \mathbf{M}_h + \varrho \mathbf{D}_h$ with $\mathbf{D}_h = \mathbf{B}_h^T \mathbf{A}_h^{-1} \mathbf{B}_h$. The spd Schur-complement system can efficiently be solved by means of the preconditioned conjugate gradient (pcg) method provided that a robust preconditioner $\mathbf{C}_h$ is available, and the matrix-by-vector multiplication $\mathbf{D}_h * \mathbf{y}_{\varrho h}^k$, which contains the application of $\mathbf{A}_h^{-1}$, can be performed efficiently. Surprisingly, for the optimal choice of the regularisation ϱ, the Schur-complement $\mathbf{S}_{\varrho h}$ is always spectrally equivalent to the mass matrix $\mathbf{M}_h$ and, therefore, to some diagonal approximation of $\mathbf{M}_h$ such as the lumped mass matrix lump($\mathbf{M}_h$). So, the lumped mass matrix lump($\mathbf{M}_h$) can serve as robust preconditioner $\mathbf{C}_h$. This result is not only true for the elliptic case [37, 39] but also for parabolic [43] and hyperbolic [47] OCPs as well as for variable regularizations ϱ locally adapted to the mesh-size. It turns out that, for the elliptic OCP (1.8)–(1.4) with H^{-1} regularization, $\mathbf{B}_h = \mathbf{A}_h = \mathbf{D}_h = \mathbf{K}_h$, and, therefore, $\mathbf{S}_{\varrho h} = \mathbf{M}_h + \varrho \mathbf{K}_h$, where $\mathbf{K}_h$ is the spd finite element Laplacian stiffness matrix. So, a fast multiplication is ensured. Similarly, for the elliptic OCP (1.3)–(1.4) with L^2 regularization, we have $\mathbf{S}_{\varrho h} = \mathbf{M}_h + \varrho \mathbf{K}_h \mathbf{M}_h^{-1} \mathbf{K}_h$. Here we can replace $\mathbf{M}_h$ by lump($\mathbf{M}_h$) without affecting the asymptotic behavior of the discretization error [39], and a fast multiplication is again ensured. In general, we have to use inner iterations for approximating the application of $\mathbf{A}_h^{-1}$. Now, the pcg with the preconditioner $\mathbf{C}_h = \text{lump}(\mathbf{M}_h)$ can be used as nested solver in a nested iteration process on a sequence of uniformly or adaptively refined meshes starting with some coarse mesh and stopping the nested iteration as soon as a prescribed accuracy for the approximation of the given desired state $\overline{y}$ is achieved without exceeding a given budget for the energy cost of the control. The reconstruction of the control from the computed state is an integral part of the nested iteration process. This allows us to solve OCPs such as (1.1)–(1.2) always in optimal, or, at least, almost optimal complexity. In the case of OCPs with state or control constraints, we have to solve variational inequalities of first kind in the state-based formulation. After the finite element discretization, these variational inequalities are living in the finite element state space, and can be reformulatated as non-differentiable non-linear systems of algebraic equations for determining the nodal solution vector $\mathbf{y}_{\varrho h}$ corresponding to the finite element state solution $y_{\varrho h}$. This non-linear system can by solved by the semi-smooth Newton method that is nothing but the primal-dual active set method [30]. Alternatively, we can use multigrid methods for variational inequaltities arising, e.g., from obstacle problems [27]; see also the overview article [20]. For ϱ robust solvers of control or state constraint OCPs, we also refer to [2, 13, 63] and the references therein.

The rest of the paper is organized as follows. Section 2 presents a theoretical framework for the analysis and numerical analysis of OCPs of the form (1.1)–(1.2) including regularization error estimates (Subsection 2.1), Galerkin discretization and error estimates (Subsection 2.2), recovering of the control from the computed Galerkin approximation to the state in a simple postprocessing procedure (Subsection 2.3), solvers and their use in nested iteration with accuracy and cost control (Subsection 2.4), the handling of additional functional box-constraints for the state and the control (Subsection 2.5). The application of this abstract framework to the distributed control of Poisson's equation (1.3)–(1.4) is presented in Section 3, where we also discuss our numerical results. The application to the distributed control of Poisson's equation deliver the blueprint for other applications such as discussed in Section 4. In Section 5, we draw some conclusions, and give an outlook on further research directions in connection with our approach.

2 Abstract optimal control problems

2.1 Abstract setting and regularization error estimates

Let $X \subset H_X \subset X^*$ and $Y \subset H_Y \subset Y^*$ be Gelfand triples of Hilbert spaces, where X^* and Y^* are the duals of X and Y with respect to H_X and H_Y, respectively. We assume that H_X and H_Y are Hilbert spaces with the inner products $\langle \cdot, \cdot \rangle_{H_X}$ and $\langle \cdot, \cdot \rangle_{H_Y}$, respectively. Moreover, the duality pairings $\langle \cdot, \cdot \rangle_{X^*,X}$ and $\langle \cdot, \cdot \rangle_{Y^*,Y}$ are defined as extension of the inner products in H_X, and in H_Y, respectively.

Let $B : Y \to X^*$ be a bounded, linear operator which is assumed to satisfy an inf-sup condition, i.e., for all $y \in Y$, we have

$$\|By\|_{X^*} \le c_2^B \, \|y\|_Y, \quad \sup_{0 \neq x \in X} \frac{\langle By, x \rangle_{X^*,X}}{\|x\|_X} \ge c_1^B \, \|y\|_Y \, .$$

In addition we assume that B is surjective. Hence, $B : Y \to X^*$ defines an isomorphism. By

$$\langle By, x\rangle_{X^*,X} =: \langle y, B^* x\rangle_{Y,Y^*} \quad \text{for all } (y,x) \in Y \times X$$

we define the adjoint operator $B^* : X \to Y^*$. For optimal control problems in which we are interested, B results from boundary value or initial-boundary value problems for PDEs or systems of PDEs. Y is the state space with the norm $\|\cdot\|_Y$, while $U = X^*$ denotes the control space with norm $\|\cdot\|_{X^*}$ which describes the cost of the control. In order to define an equivalent norm in X^* we consider a linear self-adjoint and elliptic operator $A : X \to X^*$ satisfying

$$\|Ax\|_{X^*} \leq c_2^A \, \|x\|_X, \quad \langle Ax, x\rangle_{X^*,X} \geq c_1^A \, \|x\|_X^2 \quad \text{for all } x \in X.$$

With this we define $\|x\|_A = \sqrt{\langle Ax, x\rangle_{X^*,X}}$ and $\|u\|_{A^{-1}} = \sqrt{\langle A^{-1}u, u\rangle_{X,X^*}}$ which are equivalent norms in X and X^*, respectively.

We first consider an abstract tracking type optimal control problem with neither state nor control constraints to find the minimizer $(y_\varrho, u_\varrho) \in Y \times U$ of the functional

$$\mathcal{J}(y_\varrho, u_\varrho) = \frac{1}{2}\, \|y_\varrho - \overline{y}\|_{H_Y}^2 + \frac{1}{2}\, \varrho \, \|u_\varrho\|_{A^{-1}}^2 \quad \text{s.t. } By_\varrho = u_\varrho \text{ in } U = X^*, \tag{2.1}$$

where $\varrho > 0$ is the cost or regularization parameter on which the solution depends, and $\overline{y} \in H_Y$ denotes the given target or desired state.

In the standard approach we consider the solution of the constraint equation $By_\varrho = u_\varrho$ which defines the control-to-state map $y_\varrho = B^{-1}u_\varrho$. With this we can write the reduced cost functional as

$$\widehat{J}(u_\varrho) = \frac{1}{2}\, \|B^{-1}u_\varrho - \overline{y}\|_{H_Y}^2 + \frac{1}{2}\, \varrho \, \|u_\varrho\|_{A^{-1}}^2,$$

and its minimizer $u_\varrho \in U = X^*$ is given as the unique solution of the gradient equation

$$B^{-1,*}(B^{-1}u_\varrho - \overline{y}) + \varrho \, A^{-1}u_\varrho \, = \, 0 \quad \text{in } X.$$

Since $B : Y \to X^*$ is an isomorphism, we can write the reduced cost functional as

$$\widetilde{\mathcal{J}}(y_\varrho) = \frac{1}{2}\, \|y_\varrho - \overline{y}\|_{H_Y}^2 + \frac{1}{2}\, \varrho \, \|By_\varrho\|_{A^{-1}}^2, \tag{2.2}$$

and its minimizer $y_\varrho \in Y$ is given as the unique solution of the gradient equation

$$y_\varrho + \varrho \, B^* A^{-1} B y_\varrho = \overline{y} \quad \text{in } Y^*. \tag{2.3}$$

The linear operator $S := B^* A^{-1} B : Y \to Y^*$ is self-adjoint and elliptic, i.e.,

$$\langle Sy, y\rangle_{Y^*,Y} \geq c_1^S \, \|y\|_Y^2 \quad \text{and} \quad \|Sy\|_{Y^*} \leq c_2^S \, \|y\|_Y \quad \text{for all } y \in Y,$$

with $c_1^S = c_1^A (c_1^B / c_2^A)^2$ and $c_2^S = (c_2^B)^2 / c_1^A$; see, e.g., [43, Lemma 1]. We note that, for $By_\varrho = u_\varrho$, we have

$$\|y_\varrho\|_S^2 = \langle Sy_\varrho, y_\varrho\rangle_{Y^*,Y} = \langle A^{-1}By_\varrho, By_\varrho\rangle_{X,X^*} = \langle A^{-1}u_\varrho, u_\varrho\rangle_{X,X^*} = \|u_\varrho\|_{A^{-1}}^2. \tag{2.4}$$

Moreover, for the solution $y_\varrho \in Y \subset H_Y$ of (2.3), we get

$$Sy_\varrho = \frac{1}{\varrho}\, (\overline{y} - y_\varrho) \in H_Y \, . \tag{2.5}$$

Using (2.4), we can rewrite the reduced cost functional (2.2) in the form

$$\widetilde{\mathcal{J}}(y_\varrho) = \frac{1}{2}\, \|y_\varrho - \overline{y}\|_{H_Y}^2 + \frac{1}{2}\, \varrho \, \|y_\varrho\|_S^2,$$

where the realization of $\|y_\varrho\|_S^2$ with $S = B^* A^{-1} B$ involves the inversion of A, which in general may complicate the numerical implementation. Hence we may replace $\|y_\varrho\|_S^2$ by any equivalent but more easier computable norm $\|y_\varrho\|_D^2 = \langle D y_\varrho, y_\varrho \rangle_{Y^*,Y}$ for some bounded and elliptic operator $D : X \to X^*$, and satisfying the norm equivalence inequalities

$$c_1^D \, \|y\|_S \leq \|y\|_D \leq c_2^D \, \|y\|_S \quad \text{for all } y \in Y.$$

We now minimize

$$\check{\mathcal{J}}(y_\varrho) = \frac{1}{2} \, \|y_\varrho - \overline{y}\|_{H_Y}^2 + \frac{1}{2} \, \varrho \, \|y_\varrho\|_D^2, \tag{2.6}$$

whose minimizer $y_\varrho \in Y$ is given as the unique solution satisfying

$$\langle y_\varrho, y \rangle_{H_Y} + \varrho \, \langle D y_\varrho, y \rangle_{Y^*,Y} = \langle \overline{y}, y \rangle_{H_Y} \quad \text{for all } y \in Y. \tag{2.7}$$

For the unique solution $y_\varrho \in Y$ of (2.7), and depending on the regularity of the target $\overline{y}$, we can state the following result for the regularization error $\|y_\varrho - \overline{y}\|_{H_Y}$.

Lemma 1. *Let $y_\varrho \in Y$ be the unique solution of the variational formulation* (2.7)*. For $\overline{y} \in H_Y$ there holds*

$$\|y_\varrho\|_{H_Y} \leq \|\overline{y}\|_{H_Y}, \quad \|y_\varrho\|_D \leq \varrho^{-1/2} \, \|\overline{y}\|_{H_Y}, \quad \|y_\varrho - \overline{y}\|_{H_Y} \leq \|\overline{y}\|_{H_Y},$$

while for $\overline{y} \in Y$ we have

$$\|y_\varrho - \overline{y}\|_D \leq \|\overline{y}\|_D, \quad \|y_\varrho - \overline{y}\|_{H_Y} \leq \varrho^{1/2} \, \|\overline{y}\|_D, \quad \|y_\varrho\|_D \leq \|\overline{y}\|_D.$$

If in addition $D\overline{y} \in H_Y$ is satisfied for $\overline{y} \in Y$,

$$\|y_\varrho - \overline{y}\|_{H_Y} \leq \varrho \, \|D\overline{y}\|_{H_Y}, \quad \|y_\varrho - \overline{y}\|_D \leq \varrho^{1/2} \, \|D\overline{y}\|_{H_Y}$$

follow.

Proof. For the particular test function $y = y_\varrho \in Y$ we first have

$$\langle y_\varrho, y_\varrho \rangle_{H_Y} + \varrho \, \langle D y_\varrho, y_\varrho \rangle_{Y^*,Y} = \langle \overline{y}, y_\varrho \rangle_{H_Y} \leq \|\overline{y}\|_{H_Y} \|y_\varrho\|_{H_Y},$$

i.e.,

$$\|y_\varrho\|_{H_Y} \leq \|\overline{y}\|_{H_Y}, \quad \|y_\varrho\|_D \leq \varrho^{-1/2} \, \|\overline{y}\|_{H_Y}.$$

Moreover, we also obtain

$$\varrho \, \|y_\varrho\|_D^2 = \varrho \, \langle D y_\varrho, y_\varrho \rangle_{Y^*,Y} = \langle \overline{y} - y_\varrho, y_\varrho \rangle_{H_Y} = \langle \overline{y} - y_\varrho, \overline{y} \rangle_{H_Y} - \langle \overline{y} - y_\varrho, \overline{y} - y_\varrho \rangle_{H_Y},$$

which gives

$$\|y_\varrho - \overline{y}\|_{H_Y}^2 + \varrho \, \|y_\varrho\|_D^2 = \langle \overline{y} - y_\varrho, \overline{y} \rangle_{H_Y} \leq \|\overline{y} - y_\varrho\|_{H_Y} \|\overline{y}\|_{H_Y},$$

i.e.,

$$\|y_\varrho - \overline{y}\|_{H_Y} \leq \|\overline{y}\|_{H_Y}.$$

When assuming $\overline{y} \in Y$, we can choose $y = \overline{y} - y_\varrho \in Y$ as test function to conclude

$$\begin{aligned} \|\overline{y} - y_\varrho\|_{H_Y}^2 &= \langle \overline{y} - y_\varrho, \overline{y} - y_\varrho \rangle_{H_Y} = \varrho \, \langle D y_\varrho, \overline{y} - y_\varrho \rangle_{Y^*,Y} \\ &= \varrho \, \langle D\overline{y}, \overline{y} - y_\varrho \rangle_{Y^*,Y} - \varrho \, \langle D(\overline{y} - y_\varrho), \overline{y} - y_\varrho \rangle_{Y^*,Y}, \end{aligned}$$

i.e.,

$$\|y_\varrho - \overline{y}\|_{H_Y}^2 + \varrho \, \|y_\varrho - \overline{y}\|_D^2 = \varrho \, \langle D\overline{y}, \overline{y} - y_\varrho \rangle_{H_Y} \leq \varrho \, \|\overline{y}\|_S \|y_\varrho - \overline{y}\|_D.$$

Hence,

$$\|y_\varrho - \overline{y}\|_D \le \|\overline{y}\|_D, \quad \|y_\varrho - \overline{y}\|_{H_Y} \le \varrho^{1/2}\,\|\overline{y}\|_D.$$

On the other hand, we can also write

$$\|\overline{y} - y_\varrho\|^2_{H_Y} = \varrho\,\langle D y_\varrho, \overline{y} - y_\varrho\rangle_{Y^*,Y} = \varrho\,\langle D y_\varrho, \overline{y}\rangle_{Y^*,Y} - \varrho\,\langle D y_\varrho, y_\varrho\rangle_{Y^*,Y},$$

i.e.,

$$\|\overline{y} - y_\varrho\|^2_{H_Y} + \varrho\,\|y_\varrho\|^2_D = \varrho\,\langle D y_\varrho, \overline{y}\rangle_{Y^*,Y} \le \varrho\,\|y_\varrho\|_D \|\overline{y}\|_D,$$

and hence,

$$\|y_\varrho\|_D \le \|\overline{y}\|_D.$$

Finally, if $D\overline{y} \in H_Y$,

$$\|y_\varrho - \overline{y}\|^2_{H_Y} + \varrho\,\|y_\varrho - \overline{y}\|^2_D = \varrho\,\langle D\overline{y}, \overline{y} - y_\varrho\rangle_{Y^*,Y} \le \varrho\,\|D\overline{y}\|_{H_Y} \|y_\varrho - \overline{y}\|_{H_Y},$$

i.e.,

$$\|y_\varrho - \overline{y}\|_{H_Y} \le \varrho\,\|D\overline{y}\|_{H_Y}, \quad \|y_\varrho - \overline{y}\|_D \le \varrho^{1/2}\,\|D\overline{y}\|_{H_Y}.$$

□

When using the results of Lemma 1 we obtain a bound for the costs $\|u_\varrho\|_{X^*}$ for the control $u_\varrho = B y_\varrho$, depending on the regularity of the target $\overline{y}$.

Corollary 1. *From the gradient equation $\varrho D y_\varrho + y_\varrho = \overline{y}$, see (2.7), we obtain*

$$\|D y_\varrho\|_{H_Y} = \frac{1}{\varrho}\,\|\overline{y} - y_\varrho\|_{H_Y} \le \begin{cases} \varrho^{-1}\,\|\overline{y}\|_{H_Y} & \text{for } \overline{y} \in H_Y, \\ \varrho^{-1/2}\,\|\overline{y}\|_D & \text{for } \overline{y} \in Y, \\ \|D\overline{y}\|_{H_Y} & \text{for } \overline{y} \in Y,\ D\overline{y} \in H_Y, \end{cases}$$

as well as

$$\|u_\varrho\|_{A^{-1}} = \|y_\varrho\|_S \le \frac{1}{c_1^D}\,\|y_\varrho\|_D \le \frac{1}{c_1^D} \begin{cases} \varrho^{-1/2}\,\|\overline{y}\|_{H_Y} & \text{for } \overline{y} \in H_Y, \\ \|\overline{y}\|_D & \text{for } \overline{y} \in Y. \end{cases}$$

In particular, for $\overline{y} \in Y$ the costs $\|u_\varrho\|_{A^{-1}}$ of the control $u_\varrho = B y_\varrho$ are uniformly bounded as $\varrho \to 0$. Moreover, $\|y_\varrho - \overline{y}\|_{H_Y} \to 0$ as $\varrho \to 0$ implies $u_\varrho \to \overline{u} = B\overline{y}$ in X^* in this case. However, in the more interesting case $\overline{y} \notin Y$ we conclude $B\overline{y} \notin U = X^*$, and $\|u_\varrho\|_{A^{-1}} \to \infty$ as $\varrho \to 0$. In this case we have to balance the regularization error $\|y_\varrho - \overline{y}\|_{H_Y}$ with the costs $\|u_\varrho\|_{A^{-1}}$ of the control $u_\varrho = B y_\varrho$ we are willing to pay.

In particular for less regular target functions, e.g., $\overline{y} \in Y$ but $D\overline{y} \notin H_Y$, or even $\overline{y} \notin Y$, we may include the regularization parameter ϱ in the definition of the regularization operator $D_\varrho : Y \to Y^*$. Instead of (2.7) we then consider the variational formulation to find $y_\varrho \in Y$ such that

$$\langle y_\varrho, y\rangle_{H_Y} + \langle D_\varrho y_\varrho, y\rangle_{Y^*,Y} = \langle \overline{y}, y\rangle_{H_Y} \quad \text{for all } y \in Y. \tag{2.8}$$

As in Lemma 1 we then conclude the regularization error estimates

$$\|y_\varrho - \overline{y}\|_{H_Y} \le \|\overline{y}\|_{H_Y} \quad \text{for } \overline{y} \in H_Y, \quad \|y_\varrho - \overline{y}\|_{H_Y} \le \|\overline{y}\|_{D_\varrho} \quad \text{for } \overline{y} \in Y. \tag{2.9}$$

2.2 Galerkin discretization and error estimates

For the discretization of the variational problem (2.7) we introduce a conforming finite-dimensional subspace $Y_h = \text{span}\{\varphi_i\}_{i=1}^M \subset Y$ spanned by the $M = M(h)$ basis functions $\varphi_1, \ldots, \varphi_M$. Here, h denotes some positive discretization parameter such that h tends to zero when $M = M(h)$ goes to infinity. For example, one can think about h being the mesh-size in a finite element discretization as considered in Section 3.

At this time we assume that for any $y \in Y$ there exists a projection $P_h y \in Y_h$ satisfying

$$\|y - P_h y\|_{H_Y} \leq c_1 \, h^\alpha \, \|y\|_D, \quad \|y - P_h y\|_D \, \leq \, c_2 \, \|y\|_D \, , \tag{2.10}$$

for some $\alpha > 0$, and with positive constants c_1, c_2. Moreover, if $Dy \in H_Y$ is satisfied for $y \in Y$, we also assume, for some positive constants c_3, c_4,

$$\|y - P_h y\|_{H_Y} \leq c_3 \, h^{2\alpha} \, \|Dy\|_{H_Y}, \quad \|y - P_h y\|_D \, \leq \, c_4 \, h^\alpha \, \|Dy\|_{H_Y} \, . \tag{2.11}$$

The Galerkin discretization of the variational formulation (2.7) reads as follows: Find the Galerkin approximation $y_{\varrho h} \in Y_h$ to the state $y_\varrho \in Y$ such that

$$\langle y_{\varrho h}, y_h \rangle_{H_Y} + \varrho \, \langle D y_{\varrho h}, y_h \rangle_{Y^*, Y} = \langle \overline{y}, y_h \rangle_{H_Y} \quad \text{for all } y_h \in Y_h. \tag{2.12}$$

When using standard arguments, we conclude unique solvability of (2.12) as well as Cea's lemma,

$$\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \varrho \, \|y_\varrho - y_{\varrho h}\|^2_D \leq \inf_{y_h \in Y_h} \Big[\|y_\varrho - y_h\|^2_{H_Y} + \varrho \, \|y_\varrho - y_h\|^2_D \Big], \tag{2.13}$$

and using (2.10) for $y = y_\varrho \in Y$, this gives

$$\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \varrho \, \|y_\varrho - y_{\varrho h}\|^2_D \leq \Big[c_1^2 \, h^{2\alpha} + c_2 \, \varrho \Big] \, \|y_\varrho\|^2_D \leq c \, h^{2\alpha} \, \|y_\varrho\|^2_D, \tag{2.14}$$

when choosing

$$\varrho = h^{2\alpha} \, . \tag{2.15}$$

Otherwise, if the regularisation or cost parameter ϱ is fixed, we can not expect any further convergence for small discretization parameters h satisfying $h^{2\alpha} < \varrho$.

However, depending on the regularity of the target $\overline{y}$ we can refine the error estimate (2.13) as follows:

Lemma 2. *Let $y_{\varrho h} \in Y_h$ be the unique solution of the Galerkin variational problem* (2.12)*. Then there holds the error estimate, when choosing $\varrho = h^{2\alpha}$,*

$$\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + h^{2\alpha} \, \|y_\varrho - y_{\varrho h}\|^2_D \leq c \left\{ \begin{array}{cl} \|\overline{y}\|^2_{H_Y} & \textit{for } \overline{y} \in H_Y, \\ h^{2\alpha} \, \|\overline{y}\|^2_D & \textit{for } \overline{y} \in Y, \\ h^{4\alpha} \, \|D\overline{y}\|^2_{H_Y} & \textit{for } \overline{y} \in Y, \, D\overline{y} \in H_Y. \end{array} \right. \tag{2.16}$$

Proof. For $\overline{y} \in H_Y$, the estimate follows from Cea's lemma (2.13) for the particular test function $v_h = 0$, and using Lemma 1 for $\overline{y} \in H_Y$,

$$\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \varrho \, \|y_\varrho - y_{\varrho h}\|^2_D \leq \|y_\varrho\|^2_{H_Y} + \varrho \, \|y_\varrho\|^2_D \leq 2 \, \|\overline{y}\|^2_{H_Y} .$$

For $\overline{y} \in Y$, using (2.13), the triangle inequality, Lemma 1, and (2.10), we have

$$\begin{aligned} \|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \varrho \, \|y_\varrho - y_{\varrho h}\|^2_D \quad & \leq \quad \inf_{y_h \in Y_h} \Big[\|y_\varrho - y_h\|^2_{H_Y} + \varrho \, \|y_\varrho - y_h\|^2_D \Big] \\ & \leq \; 2 \, \|y_\varrho - \overline{y}\|^2_{H_Y} + 2 \, \varrho \, \|y_\varrho - \overline{y}\|^2_D + 2 \inf_{y_h \in Y_h} \Big[\|\overline{y} - y_h\|^2_{H_Y} + \varrho \, \|\overline{y} - y_h\|^2_D \Big] \\ & \leq \; 4 \, \varrho \, \|\overline{y}\|^2_D + 2 \Big[\|\overline{y} - P_h \overline{y}\|^2_{H_Y} + \varrho \, \|\overline{y} - P_h \overline{y}\|^2_D \Big] \\ & \leq \; \Big[2 \, c_1^2 \, h^{2\alpha} + \varrho \, (4 + c_2^2) \Big] \, \|\overline{y}\|^2_D \\ & = \; c \, h^{2\alpha} \, \|\overline{y}\|^2_D \end{aligned}$$

when choosing $\varrho = h^{2\alpha}$. If in addition $S\overline{y} \in H_Y$ is satisfied, the proof of the third estimate follows the same lines. □

Since the Galerkin approximation $y_{\varrho h}$ to the state y_ϱ as solution of the minimization problem (2.6) is an approximation of the desired target $\overline{y}$, we are interested in estimates for the related error $y_{\varrho h} - \overline{y}$ in the H_Y norm.

Theorem 1. *Let $y_{\varrho h} \in Y_h$ be the unique solution of* (2.12)*, and choose $\varrho = h^{2\alpha}$. Then there hold the error estimates*

$$\|y_{\varrho h} - \overline{y}\|_{H_Y} \leq c \begin{cases} \|\overline{y}\|_{H_Y} & \text{for } \overline{y} \in H_Y, \\ h^\alpha \, \|\overline{y}\|_D & \text{for } \overline{y} \in Y, \\ h^{2\alpha} \, \|D\overline{y}\|_{H_Y} & \text{for } \overline{y} \in Y,\, D\overline{y} \in H_Y. \end{cases} \tag{2.17}$$

When using space interpolation arguments, and assuming $\overline{y} \in [H_Y, Y]_{|s}$ for some $s \in [0,1]$, we finally conclude the error estimate

$$\|y_{\varrho h} - \overline{y}\|_{H_Y} \leq c\, h^{\alpha s} \, \|\overline{y}\|_{[H_Y,Y]_{|s}} \quad \text{for } \overline{y} \in [H_Y, Y]_{|s},\; s \in [0,1]. \tag{2.18}$$

2.3 Control recovering

When an approximate optimal state $y_{\varrho h}$ is known we can compute the associated optimal control $\widetilde{u}_\varrho = B y_{\varrho h} \in U = X^*$ and an approximate control $\widetilde{u}_{\varrho h} \in U_h$ via post processing, where $U_h = \text{span}\{\psi_k\}_{k=1}^N \subset U$ is a suitable ansatz space. For this we consider the variational formulation to find $\widetilde{u}_{\varrho h} \in U_h$ such that

$$\langle \widetilde{u}_{\varrho h}, x_h \rangle_{X^*,X} = \langle B y_{\varrho h}, x_h \rangle_{X^*,X} \quad \text{for all } x_h \in X_h, \tag{2.19}$$

where $X_h = \text{span}\{\phi_k\}_{k=1}^N \subset X$ is a suitable test space. As in (2.10) we assume that there exists a projection operator $\Pi_h : X \to X_h$ satisfying the error estimate

$$\|x - \Pi_h x\|_X \leq c\, h^\alpha \, \|B^* x\|_{H_Y}. \tag{2.20}$$

In order to ensure unique solvability of the Galerkin–Petrov variational formulation (2.19) we need to assume the discrete inf-sup stability condition

$$c_S \, \|u_h\|_{X^*} \leq \sup_{0 \neq x_h \in X_h} \frac{\langle u_h, x_h \rangle_{X^*,X}}{\|x_h\|_X} \quad \text{for all } u_h \in U_h. \tag{2.21}$$

In addition to (2.19) we consider the variational formulation to find $u_{\varrho h} \in U_h$ such that

$$\langle u_{\varrho h}, x_h \rangle_{X^*,X} = \langle u_\varrho, x_h \rangle_{X^*,X} = \langle B y_\varrho, x_h \rangle_{X^*,X} \quad \text{for all } x_h \in X_h,$$

and we conclude the perturbed Galerkin orthogonality

$$\langle u_{\varrho h} - \widetilde{u}_{\varrho h}, x_h \rangle_{X^*,X} = \langle B(y_\varrho - y_{\varrho h}), x_h \rangle_{X^*,X} \quad \text{for all } x_h \in X_h.$$

Moreover, using standard arguments, we conclude the error estimate

$$\|u_\varrho - u_{\varrho h}\|_{X^*} \leq \frac{1}{c_S} \inf_{u_h \in U_h} \|u_\varrho - u_h\|_{X^*} \leq \frac{1}{c_S} \|u_\varrho\|_{X^*} = \frac{1}{c_S} \|B y_\varrho\|_{X^*} \leq \frac{c_2^B}{c_S} \|y_\varrho\|_Y.$$

Now, using the discrete inf-sup stability condition (2.21) as well as Cea's lemma (2.13) for $y_h = P_h y_\varrho$ we obtain, choosing $\varrho = h^{2\alpha}$,

$$\begin{aligned}
c_S \, \|u_{\varrho h} - \widetilde{u}_{\varrho h}\|_{X^*} &\leq \sup_{0 \neq x_h \in X_h} \frac{\langle u_{\varrho h} - \widetilde{u}_{\varrho h}, x_h \rangle_{X^*,X}}{\|x_h\|_X} \\
&= \sup_{0 \neq x_h \in X_h} \frac{\langle B(y_\varrho - y_{\varrho h}), x_h \rangle_{X^*,X}}{\|x_h\|_X} \\
&\leq \|B(y_\varrho - y_{\varrho h})\|_{X^*} \, \leq \, c_2^B \, \|y_\varrho - y_{\varrho h}\|_Y \, \leq \, \frac{c_2^B}{c_1^D} \, \|y_\varrho - y_{\varrho h}\|_D \\
&\leq \frac{c_2^B}{c_1^D} \sqrt{\frac{1}{\varrho} \, \|y_\varrho - P_h y_\varrho\|^2_{H_Y} + \|y_\varrho - P_h y_\varrho\|^2_D} \\
&\leq \frac{c_2^B}{c_1^D} \sqrt{\frac{1}{\varrho} \, c_1^2 \, h^{2\alpha} \, \|y_\varrho\|^2_D + c_2^2 \, \|y_\varrho\|^2_D} \, = \, c \, \|y_\varrho\|_D \, .
\end{aligned}$$

Therefore,

$$\|u_\varrho - \widetilde{u}_{\varrho h}\|_{X^*} \, \leq \, \|u_\varrho - u_{\varrho h}\|_{X^*} + \|u_{\varrho h} - \widetilde{u}_{\varrho h}\|_{X^*} \, \leq \, c \, \|y_\varrho\|_D \, .$$

follows.

Lemma 3. *Let the assumptions* (2.20) *and* (2.21) *hold true. Further, let $y_\varrho \in Y$ be the unique solution of the variational formulation* (2.7)*, and let $u_\varrho = B y_\varrho \in U = X^*$ be the associated control. For $y_{\varrho h} \in Y_h$ being the unique solution of the Galerkin variational formulation* (2.12) *we compute $\widetilde{u}_{\varrho h} \in U_h$ as unique solution of the Galerkin–Petrov variational formulation* (2.19)*. For this approximate control $\widetilde{u}_{\varrho h} \in U_h$ we obtain the associated state $\widetilde{y}_\varrho = B^{-1} \widetilde{u}_{\varrho h}$. For $\overline{y} \in H_Y$ we then have the error estimate*

$$\|\widetilde{y}_\varrho - \overline{y}\|_{H_Y} \leq c \, \|\overline{y}\|_{H_Y} , \tag{2.22}$$

while for $\overline{y} \in Y$ we have

$$\|\widetilde{y}_\varrho - \overline{y}\|_{H_Y} \leq c \, h^\alpha \, \|\overline{y}\|_Y \tag{2.23}$$

when choosing $\varrho = h^{2\alpha}$, and where α is given by the approximation property (2.10)*.*

Proof. For any $\psi \in H_Y \subset Y^*$ we first define $x_\psi \in X$ as unique solution of the operator equation $B^* x_\psi = \psi$. With this we obtain

$$\begin{aligned}
\|\widetilde{y}_\varrho - y_\varrho\|_{H_Y} &= \sup_{0 \neq \psi \in H_Y} \frac{\langle \widetilde{y}_\varrho - y_\varrho, \psi \rangle_{H_Y}}{\|\psi\|_{H_Y}} = \sup_{0 \neq \psi \in H_Y} \frac{\langle \widetilde{y}_\varrho - y_\varrho, B^* x_\psi \rangle_{H_Y}}{\|\psi\|_{H_Y}} \\
&= \sup_{0 \neq \psi \in H_Y} \frac{\langle B\widetilde{y}_\varrho - B y_\varrho, x_\psi \rangle_{X^*,X}}{\|\psi\|_{H_Y}} = \sup_{0 \neq \psi \in H_Y} \frac{\langle \widetilde{u}_{\varrho h} - u_\varrho, x_\psi \rangle_{X^*,X}}{\|\psi\|_{H_Y}} \\
&= \sup_{0 \neq \psi \in H_Y} \frac{\langle \widetilde{u}_{\varrho h} - u_\varrho, x_\psi - \Pi_h x_\psi \rangle_{X^*,X} + \langle \widetilde{u}_{\varrho h} - u_\varrho, \Pi_h x_\psi \rangle_{X^*,X}}{\|\psi\|_{X^*,X}} \\
&= \sup_{0 \neq \psi \in H_Y} \frac{\langle \widetilde{u}_{\varrho h} - u_\varrho, x_\psi - \Pi_h x_\psi \rangle_{X^*,X} + \langle B(y_{\varrho h} - y_\varrho), \Pi_h x_\psi \rangle_{X^*,X}}{\|\psi\|_{H_Y}} \, .
\end{aligned}$$

With

$$\begin{aligned}
\langle \widetilde{u}_{\varrho h} - u_\varrho, x_\psi - \Pi_h x_\psi \rangle_{X^*,X} &\leq \|\widetilde{u}_{\varrho h} - u_\varrho\|_{X^*} \|x_\psi - \Pi_h x_\psi\|_X \\
&\leq c \, h^\alpha \, \|y_\varrho\|_D \|B^* x_\psi\|_{H_Y} \\
&= c \, h^\alpha \, \|y_\varrho\|_D \|\psi\|_{H_Y} ,
\end{aligned}$$

and

$$
\begin{aligned}
&\langle B(y_{\varrho h}-y_\varrho), \Pi_h x_\psi\rangle_{X^*,X} \\
&\quad = \langle B(y_{\varrho h}-y_\varrho), \Pi_h x_\psi - x_\psi\rangle_{X^*,X} + \langle B(y_{\varrho h}-y_\varrho), x_\psi\rangle_{X^*,X} \\
&\quad \le \|B(y_{\varrho h}-y_\varrho)\|_{X^*}\|\Pi_h x_\psi - x_\psi\|_X + \langle y_{\varrho h}-y_\varrho, B^* x_\psi\rangle_{H_Y} \\
&\quad \le c_2^B\,\|y_{\varrho h}-y_\varrho\|_Y c\, h^\alpha \|B^* x_\psi\|_{H_Y} + \|y_{\varrho h}-y_\varrho\|_{H_Y}\|\psi\|_{H_Y} \\
&\quad = c\, h^\alpha\, \|y_{\varrho h}-y_\varrho\|_Y \|\psi\|_{H_Y} + \|y_{\varrho h}-y_\varrho\|_{H_Y}\|\psi\|_{H_Y}
\end{aligned}
$$

we conclude

$$\|\widetilde{y}_\varrho - y_\varrho\|_{H_Y} \le \|y_{\varrho h}-y_\varrho\|_{H_Y} + c\, h^\alpha \Big[\|y_\varrho\|_D + \|y_\varrho - y_{\varrho h}\|_D\Big],$$

and by the triangle inequality we have

$$\|\widetilde{y}_\varrho - \overline{y}\|_{H_Y} \le \|y_\varrho - \overline{y}\|_{H_Y} + \|\widetilde{y}_\varrho - y_\varrho\|_{H_Y}.$$

For $\overline{y} \in H_Y$ we finally conclude

$$\|\widetilde{y}_\varrho - y_\varrho\|_{H_Y} \le c\left(1 + h^\alpha\, \varrho^{-1/2}\right)\|\overline{y}\|_{H_Y},$$

while for $\overline{y} \in Y$ we have

$$\|\widetilde{y}_\varrho - y_\varrho\|_{H_Y} \le c\left(h^\alpha + h^{2\alpha}\, \varrho^{-1/2}\right)\|\overline{y}\|_D.$$

Now the assertion follows for $\varrho = h^{2\alpha}$. □

Using (2.22) and (2.23), and as in (2.18) we can use space interpolation arguments to derive the final result of this subsection.

Theorem 2. *Let the discrete state $y_{\varrho h} \in Y_h$ be the unique finite element solution of the variational formulation* (2.12) *with $\varrho = h^{2\alpha}$, where we assume $\overline{y} \in [H_Y, Y]_{|s}$ for some $s \in [0,1]$. Let the discrete control $\widetilde{u}_{\varrho h} \in U_h$ be the unique solution of* (2.19). *For the resulting state $\widetilde{y}_\varrho = B\widetilde{u}_{\varrho h} \in Y$ we then have the error estimate*

$$\|\widetilde{y}_\varrho - \overline{y}\|_{H_Y} \;\le\; c\, h^{\alpha s}\, \|\overline{y}\|_{[H_Y, Y]_{|s}}. \tag{2.24}$$

The former estimate relates the resulting state to the target, showing the accuracy of the method. In addition it is important to control the costs. Therefore we need to have a computable bound for $\|\widetilde{u}_{\varrho h}\|_{X^*}$.

Lemma 4. *Let $\widetilde{u}_{\varrho h} \in U_h$ be the unique solution of* (2.19), *and let* (2.21) *hold true. Then*

$$\|\widetilde{u}_{\varrho h}\|_{X^*} \le c\, \|y_{\varrho h}\|_D.$$

If, in addition, the discrete inf-sup stability

$$\widetilde{c}_S\, \|y_h\|_D \le \sup_{0 \ne x_h \in X_h} \frac{\langle B y_h, x_h\rangle_{X^*,X}}{\|x_h\|_X}, \quad \forall y_h \in Y_h, \tag{2.25}$$

holds, then

$$\|y_{\varrho h}\|_D \le \widetilde{c}_S\, \|\widetilde{u}_{\varrho h}\|_{X^*}.$$

Proof. We compute, using (2.21) and (2.19)

$$c_S\, \|\widetilde{u}_{\varrho h}\|_{X^*} \le \sup_{0 \ne x_h \in X_h} \frac{\langle \widetilde{u}_{\varrho h}, x_h\rangle_{X^*,X}}{\|x_h\|_X} = \sup_{0 \ne x_h \in X_h} \frac{\langle B y_{\varrho h}, x_h\rangle_{X^*,X}}{\|x_h\|_X} \le c\, \|y_{\varrho h}\|_D.$$

The second estimate follows the same lines, using (2.25) and (2.19), i.e.,

$$\widetilde{c}_S \|y_{\varrho h}\|_D \le \sup_{0 \ne x_h \in X_h} \frac{\langle B y_{\varrho h}, x_h\rangle_{X^*,X}}{\|x_h\|_X} = \sup_{0 \ne x_h \in X_h} \frac{\langle \widetilde{u}_{\varrho h}, x_h\rangle_{X^*,X}}{\|x_h\|_X} \le \|\widetilde{u}_{\varrho h}\|_{X^*}.$$

□

The last question to be answered is: How well does $\|\widetilde{u}_{\varrho h}\|_{X^*}$ approximate the actual cost $\|u_\varrho\|_{X^*}$? To deliver a satisfactory response, let us introduce the projection $Q_h : H_X \to X_h$ defined as

$$\langle Q_h u, x_h\rangle_{H_X} = \langle u, x_h\rangle_{H_X}, \quad \forall x_h \in X_h$$

and let us make the following assumptions:

i. $\dim(U_h) = \dim(X_h)$ and $Q_h : U_h \to X_h$ is uniformly bounded from below, i.e.,

$$\|Q_h u_h\|_{H_X} \geq c_Q \|u_h\|_{H_X}, \quad \forall u_h \in U_h.$$

Then, Q_h admits a bounded inverse $Q_h^{-1} : X_h \to U_h$ with $\|Q_h^{-1}\| \leq c_Q^{-1}$.

ii. The projection operator $\Pi_h : X \to X_h$ satisfies

$$\|x - \Pi_h x\|_{H_X} \leq c\, h^{2\alpha} \|x\|_D, \quad \forall x \in X \quad \text{for some } \alpha > 0.$$

iii. There holds an inverse inequality

$$\|x_h\|_D \leq c_I\, h^{-\alpha} \|x_h\|_{H_X}, \quad \forall x_h \in X_h \quad \text{for some } \alpha > 0. \tag{2.26}$$

iv. If $Dy_\varrho \in H_Y$ then $u_\varrho = By_\varrho \in H_X$ and there holds the estimate

$$\|Dy_\varrho\|_{H_Y} \leq c\, \|u_\varrho\|_{H_X}. \tag{2.27}$$

Then we can prove the following estimate.

Theorem 3. *For arbitrary but fixed $\varrho > 0$ let $u_\varrho = By_\varrho \in H_X$ and let $\widetilde{u}_{\varrho h} \in U_h$ be the unique solution of* (2.19)*. Then,*

$$\|u_\varrho - \widetilde{u}_{\varrho h}\|_{X^*} \leq c\, h^\alpha \|u_\varrho\|_{H_X} \quad \textit{for some } \alpha > 0.$$

Proof. By assumption *iv.*, $Dy_\varrho \in H_Y$, and using Cea's lemma (2.13) and (2.27) we get

$$\|y_\varrho - y_{\varrho h}\|_D \leq \left(c_4 h^{2\alpha} + c_3 \frac{h^{4\alpha}}{\varrho}\right)^{1/2} \|Dy_\varrho\|_{H_Y} \leq c\, h^\alpha \left(c_4 + c_3 \frac{h^{2\alpha}}{\varrho}\right)^{1/2} \|u_\varrho\|_{H_X}. \tag{2.28}$$

Further, it holds that

$$\begin{aligned} \|u_\varrho - \widetilde{u}_{\varrho h}\|_{X^*} &= \sup_{0 \neq x \in X} \frac{\langle u_\varrho - \widetilde{u}_{\varrho h}, x\rangle_{X^*,X}}{\|x\|_X} \\ &= \sup_{0 \neq x \in X} \left(\frac{\langle u_\varrho - \widetilde{u}_{\varrho h}, x - \Pi_h x\rangle_{X^*,X}}{\|x\|_X} + \frac{\langle u_\varrho - \widetilde{u}_{\varrho h}, \Pi_h x\rangle_{X^*,X}}{\|x\|_X} \right). \end{aligned}$$

For the second term we can estimate, using (2.28),

$$\begin{aligned} \langle u_\varrho - \widetilde{u}_{\varrho h}, \Pi_h x\rangle_{X^*,X} &= \langle B(y_\varrho - y_{\varrho h}), \Pi_h x\rangle_{X^*,X} \leq \|y_\varrho - y_{\varrho h}\|_D \|\Pi_h x\|_D \\ &\leq c\, \|y_\varrho - y_{\varrho h}\|_D \|x\|_D \leq c\, h^\alpha \left(c_4 + c_3 \frac{h^{2\alpha}}{\varrho}\right)^{1/2} \|u_\varrho\|_{H_X} \|x\|_X. \end{aligned}$$

For the first term we have

$$\langle u_\varrho - \widetilde{u}_{\varrho h}, x - \Pi_h x\rangle_{X^*,X} \leq \|u_\varrho - \widetilde{u}_{\varrho h}\|_{H_X} \|x - \Pi_h x\|_{H_X} \leq c h^\alpha \|u_\varrho - \widetilde{u}_{\varrho h}\|_{H_X} \|x\|_X,$$

and further

$$\|u_\varrho - \widetilde{u}_{\varrho h}\|_{H_X} \leq c \Big(\|u_\varrho\|_{H_X} + \|\widetilde{u}_{\varrho h}\|_{H_X} \Big).$$

So, it remains to bound

$$\|\widetilde{u}_{\varrho h}\|_{H_X} \le c\, \|u_\varrho\|_{H_X}.$$

Therefore, we first consider the case $U_h = X_h$ and estimate, using (2.19), (2.28) and the inverse inequality (2.26),

$$\begin{aligned}
\|\widetilde{u}_{\varrho h}\|_{H_X}^2 &= \langle \widetilde{u}_{\varrho h}, \widetilde{u}_{\varrho h}\rangle_{H_X} = \langle B y_{\varrho h}, \widetilde{u}_{\varrho h}\rangle_{H_X} = \langle B(y_{\varrho h} - y_\varrho), \widetilde{u}_{\varrho h}\rangle_{H_X} + \langle u_\varrho, \widetilde{u}_{\varrho h}\rangle_{H_X} \\
&\le \|y_\varrho - y_{\varrho h}\|_D \|\widetilde{u}_{\varrho h}\|_D + \|u_\varrho\|_{H_X} \|\widetilde{u}_{\varrho h}\|_{H_X} \\
&\le c\, h^\alpha \left(c_4 + c_3 \frac{h^{2\alpha}}{\varrho} \right) \|u_\varrho\|_{H_X}\, c_I h^{-\alpha} \|\widetilde{u}_{\varrho h}\|_{H_X} + \|u_\varrho\|_{H_X} \|\widetilde{u}_{\varrho h}\|_{H_X} \\
&\le c\, \|u_\varrho\|_{H_X} \|\widetilde{u}_{\varrho h}\|_{H_X}.
\end{aligned}$$

Now, if $U_h \neq X_h$, we we define $\widehat{u}_{\varrho h} = Q_h \widetilde{u}_{\varrho h} \in X_h$, which satisfies

$$\langle \widehat{u}_{\varrho h}, x_h\rangle_{H_X} = \langle Q_h \widetilde{u}_{\varrho h}, x_h\rangle_{H_X} = \langle \widetilde{u}_{\varrho h}, x_h\rangle_{X^*,X} = \langle B y_{\varrho h}, x_h\rangle_{X^*,X},\ \forall x_h \in X_h.$$

Recalling, that $Q_h : U_h \to X_h$ is boundedly invertible and $\widetilde{u}_{\varrho h} = Q_h^{-1} \widehat{u}_{\varrho h}$, we have

$$\|\widetilde{u}_{\varrho h}\|_{H_X} = \|Q_h^{-1} \widehat{u}_{\varrho h}\|_{H_X} \le \|Q_h^{-1}\| \|\widehat{u}_{\varrho h}\|_{H_X} \le c_Q^{-1}\, \|u_\varrho\|_{H_X}.$$

Thus, we can replace $\widetilde{u}_{\varrho h}$ by $\widehat{u}_{\varrho h}$ in the above derivation, which finishes the proof. □

2.4 Solvers and their use in nested iteration

Once the basis is chosen, the Galerkin variational formulation (2.12) is equivalent to the following spd linear system of algebraic equations: Find $\mathbf{y}_{\varrho h} = (y_1, \ldots, y_M)^\top \in \mathbb{R}^M$ solving the spd system

$$(\mathbf{M}_h + \varrho\, \mathbf{D}_h) \mathbf{y}_{\varrho h} = \overline{\mathbf{y}}_h, \tag{2.29}$$

where $\mathbf{M}_h = (\langle \varphi_j, \varphi_i\rangle_{H_Y})_{i,j=1,\ldots,M}$ and $\mathbf{D}_h = (\langle D\varphi_j, \varphi_i\rangle_{H_Y})_{i,j=1,\ldots,M}$ are $M \times M$ spd matrices, while the vector $y_h = (\langle \overline{y}, \varphi_i\rangle_{H_Y})_{i=1,\ldots,M} \in \mathbb{R}^M$ is defined by the given target $\overline{y} \in H_Y$. Thus, the solution $\mathbf{y}_{\varrho h} = (y_1, \ldots, y_M)^\top \in \mathbb{R}^M$ of (2.29) provides the coefficients for the Galerkin solution $y_{\varrho h} = \sum_{j=1}^M y_j \varphi_j \in Y_h \subset Y$ of (2.12) via Galerkin's isomorphism $y_{\varrho h} \leftrightarrow \mathbf{y}_{\varrho h}$.

Let us choose the optimal regularization parameter $\varrho = h^{2\alpha}$, and let $\mathbf{C}_h$ be an asymptotically optimal spd preconditioner for the spd system matrix $\mathbf{M}_h + \varrho\, \mathbf{D}_h$ of (2.29), i.e. there are positive, h respectively M_h independent spectral constants c_1 and c_2 such that the spectral equivalence inequalities

$$c_1\, \mathbf{C}_h \le \mathbf{S}_{\varrho h} = \mathbf{M}_h + \varrho\, \mathbf{D}_h \le c_2\, \mathbf{C}_h \tag{2.30}$$

hold, and the action $\mathbf{C}_h^{-1} \mathbf{r}_h$ is of asymptotically optimal algebraic complexity $O(M_h)$, where the best spectral constants c_1 and c_2 can be characterized by the minimal eigenvalue $\lambda_{\min}(\mathbf{C}_h^{-1} \mathbf{S}_{\varrho h})$ and the maximal eigenvalue $\lambda_{\max}(\mathbf{C}_h^{-1} \mathbf{S}_{\varrho h})$ of $\mathbf{C}_h^{-1} \mathbf{S}_{\varrho h}$, respectively. Then the algebraic system (2.29) can be solved by the pcg method to a given relative accuracy $\varepsilon \in (0,1)$ in the $\mathbf{S}_{\varrho h}$ energy norm with asymptotically optimal algebraic complexity $\mathcal{O}(M_h)$ provided that the multiplication of the system matrix $\mathbf{S}_{\varrho h}$ with a vector is of asymptotically optimal complexity too. More precisely, after n pcg iterations, we get the iteration error estimate

$$\|\mathbf{y}_{\varrho h} - \mathbf{y}_{\varrho h}^n\|_{\mathbf{S}_{\varrho h}} \le q_n\, \|\mathbf{y}_{\varrho h} - \mathbf{y}_{\varrho h}^0\|_{\mathbf{S}_{\varrho h}}, \tag{2.31}$$

in the $\mathbf{S}_{\varrho h}$ energy norm $\|\cdot\|_{\mathbf{S}_{\varrho h}} = (\mathbf{S}_{\varrho h}\cdot, \cdot)^{1/2}$, where $\mathbf{y}_{\varrho h}$, $\mathbf{y}_{\varrho h}^n$, and $\mathbf{y}_{\varrho h}^0$ denote the exact solution of (2.29), the nth pcg iterate, and the initial guess, respectively.

The reduction factor q_n after n pcg iterations is given by $q_n = 2q^n/(1+q^{2n})$ with $q = ((\mathrm{cond}_2(\mathbf{C}_h^{-1}\mathbf{S}_{\varrho h}))^{1/2}-1)/((\mathrm{cond}_2(\mathbf{C}_h^{-1}\mathbf{S}_{\varrho h}))^{1/2}+1) < 1$ and $\mathrm{cond}_2(\mathbf{C}_h^{-1}\mathbf{S}_{\varrho h}) = \lambda_{\max}(\mathbf{C}_h^{-1}\mathbf{S}_{\varrho h})/\lambda_{\min}(\mathbf{C}_h^{-1}\mathbf{S}_{\varrho h}) \le c_2/c_1$. The proofs of these well-known results on pcg can be found in the standard literature; see, e.g., [26, Chapter 10], or [60, Chapter 13].

The Petrov–Galerkin scheme (2.19) allows us to recover the control $\widetilde{u}_{\varrho h} \in U_h$ from the computed state $y_{\varrho h} \in Y_h$. Determining the solution $\widetilde{u}_{\varrho h} = \sum_{j=1}^N u_j \psi_j$ of the Petrov–Galerkin scheme (2.19) is equivalent to the solution of the following system of algebraic equations: Find the coefficient vector $\widetilde{\mathbf{u}}_{\varrho h} = (u_1, \ldots, u_N)^\top \in \mathbb{R}^N$ such that

$$\overline{\mathbf{M}}_h \widetilde{\mathbf{u}}_{\varrho h} = \mathbf{B}_h \mathbf{y}_{\varrho h} \tag{2.32}$$

where $\overline{\mathbf{M}}_h = (\langle \psi_j, \phi_i \rangle_{H_X})_{i,j=1,\ldots,N}$ and $\mathbf{B}_h = (\langle B\varphi_j, \phi_i \rangle_{H_X})_{i=1,\ldots,M;\, j=1,\ldots,N}$ are $N \times N$ and $M \times N$ matrices, respectively. Due to the discrete inf-sup condition (2.21), the $N \times N$ matrix $\overline{\mathbf{M}}_h$ is always regular, but in general neither symmetric nor positive definite. If we would choose $U_h = X_h = \mathrm{span}\{\phi_k\}_{k=1}^N \subset X$, then $\overline{\mathbf{M}}_h$ is spd as Gram matrix, but then the computed control is in general too smooth. Nonetheless, in our application presented in Section 3, we can choose different spaces U_h and X_h such that the inf-sup condition (2.21) is satisfied and $\overline{\mathbf{M}}_h$ is spd at the same time. Then system (2.32) can efficiently be solved by pcg.

In practice, these solvers should be used within a nested iteration procedure on a sequence of refined finite dimensional spaces with growing dimensions $M_1 < \cdots < M_\ell < \cdots < M_L$ respectively $N_1 < \cdots < N_\ell < \cdots < N_L$ related to shrinking discretization parameters (mesh sizes) $h_1 > \cdots > h_\ell > \cdots > h_L$ such that h_L goes to zero and M_L, N_L go to infinity as L tends to infinity. At some $h = h_\ell$, this nested iteration should produce

- a control $\widetilde{u}_{\varrho h}$ such that $\|\widetilde{u}_{\varrho h}\|_U = \|\widetilde{u}_{\varrho h}\|_{A^{-1}} < c_{\text{cost}}$, and
- the corresponding state $\widetilde{y}_\varrho = B^{-1}\widetilde{u}_{\varrho h}$ satisfying $\|\widetilde{y}_\varrho - \overline{y}\|_{H_Y} \le \varepsilon \, \|\overline{y}\|_{[H_Y, Y]_{|s}}$

in asymptotically optimal arithmetical complexity $\mathcal{O}(M_h)$ with a given "budget" $c_{\text{cost}} > 0$ and given accuracy $\varepsilon = 10^{-p} < 1$, where $\varrho = h^{2\alpha}$. We note that the control $\widetilde{u}_{\varrho h}$ will be recovered from the computed state $y_{\varrho h}$, and $\widetilde{y}_\varrho = B^{-1}\widetilde{u}_{\varrho h}$ satisfies the estimate (2.24). So, for sufficiently small h, we have $ch^{\alpha s} \le \varepsilon$. We further note that, in practice, $y_{\varrho h} \leftrightarrow \mathbf{y}_{\varrho h}$ and $\widetilde{u}_{\varrho h} \leftrightarrow \widetilde{\mathbf{u}}_{\varrho h}$ will be computed by solving the algebraic systems (2.29) and (2.32), respectively.

Algorithm 1 summarizes the accuracy and cost controlled nested iteration procedure described above. The subindex ℓ always indicates the refinement level, i.e. $\mathbf{M}_\ell$ stands for $\mathbf{M}_{h_\ell}$, $\mathbf{K}_\ell$ for $\mathbf{K}_{\varrho, h_\ell}$ with $\varrho = h_\ell^{2\alpha}$ etc. For $\ell = 1$, systems (2.29) and (2.32) can be solved directly as indicated in the comments at lines 7 and 8, but their iterative solution starting with zero initial guesses is also possible provided that appropriate preconditioners $\mathbf{C}_1$ and $\overline{\mathbf{C}}_1$ are available. Since good initial guesses are available for $\ell = 2, \cdots, L$, the algebraic systems (2.29) and (2.32) should be solved by preconditioned iterative methods with appropriate preconditioners $\mathbf{C}_\ell$ and $\overline{\mathbf{C}}_\ell$. In Subsection 3.6, we show that pcg can be used not only for solving (2.29) but also (2.32) with simple diagonal preconditioners $\mathbf{C}_\ell$ and $\overline{\mathbf{C}}_\ell$ obtained from lumping the corresponding mass matrices $\mathbf{M}_\ell$ and $\overline{\mathbf{M}}_\ell$.

Remark 1. *The energy cost $\|\widetilde{u}_\ell\|^2_{U=X^*} = \|\widetilde{u}_\ell\|^2_{A^{-1}} = \langle A^{-1}\widetilde{u}_\ell, \widetilde{u}_\ell \rangle_{X,X^*}$ in Line 16 of Algorithm 1 is in general not computable exactly, but we can efficiently compute a good upper bound as follows:*

$$\|\widetilde{u}_\ell\|^2_U = \langle w, \widetilde{u}_\ell \rangle_{X,X^*} = \langle w, \widetilde{u}_\ell \rangle_{H_X} \le \|w\|_{H_X} \|\widetilde{u}_\ell\|_{H_X} \le c_F^2 \|\widetilde{u}_\ell\|^2_{H_X}, \tag{2.33}$$

where we used that $w = A^{-1}\widetilde{u}_\ell \in X$ solves the variational equation

$$\langle Aw, v \rangle_{X^*,X} = \langle \widetilde{u}_\ell, v \rangle_{X^*,X} = \langle \widetilde{u}_\ell, v \rangle_{H_X} \; \forall v \in X \subset H_X,$$

Algorithm 1: Accuracy and cost controlled nested iteration.

1 **for** $\ell = 1, \ldots, L$ **do**
2   Generate $\mathbf{M}_\ell$, $\mathbf{D}_\ell$, $\overline{\mathbf{y}}_\ell$ /* (2.29) */
3   $\varrho_\ell \leftarrow h_\ell^{2\alpha}$ /* optimal regularization */
4   $\mathbf{S}_\ell \leftarrow \mathbf{M}_\ell + \varrho \mathbf{D}_\ell$ /* (2.29) */
5   Generate $\overline{\mathbf{M}}_\ell$, $\mathbf{B}_\ell$ /* (2.32) */
6   **if** $\ell = 1$ **then**
7     $\mathbf{y}_\ell \leftarrow \mathbf{S}_\ell^{-1} \overline{\mathbf{y}}_\ell$ /* solve (2.29) directly */
8     $\widetilde{\mathbf{u}}_\ell \leftarrow \overline{\mathbf{M}}_\ell^{-1} \mathbf{B}_\ell \mathbf{y}_\ell$ /* solve (2.32) directly */
9   **else**
10     $\mathbf{y}_\ell \leftarrow \mathbf{I}_{\ell-1}^{\ell} \mathbf{y}_{\ell-1}$ /* prolongation of the state */
  /* as initial guess for the iteration */
11     $\mathbf{y}_\ell \leftarrow \mathbf{S}_\ell^{-1} \overline{\mathbf{y}}_\ell$ /* solve (2.29) iteratively */
12     $\widetilde{\mathbf{u}}_\ell \leftarrow \mathbf{I}_{\ell-1}^{\ell} \widetilde{\mathbf{u}}_{\ell-1}$ /* prolongation of the control */
  /* as initial guess for the iteration */
13     $\widetilde{\mathbf{u}}_\ell \leftarrow \overline{\mathbf{M}}_\ell^{-1} \mathbf{B}_\ell \mathbf{y}_\ell$ /* solve (2.32) iteratively */
14   **end if**
15   $e_\ell \leftarrow \|\mathbf{y}_\ell - \overline{\mathbf{y}}_\ell\|_{\mathbf{M}_\ell} = \|y_\ell - \overline{y}_\ell\|_{H_Y}$ /* discretization error */
16   $c_\ell = \|\widetilde{u}_\ell\|_U^2$ /* energy cost of the control */
17   **if** $c_\ell > c_{cost}$ **then**
18   **STOP** and **return** $c_\ell, \widetilde{\mathbf{u}}_\ell, e_\ell, \mathbf{y}_\ell$ /* cost test */
19   **end if**
20   **if** $e_\ell \le \varepsilon \|\overline{\mathbf{y}}_\ell\|_{H_Y}$ **then**
21   **STOP** and **return** $c_\ell, \widetilde{\mathbf{u}}_\ell, e_\ell, \mathbf{y}_\ell$ /* accuracy test */
22   **end if**
23 **end for**
24 **return** $c_L, \mathbf{y}_L, e_L, \mathbf{y}_L$

$\|w\|_{H_X} \le c_F \|w\|_A$ (abstract Friedrichs' inequality), and $\|w\|_A \le c_F \|\widetilde{u}_\ell\|_{H_X}$. Thus, we can replace $c_\ell = \|\widetilde{u}_\ell\|_U^2$ by the easily computable cost $c_\ell = \|\widetilde{u}_\ell\|_{H_X}^2 = (\widehat{\mathbf{M}}_\ell \widetilde{\mathbf{u}}_\ell, \widetilde{\mathbf{u}}_\ell)$ in Line 16, and $c_\ell > c_{cost}$ by $c_\ell > c_{cost}/c_F^2$ in Line 17, where $\widehat{\mathbf{M}}_\ell$ denotes the spd mass matrix in H_X. In particular, for the H_X-regularization $(A = I)$, which corresponds to the L^2-regularization in the applications, the cost $c_\ell = \|\widetilde{u}_\ell\|_{U=H_X}^2 = (\widehat{\mathbf{M}}_\ell \widetilde{\mathbf{u}}_\ell, \widetilde{\mathbf{u}}_\ell)$ of the control can be calculated directly. We note that we can also use the bound $\|y_{\varrho h}\|_D$ from Lemma 4 without recovering the control $\widetilde{u}_{\varrho h}$ at the nested levels.

2.5 Constraints

To include additional constraints, e.g., on the control u_ϱ or on the state y_ϱ, we now consider the minimization of the reduced cost functional (2.6) over a non-empty, convex and closed subset $K \subset Y$, where we assume $0 \in K$ to be satisfied. The minimizer $y_\varrho \in K$ satisfying

$$\check{\mathcal{J}}(y_\varrho) = \min_{y \in K} \check{\mathcal{J}}(y)$$

is determined as the unique solution $y_\varrho \in K$ of the first kind variational inequality

$$\langle y_\varrho, y - y_\varrho \rangle_{H_Y} + \varrho \, \langle D y_\varrho, y - y_\varrho \rangle_{Y^*,Y} \ge \langle \overline{y}, y - y_\varrho \rangle_{H_Y} \quad \text{for all } y \in K. \qquad (2.34)$$

As in Lemma 1 we can state the following result on the error $\|y_\varrho - \overline{y}\|_{H_Y}$.

Lemma 5. *Let $y_\varrho \in K$ be the unique solution of the variational inequality* (2.34)*. For $\overline{y} \in H_Y$ there holds*

$$\|y_\varrho - \overline{y}\|_{H_Y} \leq \|\overline{y}\|_{H_Y}, \quad \|y_\varrho\|_D \leq \varrho^{-1/2}\,\|\overline{y}\|_{H_Y},$$

while for $\overline{y} \in K$ we have

$$\|y_\varrho - \overline{y}\|_S \leq \|\overline{y}\|_D, \quad \|y_\varrho - \overline{y}\|_{H_Y} \leq \varrho^{1/2}\,\|\overline{y}\|_D, \quad \|y_\varrho\|_D \leq \|\overline{y}\|_D.$$

If in addition $D\overline{y} \in H_Y$ is satisfied for $\overline{y} \in K$, then the estimates

$$\|y_\varrho - \overline{y}\|_{H_Y} \leq \varrho\,\|D\overline{y}\|_{H_Y} \quad \textit{and} \quad \|y_\varrho - \overline{y}\|_D \leq \varrho^{1/2}\,\|D\overline{y}\|_{H_Y}$$

follows.

Proof. From the variational inequality (2.34) we obviously have

$$\varrho\,\langle Dy_\varrho, y - y_\varrho\rangle_{Y^*,Y} \geq \langle \overline{y} - y_\varrho, y - y_\varrho\rangle_{H_Y} \quad \forall v \in K.$$

In particular for $y = 0 \in K$ this gives

$$\varrho\,\langle Dy_\varrho, y_\varrho\rangle_{Y^*,Y} + \|\overline{y} - y_\varrho\|^2_{H_Y} \leq \langle \overline{y} - y_\varrho, \overline{y}\rangle_{H_Y} \leq \|\overline{y} - y_\varrho\|_{H_Y}\|\overline{y}\|_{H_Y},$$

i.e.,

$$\|y_\varrho - \overline{y}\|_{H_Y} \leq \|\overline{y}\|_{H_Y}, \quad \|y_\varrho\|_D \leq \varrho^{-1/2}\,\|\overline{y}\|_{H_Y}.$$

When assuming $\overline{y} \in K$ we can consider $y = \overline{y}$ to obtain

$$\|y_\varrho - \overline{y}\|^2_{H_Y} + \varrho\,\|y_\varrho - \overline{y}\|^2_D \leq \varrho\,\langle D\overline{y}, \overline{y} - y_\varrho\rangle_{Y,Y^*} \leq \varrho\,\|\overline{y}\|_D\|y_\varrho - \overline{y}\|_D,$$

i.e.,

$$\|y_\varrho - \overline{y}\|_D \leq \|\overline{y}\|_D, \quad \|y_\varrho - \overline{y}\|_{H_Y} \leq \varrho^{1/2}\,\|\overline{y}\|_D\,.$$

For $y = \overline{y} \in K$ we can write (2.34) also as

$$\|y_\varrho - \overline{y}\|^2_{H_Y} + \varrho\,\|y_\varrho\|^2_D \leq \varrho\,\langle Dy_\varrho, \overline{y}\rangle_{H_Y} \leq \varrho\,\|y_\varrho\|_D\|\overline{y}\|_D,$$

i.e.,

$$\|y_\varrho\|_D \leq \|\overline{y}\|_D.$$

Finally, if $D\overline{y} \in H_Y$ for $\overline{y} \in K$, then

$$\|y_\varrho - \overline{y}\|^2_{H_Y} + \varrho\,\|y_\varrho - \overline{y}\|^2_D \leq \varrho\,\langle D\overline{y}, \overline{y} - y_\varrho\rangle_{Y^*,Y} \leq \varrho\,\|D\overline{y}\|_{H_Y}\|y_\varrho - \overline{y}\|_{H_Y},$$

implying

$$\|y_\varrho - \overline{y}\|_{H_Y} \leq \varrho\,\|D\overline{y}\|_{H_Y}, \quad \|y_\varrho - \overline{y}\|_D \leq \varrho^{1/2}\,\|D\overline{y}\|_{H_Y}\,.$$

□

Note that the results of Lemma 5 correspond to the results of Lemma 1 when no constraints are considered.

As in the unconstrained case, let $Y_h = \text{span}\{\varphi_i\}_{i=1}^M \subset Y$ be a conforming ansatz space, and let $K_h \subset Y_h$ be some non-empty, convex and closed set being an appropriate approximation of K. Then we consider the Galerkin variational inequality of (2.34) to find $y_{\varrho h} \in K_h$ such that

$$\langle y_{\varrho h}, y_h - y_{\varrho h}\rangle_{H_Y} + \varrho\,\langle Dy_{\varrho h}, y_h - y_{\varrho h}\rangle_{Y^*,Y} \geq \langle \overline{y}, y_h - y_{\varrho h}\rangle_{H_Y} \tag{2.35}$$

is satisfied for all $y_h \in K_h$, which is obviously equivalent to

$$\langle \overline{y} - y_{\varrho h}, y_h - y_{\varrho h}\rangle_{H_Y} - \varrho\,\langle Dy_{\varrho h}, y_h - y_{\varrho h}\rangle_{Y^*,Y} \leq 0 \quad \text{for all } y_h \in K_h.$$

Following [14] we can prove the following a priori error estimate:

Lemma 6. *For $y_\varrho \in K$ and $y_{\varrho h} \in K_h$ being the unique solutions of the variational inequalities* (2.34) *and* (2.35)*, respectively, there holds the error estimate*

$$\begin{aligned} &\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + 2\,\varrho\,\|y_\varrho - y_{\varrho h}\|^2_D \\ &\quad \leq 2 \inf_{y_h \in K_h} \Big[3\,\|y_\varrho - y_h\|^2_{H_Y} + \varrho\,\|y_\varrho - y_h\|^2_D \Big] + 4\,\|\varrho D y_\varrho + y_\varrho - \overline{y}\|^2_{H_Y}. \end{aligned}$$

Proof. For arbitrary $y_h \in K_h$,

$$\begin{aligned} &\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \varrho\,\|y_\varrho - y_{\varrho h}\|^2_D \\ &= \langle y_\varrho - y_{\varrho h}, y_\varrho - y_{\varrho h}\rangle_{H_Y} + \varrho\,\langle D(y_\varrho - y_{\varrho h}), y_\varrho - y_{\varrho h}\rangle_{Y^*,Y} \\ &= \langle y_\varrho - y_{\varrho h}, y_\varrho - y_h\rangle_{H_Y} + \varrho\,\langle D(y_\varrho - y_{\varrho h}), y_\varrho - y_h\rangle_{Y^*,Y} \\ &\qquad + \langle y_\varrho - y_{\varrho h}, y_h - y_{\varrho h}\rangle_{H_Y} + \varrho\,\langle D(y_\varrho - y_{\varrho h}), y_h - y_{\varrho h}\rangle_{Y^*,Y} \\ &= \langle y_\varrho - y_{\varrho h}, y_\varrho - y_h\rangle_{H_Y} + \varrho\,\langle D(y_\varrho - y_{\varrho h}), y_\varrho - y_h\rangle_{Y^*,Y} \\ &\qquad + \langle \overline{y} - y_{\varrho h}, y_h - y_{\varrho h}\rangle_{H_Y} - \varrho\,\langle D y_{\varrho h}, y_h - y_{\varrho h}\rangle_{Y^*,Y} \\ &\qquad\qquad + \langle y_\varrho - \overline{y}, y_h - y_{\varrho h}\rangle_{H_Y} + \varrho\,\langle D y_\varrho, y_h - y_{\varrho h}\rangle_{Y^*,Y} \\ &\leq \langle y_\varrho - y_{\varrho h}, y_\varrho - y_h\rangle_{H_Y} + \varrho\,\langle D(y_\varrho - y_{\varrho h}), y_\varrho - y_h\rangle_{Y^*,Y} \\ &\qquad\qquad + \langle y_\varrho - \overline{y}, y_h - y_{\varrho h}\rangle_{H_Y} + \varrho\,\langle D y_\varrho, y_h - y_{\varrho h}\rangle_{Y^*,Y} \\ &= \langle y_\varrho - y_{\varrho h}, y_\varrho - y_h\rangle_{H_Y} + \varrho\,\langle D(y_\varrho - y_{\varrho h}), y_\varrho - y_h\rangle_{Y^*,Y} \\ &\qquad\qquad + \langle \varrho D y_\varrho + y_\varrho - \overline{y}, y_h - y_{\varrho h}\rangle_{Y^*,Y} \\ &\leq \|y_\varrho - y_{\varrho h}\|_{H_Y}\|y_\varrho - y_h\|_{H_Y} + \varrho\,\|y_\varrho - y_{\varrho h}\|_D\|y_\varrho - y_h\|_D \\ &\qquad\qquad + \|\varrho D y_\varrho + y_\varrho - \overline{y}\|_{H_Y}\|y_h - y_{\varrho h}\|_{H_Y}\,. \end{aligned}$$

When using Young's inequality, we further have

$$\begin{aligned} &\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \varrho\,\|y_\varrho - y_{\varrho h}\|^2_D \\ &\leq \frac{1}{4}\,\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \|y_\varrho - y_h\|^2_{H_Y} + \frac{1}{2}\,\varrho\,\|y_\varrho - y_{\varrho h}\|^2_D + \frac{1}{2}\,\varrho\,\|y_\varrho - y_h\|^2_D \\ &\qquad + \|\varrho D y_\varrho + y_\varrho - \overline{y}\|^2_{H_Y} + \frac{1}{4}\,\|y_h - y_{\varrho h}\|^2_{H_Y} \\ &\leq \frac{1}{4}\,\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \|y_\varrho - y_h\|^2_{H_Y} + \frac{1}{2}\,\varrho\,\|y_\varrho - y_{\varrho h}\|^2_D + \frac{1}{2}\,\varrho\,\|y_\varrho - y_h\|^2_D \\ &\qquad + \|\varrho D y_\varrho + y_\varrho - \overline{y}\|^2_{H_Y} + \frac{1}{2}\,\|y_h - y_\varrho\|^2_{H_Y} + \frac{1}{2}\,\|y_\varrho - y_{\varrho h}\|^2_{H_Y}, \end{aligned}$$

i.e.,

$$\begin{aligned} &\frac{1}{4}\,\|y_\varrho - y_{\varrho h}\|^2_{H_Y} + \frac{1}{2}\,\varrho\,\|y_\varrho - y_{\varrho h}\|^2_D \\ &\qquad \leq \frac{3}{2}\,\|y_\varrho - y_h\|^2_{H_Y} + \frac{1}{2}\,\varrho\,\|y_\varrho - y_h\|^2_D + \|\varrho D y_\varrho + y_\varrho - \overline{y}\|^2_{H_Y}. \end{aligned}$$

This gives the assertion. □

3 Distributed control of the Poisson equation

In this section, we will describe the application of the abstract theory to the solution of distributed control problems for the Poisson equation, considering the control either in $L^2(\Omega)$ or in $H^{-1}(\Omega)$, and using either a constant or a variable regularization parameter ϱ.

3.1 H^{-1} regularization

First we consider the distributed optimal control problem to minimize

$$\mathcal{J}(y_\varrho, u_\varrho) = \frac{1}{2}\,\|y_\varrho - \overline{y}\|^2_{L^2(\Omega)} + \frac{1}{2}\,\varrho\,\|u_\varrho\|^2_{H^{-1}(\Omega)} \tag{3.1}$$

subject to the Dirichlet boundary value problem for the Poisson equation,

$$-\Delta y_\varrho = u_\varrho \quad \text{in } \Omega, \quad y_\varrho = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

We assume that $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded domain with either smooth boundary $\partial\Omega$, or convex. The standard variational formulation of the Dirichlet boundary value problem (3.2) is to find $y_\varrho \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla y_\varrho(x) \cdot \nabla y(x)\, dx = \langle u_\varrho, y\rangle_\Omega \tag{3.3}$$

is satisfied for all $v \in H^1_0(\Omega)$. The variational formulation (3.3) admits a unique solution $y_\varrho \in Y := H^1_0(\Omega)$ when assuming $u_\varrho \in U := H^{-1}(\Omega) = X^* = [H^1_0(\Omega)]^*$, i.e., $X = H^1_0(\Omega)$, and $H_X = H_Y = L^2(\Omega)$. When using the norm $\|y\|_Y = \|\nabla y\|_{L^2(\Omega)}$ we easily conclude the abstract assumptions for $A = B = -\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ with $c^A_1 = c^B_1 = c^A_2 = c^B_2 = 1$. Moreover, $S = B^* A^{-1} B = -\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ with $c^S_1 = c^S_2 = 1$. Hence we can rewrite the abstract variational formulation (2.7) to find $y_\varrho \in H^1_0(\Omega)$ such that

$$\langle y_\varrho, y\rangle_{L^2(\Omega)} + \varrho\,\langle \nabla y_\varrho, \nabla y\rangle_{L^2(\Omega)} = \langle \overline{y}, y\rangle_{L^2(\Omega)} \tag{3.4}$$

is satisfied for all $y \in H^1_0(\Omega)$. Note that, in fluid mechanics, the variational problem (3.4) is known as differential filter; see, e.g., [34]. The results of Lemma 1 now read

$$\|y_\varrho - \overline{y}\|_{L^2(\Omega)} \le \begin{cases} \|\overline{y}\|_{L^2(\Omega)} & \text{if } \overline{y} \in L^2(\Omega), \\ \varrho^{1/2}\,\|\nabla \overline{y}\|_{L^2(\Omega)} & \text{if } \overline{y} \in H^1_0(\Omega), \\ \varrho\,\|\Delta \overline{y}\|_{L^2(\Omega)} & \text{if } \overline{y} \in H^1_0(\Omega, \Delta), \end{cases}$$

where we have used $H^1_0(\Omega, \Delta) := \{y \in H^1_0(\Omega) : \Delta y \in L^2(\Omega)\}$. Moreover,

$$\|\nabla(y_\varrho - \overline{y})\|_{L^2(\Omega)} \le \begin{cases} \|\nabla \overline{y}\|_{L^2(\Omega)} & \text{if } \overline{y} \in H^1_0(\Omega), \\ \varrho^{1/2}\,\|\Delta \overline{y}\|_{L^2(\Omega)} & \text{if } \overline{y} \in H^1_0(\Omega, \Delta). \end{cases}$$

For the discretization of the variational formulation (3.4) we introduce the standard finite element space $Y_h = \text{span}\{\varphi_i\}_{i=1}^M \subset Y = H^1_0(\Omega)$ of piecewise linear continuous basis functions φ_i which are defined with respect to an admissible decomposition $\mathcal{T}_h$ of Ω into simplicial shape regular finite elements τ of local mesh size h_τ, and $h = \max_{\tau \in \mathcal{T}_h} h_\tau$. For $y \in Y = H^1_0(\Omega)$ let $P_h y \in Y_h$ be the unique solution of the variational formulation satisfying

$$\int_\Omega \nabla P_h y \cdot \nabla z_h\, dx = \int_\Omega \nabla y \cdot \nabla z_h\, dx \quad \text{for all } z_h \in Y_h.$$

When using standard finite element error estimates we immediately have

$$\|\nabla(y - P_h y)\|_{L^2(\Omega)} \le \|\nabla y\|_{L^2(\Omega)} \quad \text{for } y \in H^1_0(\Omega),$$

and

$$\|\nabla(y - P_h y)\|_{L^2(\Omega)} \le c\,h\,|y|_{H^2(\Omega)} \le c\,h\,\|\Delta y\|_{L^2(\Omega)} \quad \text{for } y \in H^1_0(\Omega, \Delta),$$

where for the last inequality we need that Ω is either smoothly bounded or convex, as assumed. In this case, and using the Aubin–Nitsche trick, we also have the error estimates

$$\|y - P_h y\|_{L^2(\Omega)} \leq c\, h \, \|\nabla y\|_{L^2(\Omega)}, \quad \|y - P_h y\|_{L^2(\Omega)} \leq c\, h^2 \, \|\Delta y\|_{L^2(\Omega)}.$$

Hence we have established the abstract assumptions (2.10) and (2.11), with $\alpha = 1$, implying the optimal choice $\varrho = h^2$. The finite element variational formulation of (3.4) reads to find $y_{\varrho h} \in Y_h$ such that

$$\langle y_{\varrho h}, y_h \rangle_{L^2(\Omega)} + \varrho \, \langle \nabla y_{\varrho h}, \nabla y_h \rangle_{L^2(\Omega)} = \langle \overline{y}, y_h \rangle_{L^2(\Omega)} \tag{3.5}$$

is satisfied for all $y_h \in Y_h$. For the choice $\varrho = h^2$, the error estimates (2.17) read

$$\|y_{\varrho h} - \overline{y}\|_{L^2(\Omega)} \leq c \left\{ \begin{array}{cl} \|\overline{y}\|_{L^2(\Omega)} & \text{for } \overline{y} \in L^2(\Omega), \\ h \, \|\nabla \overline{y}\|_{L^2(\Omega)} & \text{for } \overline{y} \in H^1_0(\Omega), \\ h^2 \, \|\Delta \overline{y}\|_{L^2(\Omega)} & \text{for } \overline{y} \in H^1_0(\Omega, \Delta) = H^1_0(\Omega) \cap H^2(\Omega). \end{array} \right.$$

Finally, for $s \in [0,1]$ we define the interpolation space $H^s_0(\Omega) = [L^2(\Omega), H^1_0(\Omega)]_{|s}$, and when assuming $\overline{y} \in H^s_0(\Omega)$ the error estimate (2.24) reads

$$\|y_{\varrho h} - \overline{y}\|_{L^2(\Omega)} \ \leq \ c\, h^s \, \|\overline{y}\|_{H^s_0(\Omega)}. \tag{3.6}$$

Once the basis is chosen, the finite element scheme (3.5) is equivalent to the linear system of algebraic equations

$$(\mathbf{M}_h + \varrho \, \mathbf{K}_h) \mathbf{y}_{\varrho h} = \overline{\mathbf{y}}_h, \tag{3.7}$$

where the mass and stiffness matrices are defined via their entries

$$M_h[j,i] = \int_\Omega \varphi_i(x) \varphi_j(x) \, dx \quad \text{and} \quad K_h[j,i] = \int_\Omega \varphi_i(x) \varphi_j(x) \, dx$$

for $i, j = 1, \ldots, M$, and the source vector $\overline{\mathbf{y}}_h$ via its coefficients

$$\overline{y}_j = \int_\Omega \overline{y}(x) \varphi_j(x) \, dx \quad \text{for } j = 1, \ldots, M.$$

3.2 Variable H^{-1} regularization

Instead of the variational formulation (3.4) with a constant regularization parameter ϱ, we now consider a variational formulation with a suitable regularization function $\varrho(x)$, $x \in \Omega$. For a given decomposition $\mathcal{T}_h$ of Ω into finite elements τ of local mesh size h_τ, we define the mesh dependent regularization function

$$\varrho_h(x) = h_\tau^2 \quad \text{for } x \in \tau,$$

and the mesh dependent norm

$$\|y\|^2_{H^1_0(\Omega), \varrho_h} := \int_\Omega \varrho_h(x) \, |\nabla y(x)|^2 \, dx \quad \text{for } y \in H^1_0(\Omega).$$

Then we consider the variational formulation to find $y_{\varrho_h} \in H^1_0(\Omega)$ such that

$$\int_\Omega y_{\varrho_h}(x) y(x) \, dx + \int_\Omega \varrho_h(x) \, \nabla y_{\varrho_h}(x) \cdot \nabla y(x) \, dx = \int_\Omega \overline{y}(x) y(x) \, dx \tag{3.8}$$

is satisfied for all $y \in H^1_0(\Omega)$.

As in Lemma 1 we conclude the regularization error estimates

$$\|y_{\varrho_h} - \overline{y}\|_{L^2(\Omega)} \leq \|\overline{y}\|_{L^2(\Omega)}, \quad \|y_{\varrho_h}\|_{L^2(\Omega)} \leq \|\overline{y}\|_{L^2(\Omega)}, \quad \|y_{\varrho_h}\|_{H^1_0(\Omega),\varrho_h} \leq \|\overline{y}\|_{L^2(\Omega)}$$

for $\overline{y} \in L^2(\Omega)$, and

$$\|y_{\varrho_h} - \overline{y}\|_{H^1_0(\Omega),\varrho_h} \leq \|\overline{y}\|_{H^1_0(\Omega),\varrho_h}, \quad \|y_{\varrho_h} - \overline{y}\|_{L^2(\Omega)} \leq \|\overline{y}\|_{H^1_0(\Omega),\varrho_h},$$

as well as

$$\|y_{\varrho_h}\|_{H^1_0(\Omega),\varrho_h} \leq \|\overline{y}\|_{H^1_0(\Omega),\varrho_h}$$

for $\overline{y} \in H^1_0(\Omega)$.

The finite element discretization of (3.8) reads to find $y_{\varrho_h h} \in Y_h$ such that

$$\int_\Omega y_{\varrho_h h}(x) y_h(x)\, dx + \int_\Omega \varrho_h(x)\, \nabla y_{\varrho_h h}(x) \cdot \nabla y_h(x)\, dx = \int_\Omega \overline{y}(x) y_h(x)\, dx \tag{3.9}$$

is satisfied for all $y_h \in Y_h$. In this case, Cea's lemma (2.13) reads

$$\begin{aligned}\|y_{\varrho_h} - y_{\varrho_h,h}\|^2_{L^2(\Omega)} &+ \|y_{\varrho_h} - y_{\varrho_h,h}\|^2_{H^1_0(\Omega),\varrho_h} \\ &\leq \inf_{y_h \in Y_h} \Big[\|y_{\varrho_h} - y_h\|^2_{L^2(\Omega)} + \|y_{\varrho_h} - y_h\|^2_{H^1_0(\Omega),\varrho_h} \Big],\end{aligned}$$

and when choosing $y_h \equiv 0$ this gives

$$\|y_{\varrho_h} - y_{\varrho_h,h}\|_{L^2(\Omega)} \leq \sqrt{2}\, \|\overline{y}\|_{L^2(\Omega)} \quad \text{for } \overline{y} \in L^2(\Omega).$$

Moreover, when considering some quasi-interpolation $y_h = P_h y_{\varrho_h}$ we obtain the error estimate

$$\|y_{\varrho_h} - y_{\varrho_h h}\|_{L^2(\Omega)} \leq c\, \|\overline{y}\|_{H^1_0(\Omega),\varrho_h} = c \left(\sum_{\tau \subset \mathcal{T}_h} h_\tau^2\, \|\nabla \overline{y}\|^2_{L^2(\tau)} \right)^{1/2} \quad \text{for } \overline{y} \in H^1_0(\Omega).$$

When combining these results with the regularization error estimates, we finally obtain

$$\|y_{\varrho_h h} - \overline{y}\|_{L^2(\Omega)} \leq c\, \|\overline{y}\|_{L^2(\Omega)}, \quad \|y_{\varrho_h h} - \overline{y}\|_{L^2(\Omega)} \leq c \left(\sum_{\tau \subset \mathcal{T}_h} h_\tau^2\, \|\nabla \overline{y}\|^2_{L^2(\tau)} \right)^{1/2}.$$

Instead of (3.7), we now conclude the linear system

$$(\mathbf{M}_h + \mathbf{K}_{\varrho_h h}) \mathbf{y}_{\varrho h} = \overline{\mathbf{y}}_h \tag{3.10}$$

from the finite element scheme (3.9), where the entries of the diffusion type stiffness matrix $\mathbf{K}_{\varrho_h h}$ are now given by

$$K_{\varrho_h h}[j,i] = \int_\Omega \varrho_h(x)\, \nabla \varphi_i(x) \cdot \nabla \varphi_j(x)\, dx, \quad i,j = 1, \ldots, M.$$

3.3 L^2 regularization

Instead of (3.1) we now consider the optimal control problem to minimize

$$\mathcal{J}(y_\varrho, u_\varrho) = \frac{1}{2}\, \|y_\varrho - \overline{y}\|^2_{L^2(\Omega)} + \frac{1}{2}\, \varrho\, \|u_\varrho\|^2_{L^2(\Omega)} \tag{3.11}$$

subject to the Dirichlet boundary value problem (3.2) where we now consider the source term $u_\varrho \in U = X^* = L^2(\Omega)$, i.e., $X = L^2(\Omega)$. For the solution y_ϱ of

(3.2) we therefore have $Y = H^1_0(\Omega,\Delta) := \{y \in H^1_0(\Omega) : \Delta y \in L^2(\Omega)\}$, with norm $\|y\|_{H^1_0(\Omega,\Delta)} = \|\Delta y\|_{L^2(\Omega)}$. Using $B = -\Delta : H^1_0(\Omega,\Delta) \to L^2(\Omega)$, we therefore have

$$\|By\|_{L^2(\Omega)} = \|\Delta y\|_{L^2(\Omega)} = \|y\|_{H^1_0(\Omega,\Delta)}$$

for all $0 \neq y \in H^1_0(\Omega,\Delta)$, and

$$\|y\|_{Y=H^1_0(\Omega,\Delta)} = \|\Delta y\|_{L^2(\Omega)} = \frac{\langle -\Delta y, -\Delta y\rangle_{L^2(\Omega)}}{\|\Delta y\|_{L^2(\Omega)}} \leq \sup_{0\neq q\in L^2(\Omega)} \frac{\langle -\Delta y, q\rangle_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)}},$$

i.e., $c_1^B = c_2^B = 1$. Moreover, $A = I : L^2(\Omega) \to L^2(\Omega)$, with $c_1^A = c_2^A = 1$. Thus, we define $S := B^*B : H^1_0(\Omega,\Delta) \to [H^1_0(\Omega,\Delta)]^*$ with $c_1^S = c_2^S = 1$, and where $B^* : L^2(\Omega) \to [H^1_0(\Omega,\Delta)]^*$ is the adjoint of $B : H^1_0(\Omega,\Delta) \to L^2(\Omega)$ satisfying

$$\langle B^* q, y\rangle_\Omega = \langle q, By\rangle_{L^2(\Omega)} \quad \text{for all } (q,y) \in L^2(\Omega) \times H^1_0(\Omega,\Delta).$$

Hence, we can rewrite the abstract variational formulation (2.7) to find $y_\varrho \in H^1_0(\Omega,\Delta)$ such that

$$\langle y_\varrho, y\rangle_{L^2(\Omega)} + \varrho\,\langle \Delta y_\varrho, \Delta y\rangle_{L^2(\Omega)} \;=\; \langle \overline{y}, y\rangle_{L^2(\Omega)} \tag{3.12}$$

is satisfied for all $y \in H^1_0(\Omega,\Delta)$. Note that (3.12) is the variational formulation of the Dirchlet problem for the BiLaplace equation,

$$\varrho\,\Delta^2 y_\varrho + y_\varrho = \overline{y} \quad \text{in } \Omega, \quad y_\varrho = \Delta y_\varrho = 0 \quad \text{on } \partial\Omega. \tag{3.13}$$

The results of Lemma 1 now read

$$\|y_\varrho - \overline{y}\|_{L^2(\Omega)} \leq \begin{cases} \|\overline{y}\|_{L^2(\Omega)} & \text{if } \overline{y} \in L^2(\Omega), \\ \varrho^{1/2}\,\|\Delta\overline{y}\|_{L^2(\Omega)} & \text{if } \overline{y} \in H^1_0(\Omega,\Delta), \end{cases}$$

and

$$\|\Delta(y_\varrho - \overline{y})\|_{L^2(\Omega)} \leq \|\Delta\overline{y}\|_{L^2(\Omega)} \quad \text{for } \overline{y} \in H^1_0(\Omega,\Delta).$$

For a conforming finite element discretization of the variational formulation (3.12) we need to introduce an ansatz space $Y_h \subset Y = H^1_0(\Omega,\Delta)$. At this time, and for simplicity of the presentation, let us first consider the case $d = 1$ and $\Omega = (0,1)$. In this case, we have $Y = H^1_0(0,1) \cap H^2(0,1)$, and for a conforming ansatz space we can use the space $Y_h = S^2_h(0,1) \cap H^1_0(0,1)$ of second order B splines. Since, for $d = 1$, the nodal interpolation operator $I_h : Y \to Y_h$ is well defined and bounded, we can write the abstract assumptions (2.10) as

$$\|y - I_h y\|_{L^2(0,1)} \leq c_1\,h^2\,\|y''\|_{L^2(0,1)}, \quad \|(y - I_h y)''\|_{L^2(0,1)} \leq c_2\,\|y''\|_{L^2(0,1)},$$

i.e., $\alpha = 2$, implying the optimal choice $\varrho = h^4$. The Galerkin finite element formulation of (3.12) then reads to find $y_{\varrho h} \in Y_h$ such that

$$\langle y_{\varrho h}, y_h\rangle_{L^2(0,1)} + \varrho\,\langle y''_{\varrho h}, y''_h\rangle_{L^2(0,1)} \;=\; \langle \overline{y}, y_h\rangle_{L^2(0,1)} \tag{3.14}$$

is satisfied for all $y_h \in Y_h$, and, for the error estimate (2.17), we obtain, for $\varrho = h^4$,

$$\|y_{\varrho h} - \overline{y}\|_{L^2(0,1)} \leq c \begin{cases} \|\overline{y}\|_{L^2(0,1)} & \text{for } \overline{y} \in L^2(0,1), \\ h^2\,\|\overline{y}''\|_{L^2(0,1)} & \text{for } \overline{y} \in H^1_0(0,1) \cap H^2(0,1). \end{cases}$$

Finally, when using some space interpolation arguments, we conclude the error estimate

$$\|y_{\varrho h} - \overline{y}\|_{L^2(0,1)} \leq c\,h^{2s}\,\|\overline{y}\|_{[L^2(0,1),H^1_0(0,1)\cap H^2(0,1)]_{|s}}$$

provided that $\overline{y} \in [L^2(0,1), H_0^1(0,1) \cap H^2(0,1)]_{|s}$ for some $s \in [0,1]$.

Although we can generalize the above approach to domains $\Omega = (0,1)^d \subset \mathbb{R}^d$, $d = 2,3$ by using tensor product finite element spaces or IgA spaces, the construction of conforming finite element spaces $Y_h \subset H_0^1(\Omega, \Delta)$ with respect to simplicial decompositions of Ω seems to be more challenging, e.g., [1]. Hence, we will describe an alternative non-conforming approach as follows. For $y_\varrho \in H_0^1(\Omega, \Delta)$ being the unique solution of (3.12), we define $p_\varrho = \varrho \Delta y_\varrho \in L^2(\Omega)$, and we can rewrite the Dirichlet boundary value problem for the BiLaplace equation (3.13) as system,

$$-\Delta p_\varrho = y_\varrho - \overline{y}, \quad \frac{1}{\varrho}\, p_\varrho - \Delta y_\varrho = 0 \quad \text{in } \Omega, \quad y_\varrho = p_\varrho = 0 \text{ on } \partial\Omega.$$

From this system we conclude $p_\varrho, y_\varrho \in H_0^1(\Omega)$, and in the sequel $y_\varrho \in H_0^1(\Omega, \Delta)$. The related variational formulation reads to find $(p_\varrho, y_\varrho) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\frac{1}{\varrho}\, \langle p_\varrho, q \rangle_{L^2(\Omega)} + \langle \nabla y_\varrho, \nabla q \rangle_{L^2(\Omega)} = 0, \quad \langle \nabla p_\varrho, \nabla y \rangle_{L^2(\Omega)} = \langle y_\varrho - \overline{y}, y \rangle_{L^2(\Omega)} \tag{3.15}$$

is satisfied for all $(q, y) \in H_0^1(\Omega) \times H_0^1(\Omega)$; cf. also Section 1.

For the discretization of (3.15), we now use the conforming finite element space $V_h := S_h^1(\Omega) \cap H_0^1(\Omega) = \text{span}\{\varphi_i\}_{i=1}^M$ of piecewise linear and continuous basis functions φ_i, as already used in the case of the H^{-1} regularization. This results in a coupled linear system of algebraic equations

$$\begin{pmatrix} \frac{1}{\varrho}\mathbf{M}_h & \mathbf{K}_h \\ -\mathbf{K}_h & \mathbf{M}_h \end{pmatrix} \begin{pmatrix} \mathbf{p}_{\varrho h} \\ \mathbf{y}_{\varrho h} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_h \\ \overline{\mathbf{y}}_h \end{pmatrix},$$

which is equivalent to the Schur complement system

$$(\mathbf{M}_h + \varrho \mathbf{K}_h \mathbf{M}_h^{-1} \mathbf{K}_h)\mathbf{y}_{\varrho h} = \overline{\mathbf{y}}_h\,.$$

3.4 Variable L^2 regularization

Instead of (3.12) we now consider a variational formulation to find $y_{\varrho_h} \in H_0^1(\Omega, \Delta)$ such that

$$\int_\Omega y_{\varrho_h}(x) y(x)\, dx + \int_\Omega \varrho_h(x)\, \Delta y_{\varrho_h}(x)\, \Delta y(x)\, dx \; = \; \int_\Omega \overline{y}(x)\, y(x)\, dx$$

is satisfied for all $y \in H_0^1(\Omega, \Delta)$, with the mesh dependent regularization function

$$\varrho_h(x) = h_\tau^4 \quad \text{for } x \in \tau.$$

When introducing $p_{\varrho_h} = \varrho_h \Delta y_{\varrho_h}$, we end up with a variational system to find $(p_{\varrho_h}, y_{\varrho_h}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\int_\Omega \frac{1}{\varrho_h(x)}\, p_{\varrho_h}(x)\, q(x)\, dx + \int_\Omega \nabla y_{\varrho_h}(x) \cdot \nabla q(x)\, dx = 0$$

is satisfied for all $q \in H_0^1(\Omega)$, and

$$-\int_\Omega \nabla p_{\varrho_h}(x) \cdot \nabla y(x)\, dx + \int_\Omega y_{\varrho_h}(x) y(x)\, dx = \int_\Omega \overline{y}(x) y(x)\, dx$$

is satisfied for all $y \in H_0^1(\Omega)$. The finite element discretization of this system results in a linear system of algebraic equations,

$$\begin{pmatrix} \mathbf{M}_{1/\varrho_h, h} & \mathbf{K}_h \\ -\mathbf{K}_h & \mathbf{M}_h \end{pmatrix} \begin{pmatrix} \mathbf{p}_{\varrho h} \\ \mathbf{y}_{\varrho h} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_h \\ \overline{\mathbf{y}}_h \end{pmatrix},$$

where the scaled mass matrix $\mathbf{M}_{h,1/\varrho_h}$ is given by its entries

$$M_{1/\varrho_h,h}[j,i] = \int_\Omega \frac{1}{\varrho(x)}\,\varphi_i(x)\,\varphi_j(x)\,dx \quad \text{for } i,j=1,\ldots,M.$$

When eliminating $\underline{p}$, we end up with the Schur complement system

$$(\mathbf{M}_h + \varrho \mathbf{K}_h \mathbf{M}^{-1}_{1/\varrho_h,h}\mathbf{K}_h)\mathbf{y}_{\varrho_h h} = \overline{\mathbf{y}}_h\,.$$

3.5 Control recovering

For the finite element approximation of the control $u_\varrho = By_\varrho$, we consider the abstract variational formulation (2.19) with $U = X^* = H^{-1}(\Omega)$ and $X = H^1_0(\Omega)$. In this particular situation, we can choose the finite element space $X_h = Y_h = \text{span}\{\varphi_k\}_{k=1}^M$ of piecewise linear continuous basis functions φ_k, where the assumption (2.20) coincides with the second assumption in (2.11) which was already established. When an approximate state $y_{\varrho h} \in Y_h \subset H^1_0(\Omega)$ is known, we can compute the related control $\widetilde{u}_{\varrho h} \in U_h = \text{span}\{\psi_k\}_{k=1}^M \subset L^2(\Omega) \subset H^{-1}(\Omega)$ as unique solution of the variational formulation

$$\int_\Omega \widetilde{u}_{\varrho h}(x)\phi_h(x)\,dx = \int_\Omega \nabla y_{\varrho h}(x)\cdot\nabla y_h(x)\,dx \quad \text{for all } y_h \in Y_h. \tag{3.16}$$

It remains to define U_h in order to satisfy the discrete inf-sup condition (2.21), which now reads

$$c_S\,\|u_h\|_{H^{-1}(\Omega)} \le \sup_{y_h\in Y_h\subset H^1_0(\Omega)} \frac{\langle u_h, y_h\rangle_{L^2(\Omega)}}{\|\nabla y_h\|_{L^2(\Omega)}} \quad \text{for all } u_h \in U_h.$$

A first choice is to consider the control space $U_h = Y_h \subset H^1_0(\Omega)$ of piecewise linear and continuous basis functions, i.e., we have to solve the linear system (2.32) with $\overline{\mathbf{M}}_h = \mathbf{M}_h$ and $\mathbf{B}_h = \mathbf{K}_h$. Now the discrete inf-sup condition is equivalent to the stability estimate

$$\|Q_h y\|_{H^1(\Omega)} \le \frac{1}{c_S}\,\|y\|_{H^1(\Omega)}$$

for the L^2 projection $Q_h : L^2(\Omega) \to Y_h \subset H^1_0(\Omega) \subset L^2(\Omega)$, see, e.g., [6], which also covers adaptive meshes.

For the approximate control $\widetilde{u}_{\varrho h}$ we can compute the related state $\widetilde{y}_\varrho \in H^1_0(\Omega)$ as unique solution of the variational formulation

$$\int_\Omega \nabla \widetilde{y}_\varrho(x)\cdot\nabla y(x)\,dx = \int_\Omega \widetilde{u}_{\varrho h}\,y(x)\,dx \quad \text{for all } y \in H^1_0(\Omega),$$

and from (2.24) we conclude the error estimate

$$\|\widetilde{y}_\varrho - \overline{y}\|_{L^2(\Omega)} \le c\,h^s\,\|\overline{y}\|_{H^s_0(\Omega)} \quad \text{for } \overline{y} \in H^s_0(\Omega) = [L^2(\Omega, H^1_0(\Omega))]_{|s}, s\in[0,1], \tag{3.17}$$

when choosing $\varrho = h^2$ in the case of H^{-1} regularization, and $\varrho = h^4$ in the case of L^2 regularization.

Due to the choice $U_h = Y_h \subset H^1_0(\Omega)$ the discrete control $\widetilde{u}_{\varrho h}$ is much more regular than expected, i.e., it involves boundary conditions. Although this does not effect the final error estimate (3.17), the shape of the piecewise linear and continuous control $\widetilde{u}_{\varrho h}$ may be not feasible. As an alternative we aim to construct a piecewise constant discrete control.

For a given admissible decomposition of $\Omega \subset \mathbb{R}^d$ into shape regular simplicial finite elements τ we introduce a dual mesh as follows: For any interior node $x_k \in \Omega$

we define a dual finite element ω_k satisfying $\omega_k \cap \omega_j = \emptyset$ for $x_k \neq x_j$ such that $\mathcal{T}_h = \cup_{k=1}^M \overline{\omega}_k$, see Figure 3 and, e.g., [59]. Then we define $U_h = \text{span}\{\psi_k\}_{k=1}^M \subset U = H^{-1}(\Omega)$ as ansatz space of piecewise constant basis functions ψ_k which are one in ω_k, and zero elsewhere. While the approximation assumption (2.20) remains unchanged, the discrete inf-sup condition (2.21) follows as in [59]. Moreover, the error estimate (3.17) remains true. But instead of the standard mass matrix $\overline{\mathbf{M}}_h = \mathbf{M}_h$, we now have to use a matrix $\overline{\mathbf{M}}_h = \widetilde{\mathbf{M}}_h$ defined by the entries

$$\widetilde{M}_h[j,k] = \int_\Omega \varphi_k(x)\,\psi_j(x)\,dx = \int_{\omega_j} \varphi_k(x)\,dx, \quad j,k = 1,\ldots,M.$$

Thus, the linear system (2.32) now takes the form $\widetilde{\mathbf{M}}_h \mathbf{u}_{\varrho h} = \mathbf{K}_h \mathbf{y}_{\varrho h}$. Note, that we additionally have that

$$\|\nabla y_h\|_{L^2(\Omega)} = \sup_{0 \neq x_h \in Y_h} \frac{\langle \nabla y_h, \nabla x_h \rangle_{L^2(\Omega)}}{\|\nabla x_h\|_{L^2(\Omega)}},$$

and therefore the discrete inf-sup condition (2.25) is satisfied. Thus, by Lemma 4 the cost can be estimated by

$$\|\nabla y_{\varrho h}\|_{L^2(\Omega)} \leq \|\widetilde{u}_{\varrho h}\|_{H^{-1}(\Omega)} \leq c_S \|\nabla y_{\varrho h}\|_{L^2(\Omega)}. \tag{3.18}$$

Remark 2. *For* $\mathbf{u} = (u_1, \ldots, u_M)^\top \in \mathbb{R}^M$ *we compute*

$$(\widetilde{\mathbf{M}}_h \mathbf{u}, \mathbf{u})_2 = \sum_{i,j=1}^M u_i u_j \int_\Omega \psi_j(x)\varphi_i(x)\,dx = \sum_{\ell=1}^N \sum_{i,j=1}^M u_i u_j \int_{\tau_\ell} \psi_j(x)\varphi_i(x)\,dx.$$

The local element matrices with entries

$$\widetilde{M}_{\tau_\ell}[i,j] := \int_{\tau_\ell} \psi_j(x)\varphi_i(x)\,dx, \quad \ell = 1,\ldots,N,$$

can be computed to be, see [59], for $d = 1$

$$\widetilde{\mathbf{M}}_{\tau_\ell} = \frac{|\tau_\ell|}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{with } \lambda_{min}(\widetilde{\mathbf{M}}_{\tau_\ell}) = \frac{|\tau_\ell|}{4},\ \lambda_{max}(\widetilde{\mathbf{M}}_{\tau_\ell}) = \frac{|\tau_\ell|}{2},$$

and for $d = 2$

$$\widetilde{\mathbf{M}}_{\tau_\ell} = \frac{|\tau_\ell|}{108}\begin{pmatrix} 22 & 7 & 7 \\ 7 & 22 & 7 \\ 7 & 7 & 22 \end{pmatrix} \quad \text{with } \lambda_{min}(\widetilde{\mathbf{M}}_{\tau_\ell}) = \frac{5}{36}|\tau_\ell|,\ \lambda_{max}(\widetilde{\mathbf{M}}_{\tau_\ell}) = \frac{|\tau_\ell|}{3}.$$

Therefore, we see that $\widetilde{\mathbf{M}}_h$ *is symmetric and positive definite. Moreover, by elemantary computations the spectral equivalence inequalities*

$$\underline{c}(d)\,(lump\,(\mathbf{M}_h)\mathbf{u}, \mathbf{u})_2 \;\leq\; (\widetilde{\mathbf{M}}_h \mathbf{u}, \mathbf{u})_2 \;\leq\; (lump\,(\mathbf{M}_h)\mathbf{u}, \mathbf{u})_2 \tag{3.19}$$

follow, where $lump\,(\mathbf{M}_h)$ *denotes the lumped mass matrix and*

$$\underline{c}(d) = \begin{cases} \frac{1}{2}, & d = 1, \\ \frac{5}{12}, & d = 2. \end{cases}$$

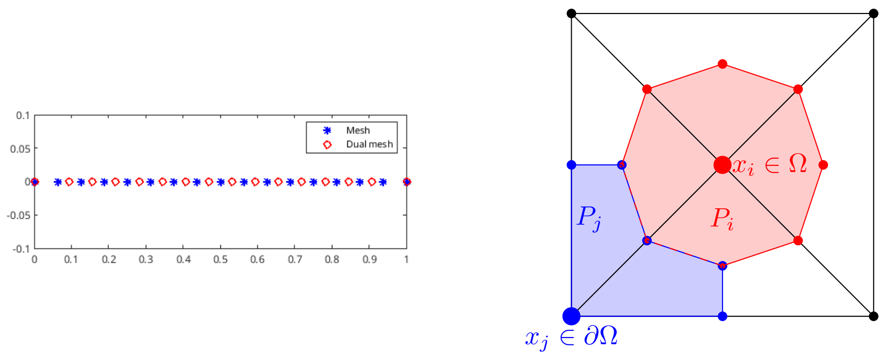


Figure 3: Meshes and dual meshes in 1D (left) and 2D (right).

3.6 Solvers and their use in nested iteration

We start this subsection with the observation that we can write all linear systems to be solved in a unified manner as

$$\mathbf{S}_{\varrho h}\mathbf{y}_{\varrho,h} = \overline{\mathbf{y}}_h \tag{3.20}$$

where $\mathbf{S}_{\varrho h} = \mathbf{M}_h + \mathbf{D}_{\varrho h}$ with

$$\mathbf{D}_{\varrho,h} = \begin{cases} \varrho\,\mathbf{K}_h & \text{for } H^{-1} \text{ regularization}, \varrho = h^2, \\ \mathbf{K}_{\varrho_h,h} & \text{for variable } H^{-1} \text{ regularization}, \varrho_h(x) = h_\tau^2, x \in \tau, \\ \varrho\,\mathbf{K}_h\mathbf{M}_h^{-1}\mathbf{K}_h & \text{for } L^2 \text{ regularization}, \varrho = h^4, \\ \mathbf{K}_h\mathbf{M}_{1/\varrho_h,h}^{-1}\mathbf{K}_h & \text{for variable } L^2 \text{ regularization}, \varrho_h(x) = h_\tau^4, x \in \tau. \end{cases}$$

We note that, for constant regularization functions $\varrho_h(x) = \varrho$ for all $x \in \Omega$, we obtain $\mathbf{K}_{\varrho_h h} = \varrho\mathbf{K}_h$ as well as $\mathbf{M}_{1/\varrho_h h} = \frac{1}{\varrho}\mathbf{M}_h$. Hence, it is sufficient to consider the variable regularizations only. For the system matrix $\mathbf{S}_{\varrho,h} = \mathbf{M}_h + \mathbf{D}_{\varrho h}$ of (3.20), we can prove the following lemma for all regularizations discussed above.

Lemma 7. *There hold the spectral equivalence inequalities*

$$c_1\,(\mathbf{C}_h\mathbf{y}_h, \mathbf{y}_h) \,\leq\, (\mathbf{S}_{\varrho h}\mathbf{y}_h, \mathbf{y}_h) \,\leq\, c_2\,(\mathbf{C}_h\mathbf{y}_h, \mathbf{y}_h) \tag{3.21}$$

for all $\mathbf{y}_h \in \mathbb{R}^M$*, cf.* (2.30)*, where the preconditioner* $\mathbf{C}_h = \mathit{lump}\,(\mathbf{M}_h)$ *is a simple diagonal matrix that is obtained from the mass matrix* $\mathbf{M}_h$ *by mass lumping, i.e.,*

$$C_h[j,k] = \mathit{lump}(M_h)[j,k] = \delta_{j,k}\sum_{i=1}^{M} M_h[j,i], \quad j,k = 1,\ldots,M;$$

$c_1 = 1/(d+2)$*, and* $c_2 = 1 + c^2$ *with the* h*-independent* $c^2 \geq \lambda_{\max}(\mathbf{M}_h^{-1}\mathbf{D}_{\varrho h})$ *being the maximal eigenvalue of the generalized eigenvalue problem* $\mathbf{D}_{\varrho h}\mathbf{v}_h = \lambda\,\mathbf{M}_h\mathbf{v}_h$ *or, at least, an upper bound of it.*

Proof. The lower estimate follows from the inequalities $\mathbf{S}_{\varrho h} = \mathbf{M}_h + \mathbf{D}_{\varrho h} \geq \mathbf{M}_h \geq (d+2)^{-1}\mathrm{lump}\,(\mathbf{M}_h)$, where the last estimate can be found in [39, Lemma 1]. The upper estimate can be obtained from $\mathbf{S}_{\varrho h} = \mathbf{M}_h + \mathbf{D}_{\varrho h} \leq (1+c^2)\mathbf{M}_h \leq (1+c^2)\,\mathrm{lump}\,(\mathbf{M}_h)$. The estimate $\mathbf{D}_{\varrho h} \leq c^2\,\mathbf{M}_h$ follows from local inverse inequalities and appropriate choices of the regularization parameter or function ϱ as given above for different regularizations. We refer to [39] for a detailed proof. We mention that $c^2 = \lambda_{\max}(\mathbf{M}_h^{-1}\mathbf{D}_{\varrho h})$ is the best possible constant. ∎

Remark 3. *The mass matrices* $\mathbf{M}_h$ *and* $\mathbf{M}_{1/\varrho_h,h}$ *in* $\mathbf{D}_{\varrho h} = \mathbf{K}_h \mathbf{M}_h^{-1} \mathbf{K}_h$ *and* $\mathbf{D}_{\varrho h} = \mathbf{K}_h \mathbf{M}_{1/\varrho_h,h}^{-1} \mathbf{K}_h$*, respectively, can be replaced by the corresponding lumped versions* *lump*$(\mathbf{M}_h)$ *and* *lump*$(\mathbf{M}_{1/\varrho_h,h})$ *without affecting the discretization error and the spectral equivalence inequalities; see [39]. We note that this replacement of the mass matrix by their lumped versions makes the matrix-vector multiplication* $\mathbf{S}_{\varrho h} * \mathbf{y}_h$ *fast. More precisely,* $\mathbf{S}_{\varrho h} * \mathbf{y}_h$ *can be performed in optimal complexity* $\mathcal{O}(h^{-d})$*. This is important when solving the system* (3.20) *by pcg as we do in the nested iteration procedure presented in Algorithm 1.*

In order to recover the control $\widetilde{u}_{\varrho h} \leftrightarrow \widetilde{\mathbf{u}}_{\varrho h} \in \mathbb{R}^M$, we have to solve the system (2.32) with $\mathbf{B}_h = \mathbf{K}_h$ and $\overline{\mathbf{M}}_h = \mathbf{M}_h$ or $\overline{\mathbf{M}}_h = \widetilde{\mathbf{M}}_h$. In both cases, $\overline{\mathbf{C}}_h = \text{lump}(\mathbf{M}_h)$ is an asymptotically optimal preconditioner, see (3.19).

Let us now specify the nested iteration and, in particular, Algorithm 1, descried in Subsection 3.6 for abstract optimal control problems, in the special case of distributed control of the Poission equation with energy regularization presented in Subsection 3.1. Let us assume that $\mathcal{T}_\ell = \mathcal{T}_{h_\ell}$ is a sequence of uniformly (or adaptively) refined simpicial, shape regular meshes with the mesh sizes $h = h_\ell$, $\ell = 1, \ldots, L$, and Y_ℓ, X_ℓ, U_ℓ are corresponding finite element spaces as described in Subsections 3.1 and 3.5. We recall that here $X_\ell = Y_\ell = \text{span}\{\varphi_i\}_{i=1}^{M=N} \subset X = Y = H_0^1(\Omega)$. Line 11: $\mathbf{y}_\ell \leftarrow \mathbf{S}_\ell^{-1} \mathbf{y}_\ell$ in Algorithm 1 now means that the system (2.29) is solved by the pcg iteration with the preconditioner $\mathbf{C}_h = \text{lump}(\mathbf{M}_h)$ and the initial guess $\mathbf{y}_\ell^0 = \mathbf{I}_{\ell-1}^\ell \mathbf{y}_{\ell-1}^n$ that is simply interpolated from the last iterate on the coarser mesh $\mathcal{T}_{\ell-1}$. It is clear that we need a constant number n of nested iterations on all levels $\ell = 2, \ldots, L$ in order to match the discretization error (3.6). The coarse mesh system $\mathbf{S}_1 \mathbf{y}_1 = \overline{\mathbf{y}}_1$ (line 7 in in Algorithm 1) is usually solved by some sparse direct method [11], but it can be solved by pcg with the same preconditioner and the initial guess $\mathbf{y}_1^0 = \mathbf{0}_1$. This immediately yields that we need $\ln h_1^{-1}$ pcg iteration in order to match the discretization error estimate (3.6) for $h = h_1$.

3.7 State constraints

We now consider the minimization of (3.1) subject to the Poisson equation (3.2) with constraints on the state $y_\varrho \in K_s := \{y \in H_0^1(\Omega) : g_- \leq y \leq g_+ \text{ a.e. in } \Omega\}$, where $g_\pm \in H_0^1(\Omega, \Delta)$ are given barrier functions, and where we assume $g_- \leq g_+$ and $0 \in K_s$ to be satisfied. The solution $y_\varrho \in K_s$ of this minimization problem is then characterized as the unique solution of the variational inequality

$$\langle y_\varrho, y - y_\varrho \rangle_{L^2(\Omega)} + \varrho \, \langle \nabla y_\varrho, \nabla (y - y_\varrho) \rangle_{L^2(\Omega)} \geq \langle \overline{y}, y - y_\varrho \rangle_{L^2(\Omega)} \quad \text{for all } y \in K_s. \quad (3.22)$$

This variational inequality completely corresponds to (2.34) as considered in the abstract setting. Hence, all results as given in Subsection 2.5 remain true. Instead of the linear system (2.29) we now have to solve a discrete variational inequality to find $\mathbf{y}_{\varrho h} \in \mathbb{R}^M \leftrightarrow y_{\varrho h} \in K_{s,h}$ such that

$$((\mathbf{M}_h + \varrho \, \mathbf{K}_h) \mathbf{y}_{\varrho h} - \overline{\mathbf{y}}_h, \mathbf{y} - \mathbf{y}_{\varrho h}) \geq 0 \quad (3.23)$$

is satisfied for all $\mathbf{y} \in \mathbb{R}^M \leftrightarrow y_h \in K_{s,h}$. We define the discrete Lagrange multiplier $\boldsymbol{\lambda} := (\mathbf{M}_h + \varrho \, \mathbf{K}_h) \mathbf{y}_{\varrho h} - \overline{\mathbf{y}}_h$, and the index set of the active nodes, $I_{s,\pm} := \{k := 1, \ldots, M : y_k = g_{\pm,k} := g_\pm(x_k)\}$. With this we then conclude the discrete complementarity conditions

$$\lambda_k = 0, \; g_{-,k} < y_k < g_{+,k} \text{ for } k \notin I_{s,\pm}, \; \lambda_k \leq 0 \text{ for } k \in I_{s,+}, \; \lambda_k \geq 0 \text{ for } k \in I_{s,-},$$

which are equivalent to

$$\lambda_k = \min\{0, \lambda_k + c(g_{+,k} - y_k)\} + \max\{0, \lambda_k + c(g_{-,k} - y_k)\}, \quad c > 0.$$

Hence we have to solve a system $\mathbf{F}(\mathbf{y}_{\varrho h}, \boldsymbol{\lambda}) = \mathbf{0}$ of (non)linear equations

$$\begin{aligned}\mathbf{F}_1(\mathbf{y}_{\varrho h}, \boldsymbol{\lambda}) &= (\mathbf{M}_h + \varrho\,\mathbf{K}_h)\mathbf{y}_{\varrho h} - \overline{\mathbf{y}}_h - \boldsymbol{\lambda} = \mathbf{0},\\ \mathbf{F}_2(\mathbf{y}_{\varrho h}, \boldsymbol{\lambda}) &= \boldsymbol{\lambda} - \min\{0, \boldsymbol{\lambda} + c(\mathbf{g}_+ - \mathbf{y})\} + \max\{0, \boldsymbol{\lambda} + c(\mathbf{g}_- - \mathbf{y})\} = \mathbf{0},\end{aligned}$$

where the latter have to be considered componentwise. For any given $\boldsymbol{\lambda}$ the system $\mathbf{F}_1(\mathbf{y}_{\varrho h}, \boldsymbol{\lambda}) = \mathbf{0}$ reads

$$(\mathbf{M}_h + \varrho\,\mathbf{K}_h)\mathbf{y}_{\varrho h} = \overline{\mathbf{y}}_h + \boldsymbol{\lambda}$$

which can be solved as in the unconstrained case, and it remains to solve the nonlinear system

$$\mathbf{F}_2((\mathbf{M}_h + \varrho\,\mathbf{K}_h)^{-1}(\overline{\mathbf{y}}_h + \boldsymbol{\lambda}), \boldsymbol{\lambda}) = \mathbf{0}. \tag{3.24}$$

For the solution of (3.24) we can apply a semi-smooth Newton method which is equivalent to a primal-dual active set strategy, see, e.g., [8, 30, 31, 33], and [15]. Instead of solving the nonlinear system (3.24) we can solve the variational inequality (3.23) by using multigrid methods, see [20] for an overview of related methods. This will be a topic of future research. When considering control constraints we replace K_s by

$$\begin{aligned}K_c \;:=\; \Big\{ y \in H_0^1(\Omega) : \langle f_-, \phi\rangle_{L^2(\Omega)} \le \langle \nabla y, \nabla\phi\rangle_{L^2(\Omega)} \le \langle f_+, \phi\rangle_{L^2(\Omega)}\\ \text{for all } \phi \in H_0^1(\Omega) \text{ with } \phi \ge 0 \text{ a.e. in } \Omega \Big\},\end{aligned}$$

where we assume $f_\pm \in L^2(\Omega)$. For a more detailed discussion we refer to [15], see also [19].

3.8 Numerical results

We first reconsider the 1d examples from the introduction. Since we can analytically solve all of these 1d OCPs, we can easily verify the numerical results for both the L^2 and the H^{-1} regularization with respect to the accuracy of the approximation of the computed finite element state to the target and the approximation of the cost of the control. Furthermore, we numerically study three multi-dimensional examples with targets possessing different features. The first two examples are taken from [10], where beside the standard L^2 regularization also other regularizations including measure and BV regularizations are studied both theoretically and numerically. These two benchmark examples from [10] are given in the two-dimensional (2d) computational domain $\Omega = (-1, 1)^2$. Here we also consider the three-dimensional counterparts given in $\Omega = (-1, 1)^3$. Finally, we numerically study a three-dimensional (3d) example with a more complicated discontinuous target that was already used in our paper [38] for numerical tests. In this example, the target is zero with exception of several small inclusions which are nothing but hot spots. For the three multi-dimensional examples, we always us the H^{-1} regularization, which is sometimes also called energy regularization, as described in Subsection 3.1. The finite element discretization, the control recovering, the solvers, and the nested iteration procedure also follow the description as given in Subsections 3.1, 3.5, and 3.6. In particular, we solve the system (3.20) by pcg preconditioned by the lumped mass matrix $\mathbf{C}_h = \text{lump}(\mathbf{M}_h)$. In the non-nested version, the pcg iterations are stopped as soon as the $\mathbf{S}_{\varrho h}^T \mathbf{C}_h^{-1} \mathbf{S}_{\varrho h}$ energy norm of the initial iteration error $\mathbf{e}_h^0 = \mathbf{u}_{\varrho h} - \mathbf{u}_{\varrho h}^0$ is reduced by a factor of 10^6. In terms of the residual $\mathbf{r}_h^n = \mathbf{S}_{\varrho h}\mathbf{e}_h^n$, the stopping criterion can be written in the form

$$(\mathbf{C}_h^{-1}\mathbf{r}_h^n, \mathbf{r}_h^n)^{1/2} \;\le\; 10^{-6}\,(\mathbf{C}_h^{-1}\mathbf{r}_h^0, \mathbf{r}_h^0)^{1/2}.$$

We always use a zero initial guess in the nonnested iterations while, in the nested iteration procedure, the initial guess is interpolated from the coarser mesh and the iteration is stopped when the discretization error is reached.

3.8.1 Numerical justification of the theoretical results in 1d

We reconsider the examples from the introduction, especially the smooth target (1.11) and the discontinuous target (1.13) for which we computed the exact solutions $y_{1,\varrho}$ and $y_{3,\varrho}$ for (3.4) and (3.13) depending on $\varrho > 0$ explicitely.

Let us consider piecewise linear finite elements, defined on the decomposition of $(0,1)$ into equidistant nodes $x_k = k/N$, $k = 0, \ldots, N$, for some $N \in \mathbb{N}$. Given the mesh size $h = 1/N$, the optimal choice of the regularization parameter is $\varrho = \varrho_{H^{-1}} = h^2$ and $\varrho = \varrho_{L^2} = h^4$ in the case of the H^{-1} regularization and L^2 regularization, respectively. Now let us fix $\varrho = 2^{-\ell}$ for some $\ell \in \mathbb{N}$ and choose two decompositions of mesh sizes $h_{H^{-1}}$ and h_{L^2}, such that $\varrho = h_{H^{-1}}^2 = h_{L^2}^4$. On these meshes we compute the corresponding states $y_{i,\varrho h_{H^{-1}}}$ and $y_{i,\varrho h_{L^2}}$, $i = 1, 3$ of the H^{-1} regularization and the L^2 regularization, solving (3.5) and (3.15), respectively. The states are plotted in Figure 4. We clearly observe, that the reconstructed states approximate the exact states very well. We note that the larger distance of the target to the state reconstructed using the L^2 regularization with respect to the state reconstructed by the H^{-1} regularization does not stem from the state being computed on a coarser mesh, but is rather inherited from the continuous problem. This is further supported by Figure 5, where the exact errors $\|\overline{y} - y_\varrho\|_{L^2(0,1)}$ are plotted against the finite element errors $\|\overline{y} - y_{\varrho h}\|_{L^2(0,1)}$.

In a post processing step, we now compute the reconstruction of the control $\widetilde{u}_{\varrho h} \in X_h$ on the primal mesh and $\widetilde{u}_{\varrho h}^d \in U_h^d$ on the dual mesh, solving (3.16). The results are presented in Figure 4. In the one dimensional case $\Omega = (0,1)$, we can compute the exact cost, by introducing $\widetilde{y}_\varrho \in H_0^1(0,1)$ given as

$$\widetilde{y}_\varrho(x) = \int_0^1 G(x,y)\widetilde{u}_{\varrho h}(y)\, dy,$$

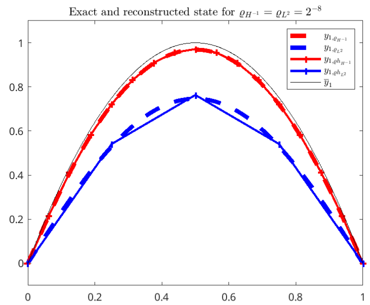
where

$$G(x,y) = \begin{cases} y\,(1-x), & y \in (0,x) \\ x\,(1-y), & y \in (x,1), \end{cases}$$

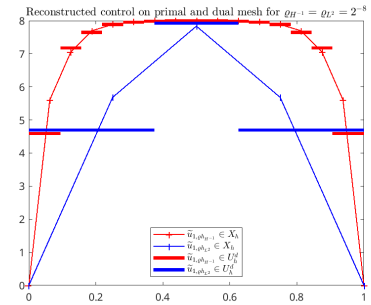
denotes the Greens function, i.e., $-\widetilde{y}_\varrho'' = \widetilde{u}_{\varrho h}$ in $(0,1)$. Then we compute

$$\begin{aligned} \|\widetilde{u}_{\varrho h}\|_{H^{-1}(0,1)} &= \sup_{0 \neq v \in H_0^1(0,1)} \frac{\langle \widetilde{u}_{\varrho h}, v\rangle_{L^2(0,1)}}{\|v'\|_{L^2(0,1)}} = \sup_{0 \neq v \in H_0^1(0,1)} \frac{\langle -\widetilde{y}_\varrho'', v\rangle_{L^2(0,1)}}{\|v'\|_{L^2(0,1)}} \\ &= \sup_{0 \neq v \in H_0^1(0,1)} \frac{\langle \widetilde{y}_\varrho', v'\rangle_{L^2(0,1)}}{\|v'\|_{L^2(0,1)}} = \|\widetilde{y}_\varrho'\|_{L^2(0,1)} = \sqrt{\langle \widetilde{u}_{\varrho h}, \widetilde{y}_\varrho\rangle_{L^2(0,1)}}. \end{aligned}$$

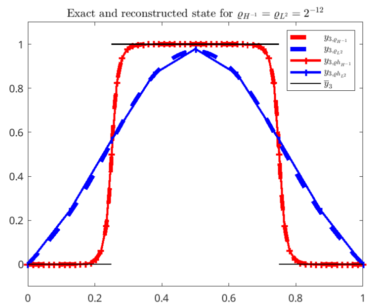
The cost of the different control reconstructions is compared with the exact cost $\|u_\varrho\|_{H^{-1}(0,1)}$ and $\|u_\varrho\|_{L^2(0,1)}$ in Figures 6 and 7, respectively. We note that all the computations align very well and fit the exact cost. Although it might seem that the L^2 regularization comes with lower cost, we again stress that for fixed $\varrho > 0$ the error $\|\overline{y} - y_\varrho\|_{L^2(0,1)}$ for the L^2 regularization is larger compared to the H^{-1} regularization. Thus, to achieve the same level of accuracy one needs to consider a smaller regularization parameter ϱ leading to the same cost for the L^2 regularization. Thus, for the implementation the H^{-1} regularization is beneficial, as the regularization parameter does not need to be chosen too small. Hence, if one is interested in the L^2 cost, we propose to compute the state via the H^{-1} regularization, then reconstruct the control and compute the L^2 cost and check if it is still below the threshold given by the application.



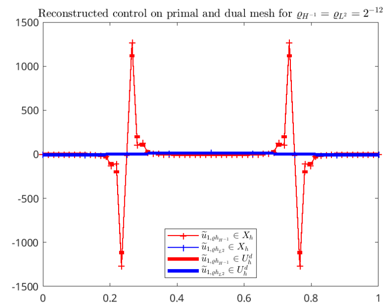
(a) $\overline{y}_1$, $y_{1,\varrho}$ and $y_{1,\varrho h}$



(b) $\widetilde{u}_{1,\varrho h}$



(c) $\overline{y}_3$, $y_{3,\varrho}$ and $y_{3,\varrho h}$



(d) $\widetilde{u}_{3,\varrho h}$

Figure 4: Targets $\overline{y}_1$, $\overline{y}_3$, (exact) reconstructed states using the H^{-1} and L^2 regularization $y_{i,\varrho}$ and $y_{i,\varrho h}$, respectively. And reconstruction of the controls $\widetilde{u}_{i,\varrho h}$ on the primal and dual mesh.

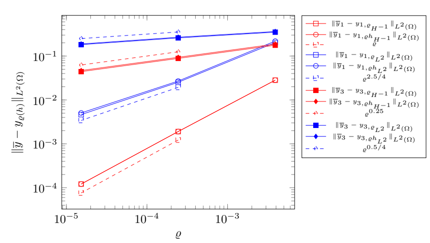
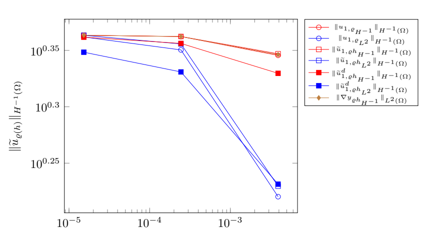
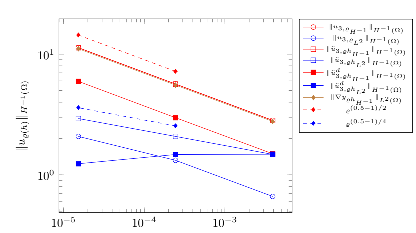


Figure 5: Errors $\|\overline{y} - y_\varrho\|_{L^2(\Omega)}$ and $\|\overline{y} - y_{\varrho h}\|_{L^2(\Omega)}$ for the different targets $\overline{y}_1$ and $\overline{y}_3$ and for the H^{-1} and the L^2 regularization.

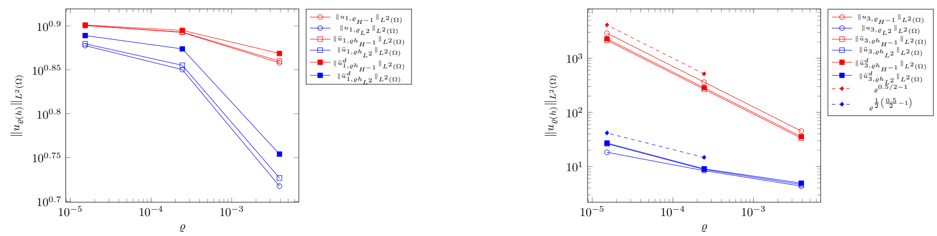


(a) Cost $\|u_{1,\varrho}\|_{H^{-1}(\Omega)}$ and $\|\widetilde{u}_{1,\varrho h}\|_{H^{-1}(\Omega)}$



(b) Cost $\|u_{3,\varrho}\|_{H^{-1}(\Omega)}$ and $\|\widetilde{u}_{3,\varrho h}\|_{H^{-1}(\Omega)}$

Figure 6: Cost $\|u_\varrho\|_{H^{-1}(\Omega)}$ and cost of the reconstructed control $\|\widetilde{u}_{\varrho h}\|_{H^{-1}(\Omega)}$ for the targets $\overline{y}_1$ and $\overline{y}_3$ when choosing $\varrho = \varrho_{H^{-1}} = h^2_{H^{-1}} = \varrho_{L^2} = h^4_{L^2}$.



(a) Cost $\|u_{1,\varrho}\|_{L^2(\Omega)}$ and $\|\widetilde{u}_{1,\varrho h}\|_{L^2(\Omega)}$ (b) Cost $\|u_{3,\varrho}\|_{L^2(\Omega)}$ and $\|\widetilde{u}_{3,\varrho h}\|_{L^2(\Omega)}$

Figure 7: Cost $\|u_\varrho\|_{L^2(\Omega)}$ and cost of the reconstructed control $\|\widetilde{u}_{\varrho h}\|_{L^2(\Omega)}$ for the targets $\overline{y}_1$ and $\overline{y}_3$ when choosing $\varrho = \varrho_{H^{-1}} = h^2_{H^{-1}} = \varrho_{L^2} = h^4_{L^2}$.

3.8.2 Peak

We now consider the smooth target

$$\overline{y}(x) = e^{-50[(x_1-0.2)^2+(x_2+0.1)^2]}$$

with $x = (x_1, x_2) \in \Omega = (-1,1)^2 \subset \mathbb{R}^2$; see [10]. We note that this target does not vanish on the boundary $\partial\Omega$ of Ω. Therefore, it does not belong to the state space $Y = H^1_0(\Omega)$. The violation of the homogeneous boundary conditions may cause boundary layers which affect the convergence of the finite element approximation to the state. Furthermore, we consider a 3d version of this example with the target function

$$\overline{y}(x) = e^{-50[(x_1-0.2)^2+(x_2+0.1)^2+(x_3+0.3)^2]}$$

with $x = (x_1, x_2, x_3) \in \Omega = (-1,1)^3 \subset \mathbb{R}^3$. Table 1 presents the numerical results for the nonnested and nested iteration regimes. We obtain the full experimental order of convergence (eoc) as we would get for smooth targets satisfying homogeneous Dirichlet conditions. The nested iteration procedure produces approximations with the same accuracy as the nonnested iteration but several times faster.

ℓ	#Dofs	Non-nested			Nested		
		error	eoc	its (time)	error	eoc	its (time)
1	4,913	3.34e−2	–	10 (2.1e−3 s)	3.34e−2	–	10(2.1e−3 s)
2	35,937	1.25e−2	1.41	11 (3.8e−3 s)	1.28e−2	1.39	2 (8.5e−4 s)
3	274,625	3.48e−3	1.85	11 (5.5e−3 s)	3.74e−3	1.77	2 (1.2e−3 s)
4	2,146,689	8.87e−4	1.97	11 (2.9e−2 s)	9.91e−4	1.92	2 (6.5e−3 s)
5	16,974,593	2.22e−4	2.00	11 (2.1e−1 s)	2.54e−4	1.96	2 (4.7e−2 s)
6	135,005,697	5.56e−5	2.00	11 (1.5e−0 s)	6.43e−5	1.98	2 (3.7e−1 s)

Table 1: Peak ($d = 3$): Comparison of Nonnested and Nested iterations: L^2 error, experimental order of convergence eoc, number its of pcg iterations, and computational time (time) in seconds on uniform mesh refinements, using 256 cores.

3.8.3 Pedestal

The next example is also inspired by a 2d example used in the numerical experiments presented in [10]. The target

$$\overline{y}(x) = \begin{cases} 1 & \text{if } x \in (-1/2, 1/2)^d, \\ 0 & \text{else,} \end{cases}$$

is nothing than a pedestal with a plateau of the high 1. This target $\overline{y}$ is discontinuous, and, therefore, it does not belong the state space $Y = H^1_0(\Omega)$, but to the spaces $H^s(\Omega)$ with $s < 1/2$. Thus, we can only expect reduced convergence rates in the case of uniform mesh refinement. Table 2 presents some numerical results for the three-dimensional case, i.e., $d = 3$. We again compare the nonnested and nested iteration regimes. Since the target does only belong to $H^s(\Omega)$ with $s < 0.5$, we only see the reduced eoc of about 0.5. The nested iteration procedure again produces approximations with the same accuracy as the nonnested iteration but several times faster.

ℓ	#Dofs	Non-nested			Nested		
		error	eoc	Its (Time)	error	eoc	Its (Time)
1	4,913	3.66e−1	–	10 (2.2e−3 s)	3.66e−0	–	10(2.2e−3 s)
2	35,937	2.67e−1	0.45	11 (3.2e−3 s)	2.73e−1	0.43	1 (5.2e−4 s)
3	274,625	1.87e−1	0.52	11 (5.5e−3 s)	1.93e−1	0.50	1 (7.5e−4 s)
4	2,146,689	1.31e−1	0.51	11 (2.8e−2 s)	1.35e−1	0.52	1 (4.5e−3 s)
5	16,974,593	9.24e−2	0.51	11 (2.1e−1 s)	9.43e−2	0.52	1 (3.5e−2 s)
6	135,005,697	6.52e−2	0.50	11 (1.5e−0 s)	6.62e−2	0.51	1 (2.5e−1 s)

Table 2: Pedestal ($d = 3$): Comparison of Non-nested and Nested iterations: L^2 error, experimental order of convergence eoc, number its of pcg iterations, and computational time (time) in seconds on uniform mesh refinements, using 256 cores.

3.8.4 Inclusions

Finally, we consider the target

$$\overline{y}(x) = \begin{cases} 1 & \text{if } (x_1 - 0.5)^2 + (x_2 - 0.5)^2 + (x_3 - 0.5)^2 \leq 0.05^2, \\ 2 & \text{if } (x_1 - 0.5)^2 + (x_2 - 0.25)^2 + (x_3 - 0.75)^2 \leq 0.0625^2, \\ 3 & \text{if } (x_1 - 0.5)^2 + (x_2 - 0.75)^2 + (x_3 - 0.75)^2 \leq 0.0625^2, \\ 4 & \text{if } (x_1 - 0.5)^2 + (x_2 - 0.75)^2 + (x_3 - 0.25)^2 \leq 0.075^2, \\ 5 & \text{if } x_1 \in [0.25, 0.75] \text{ and } x_2 \in [0.45, 0.5] \text{ and } x_3 \in [0.125, 0.375], \\ 6 & \text{if } (x_1 - 0.5)^2 + (x_2 - 0.25)^2 + (x_3 - 0.25)^2 \leq 0.0625^2, \\ 0 & \text{else,} \end{cases}$$

with piecewise constant, positive values inside small inclusions in the 3d domain $\Omega = (0,1)^3$. Outside of these hot spots the target is zero. Again, we expect interface boundary layers and reduced convergence rate in the case of uniform mesh refinement. Table 3 provides the numerical results for non-nested and nested iterations. First we observe that the eoc is about 0.5 that perfectly corresponds to the regularity of the target. The nested iteration procedure again produces approximations with the same accuracy as the non-nested iteration but several times faster. More precisely, at the finest refinement level $\ell = 6$ with 135,005,697 unknowns (#Dofs), the nested iteration reaches the same accuracy (error) as the nonnested iteration within 0.18 seconds in comparison with 1.50 seconds needed for the nonnested iteration. We note that we stopped the non-nested iteration at the relative accuracy 10^{-6}. This can be relaxed, and the relative accuracy can be adapted to the discretization error. Figure 8 shows the computed finite element approximation to the state at level $\ell = 4$.

4 An overview on other applications

In this section, we are going to discuss some selected further applications of the abstract theory as presented in Section 2.

ℓ	#Dofs	Non-nested			Nested		
		error	eoc	Its (Time)	error	eoc	Its (Time)
1	$4,913$	3.50e−1	–	22 (5.3e−3 s)	3.50e−1	–	22 (5.1e−3 s)
2	$35,937$	3.26e−1	0.10	25 (6.8e−3 s)	3.31e−1	0.08	1 (6.4e−4 s)
3	$274,625$	2.35e−1	0.47	24 (9.3e−3 s)	2.35e−1	0.50	2 (1.0e−3 s)
4	$2,146,689$	1.60e−1	0.55	24 (2.2e−2 s)	1.61e−1	0.55	2 (2.6e−3 s)
5	$16,974,593$	1.13e−1	0.51	24 (2.2e−1 s)	1.12e−1	0.52	2 (2.7e−2 s)
6	$135,005,697$	7.95e−2	0.50	24 (1.5e−0 s)	7.95e−2	0.50	2 (1.8e−1 s)

Table 3: Inclusions ($d = 3$): Comparison of non-nested and nested iterations: L^2 error, experimental order of convergence eoc, number its of pcg iterations, and computational time (time) in seconds on uniform mesh refinements, using 512 cores.

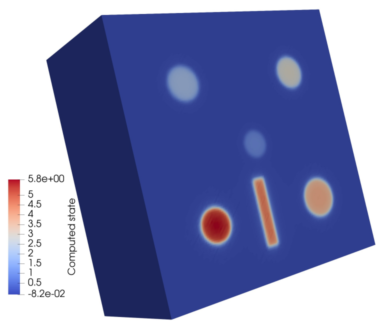


Figure 8: Computed state solution on the uniform refined mesh with $2,146,689$ Dofs at the cut $x_3 = 1/2$.

4.1 Dirichlet boundary control of the Laplace equation

As a first example, we consider the Dirichlet boundary control problem to minimize

$$\mathcal{J}(y_\varrho, u_\varrho) = \frac{1}{2}\,\|y_\varrho - \overline{y}\|^2_{L^2(\Omega)} + \frac{1}{2}\,\varrho\,|u_\varrho|^2_{H^{1/2}(\Gamma)} \tag{4.1}$$

subject to the Dirichlet boundary value problem for the Laplace equation

$$-\Delta y_\varrho = 0 \quad \text{in } \Omega, \quad y_\varrho = u_\varrho \quad \text{on } \Gamma := \partial\Omega, \tag{4.2}$$

where the control u_ϱ is now nothing but the Dirichlet data of the state y_ϱ on the boundary Γ of Ω. In this case, we again have $H_Y = L^2(\Omega)$, but the state space Y now is the space of all harmonic functions in $H^1(\Omega)$, i.e., $Y := \{y \in H^1(\Omega) : \langle \nabla y, \nabla v\rangle_{L^2(\Omega)} = 0 \ \forall v \in H^1_0(\Omega)\}$. The state to control map $u_\varrho = \gamma_0^{\text{int}} y_\varrho$ is then given by the interior Dirichlet trace operator $\gamma_0^{\text{int}} : H^1(\Omega) \to H^{1/2}(\Gamma)$, i.e., $B = \gamma_0^{\text{int}} : Y \to U = H^{1/2}(\Gamma)$. Moreover, we introduce $X = H_*^{-1/2}(\Gamma) := \{\psi \in H^{-1/2}(\Gamma) : \langle \psi, 1\rangle_\Gamma = 0\}$. A semi-norm in the control space $U = H^{1/2}(\Gamma)$ is induced by the Steklov–Poincaré operator $S : H^{1/2}(\Gamma) \to H_*^{-1/2}(\Gamma)$ which is defined via

$$|u_\varrho|^2_{H^{1/2}(\Gamma)} = \langle S u_\varrho, u_\varrho\rangle_\Gamma = \int_\Gamma \frac{\partial}{\partial n_x} y_\varrho(x)\, y_\varrho(x)\, dx = \int_\Omega |\nabla y_\varrho(x)|^2\, dx,$$

where $y_\varrho \in Y$ is the harmonic extension of $u_\varrho \in U$. With this we can write the minimization problem (4.1)-(4.2) as in (2.6) to minimize

$$\check{\mathcal{J}}(y_\varrho) = \frac{1}{2}\,\|y_\varrho - \overline{y}\|^2_{L^2(\Omega)} + \frac{1}{2}\,\varrho\,\|\nabla y_\varrho\|^2_{L^2(\Omega)} \tag{4.3}$$

on the space Y of harmonic functions. The minimizer $y_\varrho \in Y$ of the reduced cost functional (4.1) is then charcterized as the unique solution of the gradient equation in variational form, satisfying

$$\langle y_\varrho, y\rangle_{L^2(\Omega)} + \varrho\,\langle \nabla y_\varrho, \nabla y\rangle_{L^2(\Omega)} = \langle \overline{y}, y\rangle_{L^2(\Omega)} \quad \text{for all } y \in Y. \tag{4.4}$$

Moreover, in this particular case, we can formulate state and control constraints at once by defining $Y_{s/c} := \{y \in Y : g_- \le y \le g_+\}$, where $g_\pm \in Y$ are given (constant) barrier functions. For $y_\varrho \in Y_{s/c}$, the minimizer of (4.3) is then determined as the unique solution of the variational inequality (2.34) satisfying

$$\langle y_\varrho, y-y_\varrho\rangle_{L^2(\Omega)} + \varrho\,\langle \nabla y_\varrho, \nabla(y-y_\varrho)\rangle_{L^2(\Omega)} \ge \langle \overline{y}, y-y_\varrho\rangle_{L^2(\Omega)} \quad \text{for all } y \in Y_{s/c}. \tag{4.5}$$

It is obvious that all regularization error estimates as given in the abstract setting remain true. In order to incorporate the constraints in the definition of the state space Y, instead of (4.4) we can introduce a Lagrange multiplier $p_\varrho \in H^1_0(\Omega)$ and solve a saddle point variational formulation for $(y_\varrho, p_\varrho) \in H^1(\Omega) \times H^1_0(\Omega)$ satisfying

$$\begin{array}{rcl} \langle y_\varrho, y\rangle_{L^2(\Omega)} + \varrho\,\langle \nabla y_\varrho, \nabla y\rangle_{L^2(\Omega)} + \langle \nabla p_\varrho, \nabla y\rangle_{L^2(\Omega)} & = & \langle \overline{y}, y\rangle_{L^2(\Omega)}, \\ \langle \nabla y_\varrho, \nabla q\rangle_{L^2(\Omega)} & = & 0 \end{array} \tag{4.6}$$

for all $(y, q) \in H^1(\Omega) \times H^1_0(\Omega)$. Finite element error estimates for the numerical solution of (4.6) follow when using standard arguments.

Remark 4. *The Dirichlet boundary control problem* (4.1)-(4.2) *in the control space* $U = H^{1/2}(\Gamma)$ *was first considered in [52], see also [9, 17, 66]; for the consideration of control or state constraints, see [15, 19, 22]. Further extensions include Dirichlet control for Stokes flow [18], or for parabolic evolution equations [21].*

4.2 Distributed control of parabolic evolution equations

The abstract theory as given in Section 2 is not restricted to elliptic state equations, but can also be applied to time-dependent PDEs. As an example for an parabolic evolution equation we consider the minimization of

$$\mathcal{J}(y_\varrho, u_\varrho) = \frac{1}{2}\,\|y_\varrho - \overline{y}\|^2_{L^2(Q)} + \frac{1}{2}\,\|u_\varrho\|^2_{L^2(0,T;H^{-1}(\Omega))}$$

subject to the initial-boundary value problem for the heat equation

$$\partial_t y_\varrho - \Delta_x y_\varrho = u_\varrho \quad \text{in } Q, \quad y_\varrho = 0 \text{ on } \Sigma, \quad y_\varrho = 0 \text{ on } \Sigma_0,$$

where, for a given time horizon $T > 0$, $Q := \Omega \times (0,T)$ is the space-time cylinder with the lateral boundary $\Sigma = \partial\Omega \times (0,T)$ and the bottom $\Sigma_0 = \Omega \times \{0\}$. In this case, we have $H_X = H_Y = L^2(Q)$, as well as $X = L^2(0,T;H^1_0(\Omega))$ and $U = X^* = L^2(0,T;H^{-1}(\Omega))$. Moreover, the related state space is defined as $Y = \{y \in X : \partial_t y \in X^*, y(0) = 0\}$. Within this setting we have $B = \partial_t - \Delta_x : Y \to X^*$, and $A = -\Delta_x : X \to X^*$. The reduced optimality system then reads to find $(p_\varrho, y_\varrho) \in X \times Y$ such that

$$\begin{array}{rcl} \frac{1}{\varrho}\,\langle \nabla_x p_\varrho, \nabla_x q\rangle_{L^2(\Omega)} + \langle \partial_t y_\varrho, q\rangle_Q + \langle \nabla_x y_\varrho, \nabla_x y\rangle_{L^2(\Omega)} & = & 0, \\ -\langle p_\varrho, \partial_t y\rangle_Q - \langle \nabla_x p_\varrho, \nabla_x y\rangle_{L^2(\Omega)} + \langle y_\varrho, y\rangle_{L^2(\Omega)} & = & \langle \overline{y}, y\rangle_{L^2(\Omega)} \end{array}$$

is satisfied for all $(q, y) \in X \times Y$. This variational formulation and its space-time finite element discretization was analysed in [43], for other approaches, see, e.g., [3, 16, 25, 41, 42, 48, 65]. While all of these approaches require the solution of a coupled forward-backward system, a new approach related to the abstract variational formulation (2.7) was recently considered in [46], where D is induced by the norm of the anisotropic Sobolev space $H^{1,1/2}_{0;0,}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^{1/2}_{0,}(0,T;L^2(\Omega))$.

4.3 Distributed control of hyperbolic evolution equations

As an example for a hyperbolic evolution equation as state equation we consider the minimization of

$$\mathcal{J}(y_\varrho, u_\varrho) = \frac{1}{2}\,\|y_\varrho - \overline{y}\|^2_{L^2(Q)} + \frac{1}{2}\,\|u_\varrho\|^2_{L^2(0,T;H^{-1}(\Omega))}$$

subject to the initial-boundary value problem for the wave equation

$$\partial_{tt} y_\varrho - \Delta_x y_\varrho = u_\varrho \quad \text{in}\, Q, \quad y_\varrho = 0 \text{ on } \Sigma, \quad y_\varrho = \partial_t y_\varrho = 0 \text{ on } \Sigma_0. \qquad (4.7)$$

When assuming $u_\varrho \in L^2(Q)$, the initial boundary value problem (4.7) admits a unique solution $y_\varrho \in H^{1,1}_{0;0,}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;L^2(\Omega))$; see, e.g., [36, 61]. Hoeever, this does not define an isomorphism. When using $X = H^{1,1}_{0;,0}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;L^2(\Omega))$, i.e., $U = [H^{1,1}_{0;,0}(Q)]^*$, and in order to ensure that the wave operator $B := \Box := \partial_{tt} - \Delta_x : Y \to X^*$ is an isomorphism, we need to define the state space accordingly. Therefore, and following [62], we introduce

$$Y := \mathcal{H}_{0;0,}(Q) := \overline{H^{1,1}_{0;0,}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}}, \quad \|u\|_{\mathcal{H}(Q)} := \sqrt{\|u\|^2_{L^2(Q)} + \|\Box\widetilde{u}\|^2_{[H^1_0(Q_-)]^*}},$$

where $\widetilde{u}$ is the zero extension of $u \in L^2(Q)$ to $Q_- := \Omega \times (-T,T)$. In this setting we can define $A : X \to X^*$, satisfying

$$\langle Ap, q\rangle_Q := \langle \partial_t p, \partial_t q\rangle_{L^2(Q)} + \langle \nabla_x p, \nabla_x q\rangle_{L^2(Q)} \quad \text{for all } p, q \in X.$$

Then we can write the abstract gradient equation (2.3) as variational problem to find $(y_\varrho, p_\varrho) \in \mathcal{H}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q)$ such that

$$\varrho^{-1}\,\langle Ap_\varrho, q\rangle_Q + \langle \Box\widetilde{y}_\varrho, \mathcal{E}q\rangle_{Q_-} = 0, \quad -\langle \Box\widetilde{y}, \mathcal{E}p_\varrho\rangle_{Q_-} + \langle y_\varrho, y\rangle_{L^2(Q)} = \langle \overline{y}, y\rangle_{L^2(Q)} \quad (4.8)$$

is satisfied for all $(y,q) \in \mathcal{H}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q)$, where $\mathcal{E} : H^{1,1}_{0;,0}(Q) \to H^1_0(Q_-)$ is a suitable extension operator, e.g., reflection in time with respect to $t = 0$. In any case, for a space-time finite element Galerkin discretization of (4.8), we introduce the standard finite element spaces $Y_h = S^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ and $X_h = S^1_h(\Omega) \cap H^{1,1}_{0;,0}(Q)$ of piecewise linear continuous basis functions. For $(y_h, q_h) \in Y_h \times X_h$ and following [62, Lemma 3.5] we have

$$\langle \Box y_h, \mathcal{E}q_h\rangle_{Q_-} = -\langle \partial_t y_h, \partial_t q_h\rangle_{L^2(Q)} + \langle \nabla_x y_h, \nabla_x q_h\rangle_{L^2(Q)}.$$

Hence we have to find $(y_{\varrho h}, p_{\varrho h}) \in Y_h \times X_h$ such that

$$\begin{aligned} \langle \partial_t p_{\varrho h}, \partial_t q_h\rangle_{L^2(Q)} &+ \langle \nabla_x p_{\varrho h}, \nabla_x q_h\rangle_{L^2(Q)} \\ &- \langle \partial_t y_{\varrho h}, \partial_t q_h\rangle_{L^2(Q)} + \langle \nabla_x y_{\varrho h}, \nabla_x q_h\rangle_{L^2(Q)} \;=\; 0, \\ \langle \partial_t y_h, \partial_t p_{\varrho h}\rangle_{L^2(Q)} - \langle \nabla_x y_h, \nabla_x p_{\varrho h}\rangle_{L^2(Q)} &+ \langle y_{\varrho h}, y_h\rangle_{L^2(Q)} \;=\; \langle \overline{y}, y_h\rangle_{L^2(Q)} \end{aligned}$$

is satisfied for all $(y_h, q_h) \in Y_h \times X_h$. For a more detailed analysis of this approach, see [47]; for other approaches we refer to, e.g., [24, 35, 50, 53, 67].

5 Conclusions and outlook

We have first considered abstract tracking-type OCPs subject to some state equation $By = u$ where we think of linear elliptic, parabolic, and hyperbolic PDEs or PDE systems. If the state operator B is an isomorphism between the state space Y

and the control space U, then the OCP can be reduced to a state-based optimality condition that is nothing but a state-based operator equation, or to a state-based variational inequality in the case of additional abstract constraints imposed on the state or the control. We have started with the investigation of the case without any box constraints. For this case, we have provided estimates of the error $\|y_\varrho - \overline{y}\|_{H_Y}$ between the optimal state y_ϱ and the desired state $\overline{y}$ in the tracking norm $\| \cdot \|_{H_Y}$ in terms of the regularization parameter ϱ and the regularity of the desired state $\overline{y}$. After the Galerkin discretization of the state-based operator equation, we have analysed the error $\|y_{\varrho h} - \overline{y}\|_{H_Y}$ between the Galerkin solution $y_{\varrho h}$ and the desired state $\overline{y}$. We have observed that the finite element mesh size h is strongly related to the regularization parameter ϱ in order to obtain the asymptotically optimal convergence rate. The optimal control $\widetilde{u}_{\varrho h}$ can be computed from the Galerkin state solution $y_{\varrho h}$ in a postprocessing procedure. Moreover, we have presented efficient (parallel) iterative solvers that can be used in a smart nested iteration process where we can control the accuracy of the computed state and the energy cost of the control. Furthermore, we have shown that one can easily add and analyse constraints for the state or the control.

In the second part of the paper, we have applied the abstract theoretical framework to the distributed control of Poisson's equation as blueprint for other applications. We have presented numerical results for 1d, 2d, and 3d benchmarks with different features concerning the regularity of the target. These numerical results not only illustrate our theoretical error estimates quantitatively, but also demonstrate that the iterative solvers are very efficient, in particular, in a nested iteration setting on parallel computers.

Finally, we have briefly discussed some other applications of the abstract framework to PDE constrained OCP like the Dirichlet boundary control of the Laplace equation and OCPs subject to parabolic or hyperbolic state equations where space-time finite elements are used for their discretization.

Further extensions include problem classes for other boundary control problems such as Neumann or Robin type problems; OCPs with either partial observations or partial controls, or the optimal control of non-linear state equations such as the Navier–Stokes system.

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