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Boundary integral equation methods for inverse problems in electrical engineering

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Abstract

Solving inverse problems gets more and more important in the computer simulation of technical processes, as for example in electrical engineering. Here, we present some methods of shape reconstruction from electrical capacitance tomography measurement data. To minimize the cost functional,we need to calculate the corresponding shape derivatives. Fast boundary element methods are used for solving the appropriate forward problems to reduce the quadratic effort to an almost linear one.

1 Introduction

In electrical capacitance tomography (ECT), the aim is to determine interior objects inside a surrounding domain from measured data (Huang et al. 1988). These measurements are done by using multi-electrode assemblies (Wegleiter et al. 2005, Kortschak et al. 2005). Mathematically, the direct problem corresponds to mixed boundary value problems with predescribed potentials at the electrodes and zero fluxes elsewhere. The measured data are then obtained from the fluxes at the electrodes. To find the shape of the inscribed objects, we therefore have to solve the corresponding inverse problem.

The inverse problem to be solved corresponds to the minimization of a cost functional for which the associated shape derivative is needed (Sokołowski et al. 1992, Delfour et al. 2001). The application of the Fréchet derivative corresponds to an associated transmission boundary value problem (Kirsch 1992, Hettlich et al. 1996, Hettlich et al. 1998). By using the adjoint state, the shape derivative itself can be described as the unique solution of a related transmission problem (Pironneau 1984, Ito et al. 2001). The unknown shape can be described either by using a suitable parameterization (Kirsch 1992) or by using level set functions (Osher et al. 1988). The latter allows to handle changes in the topology and to identify objects without any a priori information, see (Santosa 1996, Litman et al. 1998, Dorn et al. 2000, Burger 2001, Ito et al. 2001, Hintermüller et al. 2003, Osher et al. 2003, Dorn et al. 2006).

Since the permittivity is assumed to be piecewise constant, boundary integral equation methods (Sauter et al. 2004, Steinbach 2003) can be used to solve the involved transmission problems by applying domain decomposition techniques (Steinbach 2003, Toselli et al. 2005, Wohlmuth 2001). For the use of boundary element methods in the context of inverse problems see, e.g., (Colton et al. 1983, Eppler et al. 2006, Harbrecht et al. to appear). Since the discretization of boundary integral equations leads to dense stiffness matrices, fast boundary element methods are required (Rjasanow et al. 2007), see also (Greengard et al. 1987) for the fast multipole method, (Hackbusch et al. 1989) for panel clustering, (Dahmen et al. 1993) for the use of wavelets, (Bebendorf et al. 2003) for the adaptive cross approximation and (Hackbusch 1999) for the use of $\mathcal{H}$ matrix arithmetics.

The paper is organized as follows. In Section 2, we present a model problem for electrical capacitance tomography and formulate the related shape identification problem. Section 3 is devoted to the derivation of algorithms to determine the interior objects by using the shape derivative and the adjoint variable method in combination with level set functions. In Section 4, we describe a boundary integral formulation to solve the involved transmission problems by using a domain decomposition approach, we discuss the Galerkin discretization of boundary integral operators, and comment on the use of the fast multipole method.

2 Model Problem

2.1 Electrical capacitance tomography

We are interested in the shape identification of some unknown bounded domain $D \subset \Omega$ where $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$, and Ω is decomposed by $\Omega = \Omega_0 \cup \partial D \cup D$, $\Omega_0 := \Omega \backslash \overline{D}$.

In electrical capacitance tomography (ECT) (Huang et al. 1988), the shape ∂D is identified from measurements of the coupling capacitances of a multi-electrode assembly (Wegleiter et al. 2005, Kortschak et al. 2005). In the direct simulation, this corresponds to a mixed boundary value problem with predescribed potentials at the electrodes and zero fluxes elsewhere. Instead, here we will consider a model problem where the potential is given along the boundary $\Gamma = \partial\Omega$ and the fluxes are considered as a result of the measurement. However, the boundary integral approach as described for this model problem can be applied to the more general case straightforwardly.

2.2 Mathematical model problem

Neglecting the wave propagation, the potential u satisfies the electrostatic field equation

$$\begin{aligned} -\operatorname{div}\left(\varepsilon(x)\nabla u(x)\right) &= 0 && \text{for } x \in \Omega, \\ u(x) &= g(x) && \text{for } x \in \Gamma \end{aligned} \tag{1}$$

for given Dirichlet boundary data g, where the permittivity ε is assumed to be piecewise constant, i.e., $\varepsilon(x) = \varepsilon_0$ for $x \in \Omega_0$, $\varepsilon(x) = \varepsilon_D$ for $x \in D$. Defining $u_0(x) = u(x)$ for $x \in \Omega_0$ and $u_D(x) = u(x)$ for $x \in D$, the direct problem (1) corresponds to the transmission boundary value problem

$$\begin{array}{rcll} -\Delta u_0 &=& 0 & \text{in } \Omega_0, \\ -\Delta u_D &=& 0 & \text{in } D, \\ u_0 &=& g & \text{on } \Gamma, \\ u_0 - u_D &=& 0 & \text{on } \partial D, \\ \varepsilon_0 \dfrac{\partial}{\partial \nu} u_0 - \varepsilon_D \dfrac{\partial}{\partial \nu} u_D &=& 0 & \text{on } \partial D, \end{array} \tag{2}$$

where ν denotes the exterior normal vector of ∂D. Note that the transmission boundary value problem (2) is rather standard in domain decomposition methods, see e.g. (Steinbach 2003, Toselli et al. 2005, Wohlmuth 2001).

The solution of the direct problems (1) and (2), respectively, defines the corresponding Neumann data, i.e. the flux

$$t(x) = \varepsilon_0 \frac{\partial}{\partial n} u_0(x) \quad \text{for } x \in \Gamma, \tag{3}$$

where n denotes the exterior normal vector of Γ, satisfying, by using Green's formula,

$$\int_\Gamma t(x) v(x) ds_x = \int_\Omega \varepsilon(x) \nabla u(x) \cdot \nabla v(x) dx$$

for some suitable test functions $v \in H^1(\Omega)$. In particular, by solving the direct problem, this defines a formal map G of the shape ∂D describing the interior domain D onto the related Neumann datum,

$$G(\partial D) = \varepsilon_0 \frac{\partial}{\partial n} u_0 \quad \text{on } \Gamma. \tag{4}$$

The inverse problem consists in finding the shape ∂D and the interior domain D, respectively, from some measured flux f, in particular we have to solve the operator equation

$$G(\partial D) = \varepsilon_0 \frac{\partial}{\partial n} u_0 = f \quad \text{on } \Gamma. \tag{5}$$

Taking into account the physical meaning of the flux t, the appropriate cost functional Ψ is induced by the energy norm in $H^{-1/2}(\Gamma)$ where

$$\begin{aligned} \Psi(\partial D) &= \frac{1}{2} \|G(\partial D) - f\|^2_{H^{-1/2}(\Gamma)} \\ &= \frac{1}{2} \|G(\partial D) - f\|^2_V \end{aligned} \tag{6}$$

and

$$\|w\|_V^2 = \frac{1}{4\pi} \int_\Gamma w(x) \int_\Gamma \frac{1}{|x-y|} w(y)\, ds_y\, ds_x,$$

is an equivalent norm in $H^{-1/2}(\Gamma)$ which is induced by the single layer potential V of the Laplace operator. Therefore, the operator equation (5) corresponds to finding ∂D such that

$$\Psi(\partial D) = \frac{1}{2} \left\| G(\partial D) - f \right\|_V^2 = 0 \tag{7}$$

is satisfied, i.e., we have to find a minimizer of $\Psi(\partial D)$. Note that in most cases the minimization of the cost functional $\Psi(\partial D)$ has to be done by adding an adequate regularization term, see e.g. (Engl et al. 1996) and the references given therein.

3 Solution algorithms

In this section, we want to describe some algorithms to solve the inverse problem (5), i.e., to minimize the cost functional (6) over a set X of admissible shapes. In general, these algorithms may depend on the definition of X which can be given either via parameterizations, or via the use of level set functions.

3.1 Newton algorithm

Starting from a given representation of the shape ∂D_k, a new iterate can be constructed by introducing the formal Newton algorithm

$$\partial D_{k+1} = \partial D_k - G'[\partial D_k]^{-1} \left(G(\partial D_k) - f \right), \tag{8}$$

where $G'[\partial D_k]$ is the Fréchet derivative of G in the sense of $H^{-1/2}(\Gamma)$. Therefore, by introducing the update

$$\partial D_{k+1} = \partial D_k + \boldsymbol{h}_k,$$

we have to solve the linearized system

$$G'[\partial D_k]\boldsymbol{h}_k = f - G(\partial D_k) \tag{9}$$

to realize one Newton step.

3.2 Shape derivatives

For a given vector $\boldsymbol{h}$, the application of the Fréchet derivative is given by

$$G'[\partial D]\boldsymbol{h} = \varepsilon_0 \frac{\partial}{\partial n} u_0',$$

where u_0' is the unique solution of the transmission problem (Hettlich et al. 1996, Hettlich et al. 1998)

$$\begin{array}{rcll} -\Delta u_0' & = & 0 & \text{in } \Omega_0, \\ -\Delta u_D' & = & 0 & \text{in } D, \\ u_0' & = & 0 & \text{on } \Gamma, \end{array} \tag{10}$$

with the transmission conditions

$$\begin{aligned} u_0' - u_D' &= -\frac{\varepsilon_D - \varepsilon_0}{\varepsilon_0} h_\nu \frac{\partial}{\partial \nu} u_D, \\ \varepsilon_0 \frac{\partial}{\partial \nu} u_0' - \varepsilon_D \frac{\partial}{\partial \nu} u_D' &= -(\varepsilon_D - \varepsilon_0) \nabla_\tau \cdot (h_\nu \nabla_\tau u_D) \end{aligned}$$

on ∂D. Note that u_D is the solution of the direct transmission problem (2), $h_\nu = \boldsymbol{h} \cdot \boldsymbol{\nu}$ and ∇_τ is the surface gradient on ∂D.

In the case of a two-dimensional star-like interior domain D and by using a suitable parameter representation of ∂D, for example by polar coordinates, one can derive numerical algorithms to solve the linearized systems (9), see e.g. (Hettlich et al. 1996, Hettlich et al. 1998) and (Kirsch 1992).

3.3 Adjoint variable method

Instead of a direct evaluation of the Fréchet derivative $G'[\partial D]\boldsymbol{h}$ by solving the transmission problem (10), we are interested in alternative representations to be used in the minimization of the cost functional $\Psi(\partial D)$ in (6). To minimize the cost functional, we first note that the corresponding shape derivative reads

$$\begin{aligned} \Psi'(\partial D) &= \int_\Gamma V\,(G(\partial D) - f)\;(G(\partial D))'\,ds_x \\ &= \int_\Gamma p_0(x)\,\left(\varepsilon_0 \frac{\partial}{\partial n} u_0\right)'\,ds_x \end{aligned} \tag{11}$$

where V is the single layer potential and p_0 is the adjoint state function satisfying

$$\begin{aligned} -\Delta p_0 &= 0 && \text{in } \Omega_0, \\ -\Delta p_D &= 0 && \text{in } D, \\ p_0 &= V\left(\varepsilon_0 \frac{\partial}{\partial n} u_0 - f\right) && \text{on } \Gamma. \end{aligned} \tag{12}$$

If we require the transmission conditions

$$\begin{aligned} p_0 - p_D &= 0 && \text{on } \partial D, \\ \varepsilon_0 \frac{\partial}{\partial \nu} p_0 - \varepsilon_D \frac{\partial}{\partial \nu} p_D &= 0 && \text{on } \partial D \end{aligned} \tag{13}$$

in addition, we finally obtain

$$\Psi'(\partial D) = (\varepsilon_D - \varepsilon_0) \int_{\partial D} (\nabla u_D \cdot \nabla p_0)(\boldsymbol{h} \cdot \boldsymbol{\nu})\,ds_x.$$

Therefore, the steepest descent direction $\boldsymbol{h}^*$ to minimize the cost functional $\Psi(\partial D)$ is given by, when assuming $\varepsilon_D > \varepsilon_0$,

$$\boldsymbol{h}^* = -(\nabla u_D \cdot \nabla p_0)\boldsymbol{\nu} \quad \text{on } \partial D.$$

To compute this search direction for a given shape ∂D_k, we have to solve both the direct transmission problem (2) and the adjoint transmission problem (12)–(13). For an approximate solution of these transmission problems, one may use, e.g., either finite element methods or boundary element methods.

Again, this approach is independent of the chosen representation of ∂D_k. However, a line search algorithm has to be applied to determine an appropriate step size. This can be done for example in connection with a level set description of the interface ∂D_k.

3.4 Level set functions

For $\Omega \subset \mathbb{R}^3$, we define (Osher et al. 2003) the level set function

$$\phi(\boldsymbol{x}, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$$

where $\phi(x,t) = 0$ describes the shape ∂D while the interior domain D is characterized by $\phi(x,t) < 0$. By using level set methods, it is quite feasible to describe the change of the boundary ∂D_k in a given direction $\boldsymbol{h}^*$.

If the shape ∂D_k is perturbed by a small vector field $t\boldsymbol{h}$, the level set function $\phi(x,t) = 0$ is changed to

$$\phi(x + t\boldsymbol{h}(x), t) = 0.$$

Taking the derivative with respect to t results in

$$\phi_t(x,t) + \boldsymbol{h}(x) \cdot \nabla_x \phi(x,t) = 0.$$

By using $\boldsymbol{\nu} = -\frac{\nabla\phi}{|\nabla\phi|}$ to describe the exterior normal vector ν on ∂D, we obtain for the deepest descent direction

$$\boldsymbol{h}^* = -(\nabla u_D \cdot \nabla p_0)\boldsymbol{\nu} = (\nabla u_D \cdot \nabla p_0)\frac{\nabla\phi}{|\nabla\phi|} \quad \text{on } \partial D,$$

and therefore, choosing $\boldsymbol{h} = \boldsymbol{h}^*$ gives an Hamilton–Jacobi equation for the level set function $\phi(\boldsymbol{x}, t)$,

$$\phi_t + F(\boldsymbol{x}, t)|\nabla\phi| = 0, \tag{14}$$

where

$$F(\boldsymbol{x}, t) := \nabla u_D(x) \cdot \nabla p_0(x), \quad \text{for } x \in \partial D. \tag{15}$$

Note that the Hamilton–Jacobi equation (14) can be solved by some standard numerical scheme, e.g. by an explicit Euler method.

Summarizing this chapter, we can formulate an algorithm to determine the unknown shape ∂D, see also (Ito et al. 2001).

Algorithm 1 Level set algorithm for shape identification

Set $k = 0$ and choose an initial level set function $\phi^0(x)$.

repeat

Determine $\partial D_k = \{x \ \in \Omega : \phi^k(x) = 0\}$.

Solve the direct transmission problem (2) for D_k and Ω_k to compute u^k.

Solve the adjoint problem (12)–(13) for D_k and Ω_k to compute p^k.

Calculate the normal velocity

$$F^k(x) = \nabla u^k(x) \cdot \nabla p^k(x).$$

Update the level set function

$$\phi_t(x,t) + F^k(x)|\nabla_x \phi(x,t)| = 0, \quad \phi(x, t_k) = \phi_k(x)$$

for $t = t_k$ by an explicit Euler step

$$\frac{\phi^{k+1}(x) - \phi^k(x)}{\Delta t} + F^k(x)|\nabla \phi^k(x)| = 0,$$

where

$$\Delta t = \min\left\{\frac{h}{2\|F^k\|_\infty}, \frac{h}{2}\right\}.$$

Set $k = k + 1$.

until convergence is reached, i.e.,

$$\Psi(\partial D_k) = \frac{1}{2}\left\|\varepsilon_0 \frac{\partial}{\partial n} u_0^k - f\right\|_V^2 < \delta$$

is satisfied where δ is a predescribed accuracy.

4 Fast boundary element methods

The realization of Algorithm 1 is essentially based on the solution of the direct transmission problem (2), and of the adjoint transmission problem (12)–(13). These transmission problems have to be solved many times within the level set algorithm. Therefore, efficient iterative solvers for these direct problems are mandatory. For the solution of transmission problems, we will use a domain decomposition approach which is based on a strong coupling of the Dirichlet data (potential) and on a weak coupling of the associated Neumann data (flux). To describe the local Dirichlet to Neumann maps, i.e. the solution of local boundary value problems, we will use boundary integral equation methods (Sauter et al. 2004, Steinbach 2003). Note that all required data of Algorithm 1 are provided when using this approach. Since the discretization of boundary integral equations leads to dense stiffness matrices, fast boundary element methods are required (Rjasanow et al. 2007). Here, we will describe a fast multipole approach (Greengard et al. 1987, Of et al. 2005, Of et al. 2006).

4.1 Dirichlet domain decomposition method

According to (2) and (12), we consider the general transmission boundary value problem

$$\begin{aligned} -\Delta u_0(x) &= 0 && \text{for } x \in \Omega_0, \\ -\Delta u_D(x) &= 0 && \text{for } x \in D, \\ u_0(x) &= g(x) && \text{for } x \in \Gamma, \end{aligned} \tag{16}$$

with the inhomogeneous transmission conditions

$$\begin{aligned} u_0(x) - u_D(x) &= g_I(x) && \text{for } x \in \partial D, \\ \varepsilon_0 \frac{\partial}{\partial \nu} u_0(x) - \varepsilon_D \frac{\partial}{\partial \nu} u_D(y) &= f_I(x) && \text{for } x \in \partial D. \end{aligned} \tag{17}$$

Let u be a given continuous function which is defined on ∂D. By reformulating the Dirichlet boundary condition using a function u such that

$$\begin{aligned} u_D(x) &= u(x) - g_I(x) && \text{for } x \in \partial D, \\ u_0(x) &= u(x) && \text{for } x \in \partial D, \\ u_0(x) &= g(x) && \text{for } x \in \Gamma, \end{aligned}$$

we can ensure both the Dirichlet boundary condition in (16) and the Dirichlet transmission condition in (17). Moreover, we can solve the local Dirichlet boundary value problems

$$\begin{aligned} -\Delta u_D(x) &= 0 && \text{for } x \in D, \\ u_D(x) &= u(x) - g_I(x) && \text{for } x \in \partial D \end{aligned}$$

and

$$\begin{aligned} -\Delta u_0(x) &= 0 && \text{for } x \in \Omega, \\ u_0(x) &= u(x) && \text{for } x \in \partial D, \\ u_0(x) &= g(x) && \text{for } x \in \Gamma, \end{aligned}$$

to compute the associated Neumann data t_D and t_0, compare (3). Hence, we can write the local Dirichlet to Neumann maps as

$$\begin{aligned} t_D(x) &= S_D u(x) - S_D g_I(x) \qquad \text{for } x \in \partial D, \\ t_0(x) &= S_0 u(x) + S_0 g(x) \qquad \text{for } x \in \partial D, \end{aligned}$$

where S_D and S_0 are called Steklov–Poincaré operators which are related to the solution of a local Dirichlet boundary value problem. It remains to ensure the Neumann transmission condition in (17), in particular u has to be a solution of

$$\varepsilon_0 S_0 u - \varepsilon_D S_D u = f_I - \varepsilon_D S_D g_I - \varepsilon_0 S_0 g \quad \text{on } \partial D.$$

The weak formulation of this operator equation to find the correct transmission data u results in a variational problem which can be discretized as usual. In particular, when considering piecewise linear continuous basis functions φ_i which are defined with respect to some regular boundary element mesh of ∂D with mesh size h, this gives a linear system of algebraic equations

$$(\varepsilon_0 \mathbf{S}_{0,h} - \varepsilon_D \mathbf{S}_{D,h})\boldsymbol{u} = \boldsymbol{b}.$$

The stiffness matrices are given by

$$\begin{aligned} \mathbf{S}_{0,h}[i,j] &= \int_{\partial D} S_0 \varphi_i(x)\varphi_j(x) ds_x, \\ \mathbf{S}_{D,h}[i,j] &= \int_{\partial D} S_D \varphi_i(x)\varphi_j(x) ds_x, \end{aligned}$$

and the vector of the right-hand side by

$$\begin{aligned} b_j &= \int_{\partial D} f_I(x)\varphi_j(x) ds_x - \int_{\partial D} \varepsilon_D S_D g_I(x)\varphi_j(x) ds_x \\ &\quad - \int_{\partial D} \varepsilon_0 S_0 g(x)\varphi_j(x) ds_x \end{aligned}$$

for $i, j = 1, \ldots, M_D$, where M_D is the number of degrees of freedom. The computation of the above system matrices involves the solution of local Dirichlet boundary value problems. In general, this has to be done either by a finite or by a symmetric boundary element approximation (Steinbach 2003). To construct efficient iterative solvers for this linear system, one may consider finite and boundary tearing and interconnecting methods (Langer et al. 2005). Note that these methods are also robust with respect to large jumps in the coefficients ε_0 and ε_D. Moreover, boundary element tearing and interconnecting methods can easily be combined with fast boundary element methods (Langer et al. 2007) or any other local discretization scheme.

4.2 Boundary element methods

Here, we will describe the realization of the Steklov–Poincaré operators S_0 and S_D by a symmetric boundary element method. For simplicity, we drop the indices of the subdomains and describe the boundary element method for a single subdomain Ω with boundary Lipschitz $\Gamma = \partial\Omega$.

The solution of the Laplace equation $\Delta u = 0$ is given by the representation formula

$$u(x) = \int_\Gamma U^*(x,y)t(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y)u(y)ds_y$$

for $x \in \Omega$, where n_y denotes the exterior normal vector for $y \in \Gamma$, and $U^*(x,y)$ is the fundamental solution given by

$$U^*(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|}.$$

The knowledge of the Cauchy data $u(y)$ and $t(y) := \frac{\partial}{\partial n_y} u(y)$ for $y \in \Gamma$ is sufficient to describe the solution of the boundary value problem. So, it remains to determine the complete Cauchy data by the help of the first boundary integral equation

$$u(x) = \left(\frac{1}{2}I - K\right)u(x) + (Vt)(x) \tag{18}$$

for almost all $x \in \Gamma$, and of the hypersingular boundary integral equation

$$t(x) = (Du)(x) + \left(\frac{1}{2}I + K'\right)t(x) \tag{19}$$

for almost all $x \in \Gamma$. Note that both boundary integral equations result from the representation formula when considering the Dirichlet and Neumann traces, respectively. The involved boundary integral operators are the single layer potential

$$(Vt)(x) = \int_\Gamma U^*(x,y)t(y)ds_y,$$

the double layer potential

$$(Ku)(x) = \int_{\Gamma\setminus\{x\}} \frac{\partial}{\partial n_y} U^*(x,y)u(y)ds_y,$$

its adjoint operator

$$(K't)(x) = \int_{\Gamma\setminus\{x\}} \frac{\partial}{\partial n_x} U^*(x,y)t(y)ds_y,$$

and the hypersingular operator

$$(Du)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y)u(y)ds_y$$

for $x \in \Gamma$. By solving the first boundary integral equation (18), we obtain with

$$t(x) = (Su)(x) := V^{-1}\left(\frac{1}{2}I + K\right)u(x) \tag{20}$$

a first representation of the Steklov–Poincaré operator. Inserting (20) into the hypersingular boundary integral equation (19) gives a second possible definition of the Steklov–Poincaré operator,

$$(Su)(x) = \left[D + \left(\frac{1}{2}I + K'\right) V^{-1} \left(\frac{1}{2}I + K\right)\right] u(x). \tag{21}$$

The symmetric boundary integral representation (21) is now well suited for a Galerkin approximation. For this we consider piecewise constant basis functions $\{\psi_k\}_{k=1}^N$ to approximate the flux, and piecewise linear continuous basis functions $\{\varphi_i\}_{i=1}^M$ to approximate the potential. Then we obtain a symmetric boundary element approximation

$$\mathbf{S}_h = \mathbf{D}_h + \left(\frac{1}{2}\mathbf{M}_h^\top + \mathbf{K}_h^\top\right) \mathbf{V}_h^{-1} \left(\frac{1}{2}\mathbf{M}_h + \mathbf{K}_h\right), \tag{22}$$

where

$$\begin{aligned} \mathbf{V}_h[\ell,k] &= \langle V\psi_k, \psi_\ell\rangle_\Gamma, & \mathbf{K}_h[\ell,i] &= \langle K\varphi_i, \psi_\ell\rangle_\Gamma, \\ \mathbf{D}_h[j,i] &= \langle D\varphi_i, \varphi_j\rangle_\Gamma, & \mathbf{M}_h[\ell,i] &= \langle \varphi_i, \psi_\ell\rangle_\Gamma \end{aligned}$$

for $k, \ell = 1, \ldots, N$ and $i, j = 1, \ldots, M$.

4.3 Fast multipole method

The stiffness matrices of the boundary element method are fully populated due to the nonlocal character of the fundamental solution $U^*(x,y)$. Therefore, $\mathcal{O}(N^2)$ matrix entries have to be computed and stored, and a single matrix times vector product is of order $\mathcal{O}(N^2)$, too. These facts limit the use of standard boundary element methods to 10000–20000 boundary elements for todays personal computers. Fast boundary element methods, like the fast multipole method (Greengard et al. 1987), overcome these limits by data-sparse approximations of the matrices. Most of these methods are quite related. Here, we will describe the main ideas of the fast multipole method for the single layer potential. For details please see, e.g., (Greengard et al. 1987, Nishimura 2002, Of et al. 2006).

The matrix times vector product $\boldsymbol{w} = \mathbf{V}_h\boldsymbol{t}$ of the Galerkin matrix of the single layer potential V reads for the ℓ-th component, $\ell = 1, \ldots, N$,

$$w_\ell = \sum_{k=1}^N \mathbf{V}_h[\ell,k]t_k = \frac{1}{4\pi}\sum_{k=1}^N t_k \int_{\tau_\ell}\int_{\tau_k} \frac{1}{|x-y|} ds_y ds_x$$

where τ_k denotes a boundary element. The effort of such a matrix times vector product is of order $\mathcal{O}(N^2)$ as none of the matrix entries vanishes. This effort can be reduced by separation of variables using a Taylor series expansion of the fundamental solution. An expansion by spherical harmonics is more suitable,

$$U^*(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|} \approx \frac{1}{4\pi}\sum_{n=0}^p \sum_{m=-n}^n |x|^n Y_n^{-m}(\hat{x}) \frac{Y_n^m(\hat{y})}{|y|^{n+1}}$$

with spherical harmonics

$$Y_n^{\pm m}(\hat{x}) = \sqrt{\frac{(n-m)!}{(n+m)!}}(-1)^m \frac{d^m}{d\hat{x}_3^m} P_n(\hat{x}_3)(\hat{x}_1 \pm i\hat{x}_2)^m$$

for $m \geq 0$ and $\hat{x} = x/|x|$. P_n denotes the Legendre polynomial of n–th order and these polynomials can be computed efficiently by recursion formulas. The expansion has been truncated for the computational realization. The induced error can be controlled to guarantee optimal convergence rates of the fast method, see, e.g., (Of et al. 2006). The expansion is only valid for $\frac{|x|}{|y|} < \frac{1}{d}$ where $d > 1$. Therefore, the expansion can be applied only for well separated boundary elements. This leads to a separation of the matrix into a farfield (FF) and a nearfield (NF) part where the expansion can be applied or not, respectively. The approximation of the matrix times vector product now reads as

$$\widetilde{w}_\ell = \sum_{k \in \mathrm{NF}(\ell)} \mathbf{V}_h[\ell, k]\, t_k + \sum_{n=0}^{p} \sum_{m=-n}^{n} \int_{\tau_\ell} |x|^n Y_n^{-m}(\hat{x}) ds_x L_n^m(\ell)$$

with the coefficients

$$L_n^m(\ell) = \sum_{k \in \mathrm{FF}(\ell)} \frac{t_k}{4\pi} \int_{\tau_k} \frac{Y_n^m(\hat{y})}{|y|}^{n+1} ds_y.$$

If the coefficients $L_n^m(\ell)$ are known, a fast evaluation of the matrix times vector product is given. Unfortunately, the coefficients $L_n^m(\ell)$ depend on the index ℓ of the evaluation element τ_ℓ. Further, these coefficients have to be recomputed in each matrix times vector product.

The second main ingredient of the fast multipole method is a hierarchical structure which is used to compute the coefficients $L_n^m(\ell)$ efficiently (Greengard et al. 1987). To this end, neighbored elements are clustered, and an appropriate cluster tree is defined. The contribution of each cluster is expressed by a truncated expansion. The separation into nearfield and farfield is also applied to the cluster tree defining admissible cluster pairs. Finally, all the farfield sums for τ_ℓ and τ_k belonging to admissible pairs of clusters are assembled efficiently by exploiting the hierarchical tree structure. As a consequence, one matrix times vector multiplication is performed with the almost linear effort of $\mathcal{O}(N \log^2 N)$ and the approximated solution of the discrete system will provide an approximation whose error is of the same order as in the a priori error estimate. For details and numerical examples of complex structures for industrial applications please see, e.g., (Of et al. 2005, Of et al. 2006).

Conclusions

We have described a fast symmetric boundary element method for ECT inverse problems, in particular for a model problem. The numerical implementation will be done by using available fast boundary element tools. This will be extended to the more complicated configuration of the sensor assembly (Wegleiter et al. 2005). Moreover, several regularization techniques have to be considered.

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