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Stabilized Boundary Element Methods for Exterior Helmholtz Problems

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Abstract

In this paper we describe and analyze some modified boundary element methods to solve exterior boundary value problems for the Helmholtz equation with either Dirichlet or Neumann boundary conditions. The proposed approach avoids spurious modes even in the case of Lipschitz boundaries. Moreover, the regularisation is done based on boundary integral operators which are already available in standard boundary element formulations. Numerical examples are given to compare the proposed approach with other already existing regularized formulations.

1 Introduction

The boundary integral formulation [2, 9, 13] of exterior boundary value problems for the Helmholtz equation with either Dirichlet or Neumann boundary conditions may lead to boundary integral equations which are either not uniquely solvable, or not solvable at all. In particular we will have spurious modes if the wave number corresponds to an eigenvalue of the interior eigenvalue problem for the Laplace operator with either homogeneous Dirichlet or Neumann boundary conditions, respectively.

Hence one may use combined boundary integral formulations such as the indirect Brakhage–Werner approach [5], or the direct Burton–Miller formulation [8] to obtain boundary integral equations which are uniquely solvable for all wave numbers. The above mentioned combined boundary integral formulations are usually analyzed in $L_2(\Gamma)$ by using some compactness arguments for the Laplace double layer potential. Hence this approach is restricted to the case of sufficiently smooth boundaries.

When introducing appropriate regularization operators one can formulate modified boundary integral equations where the unique solvability can be ensured even for Lipschitz domains when considering the resulting integral equations in the energy spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively, see, e.g., [6, 7]. For other regularisation procedures, see for example [3, 4, 21].

In [10] we have proposed an alternative modified boundary integral equation which is based on the use of standard boundary integral operators only. The unique solvability of the proposed combined boundary integral equation follows from a Gårdings inequality while the injectivity is based on some mapping properties of the underlying boundary integral operators. Here we extend our previous approach to a coupled system which is equivalent to the above mentioned combined boundary integral equation. In particular we prove a Gårding's inequality also for the system which enables us to use standard arguments [16, 19, 20, 22] to analyze a Galerkin discretization scheme for the system. Note that the Galerkin discretization of the coupled system also defines an approximation of the Galerkin discreitization of the combined boundary integral equation.

This paper is organized as follows: In Section 2 we recall different combined and regularized boundary integral formulations for the exterior Dirichlet boundary value problem of the Helmholtz equation. In particular we prove the unique solvability of a coupled system of boundary integral equations. Galerkin boundary element method is formulated and analyzed in Section 3, again for the Dirichlet problem. In Section 4 we summarize the corresponding modified boundary element method for the exterior Neumann boundary value problem. Numerical results for the proposed modified boundary element methods and comparisons with other existing approaches are finally discussed in Section 5.

2 Boundary Integral Equations

As a model problem we consider the exterior Dirichlet boundary value problem for the Helmholtz equation,

$$\Delta u(x) + \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega^c = \mathbb{R}^3 \backslash \overline{\Omega}, \quad u(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega, \tag{2.1}$$

where in addition

$$\lim_{R\to\infty} \int\limits_{\partial B_R} \left| \frac{\partial}{\partial n_y} u(y) - i\kappa u(y) \right|^2 ds_y = 0 \tag{2.2}$$

is the Sommerfeld radiation condition. Moreover, $\kappa \in \mathbb{R}_+$ is the wave number, $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, and $g \in H^{1/2}(\Gamma)$ are some prescribed Dirichlet data.

By using the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$,

$$U^*_\kappa(x,y) = \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x-y|} \quad \text{for } x \neq y,$$

we can find different representations to describe the unique solution of the boundary value problem (2.1). In particular when using an indirect approach we can represent the solution either by means of the single layer potential

$$u(x) = (\widetilde{V}_\kappa w)(x) = \int\limits_{\Gamma} U^*_\kappa(x,y) w(y) ds_y \quad \text{for } x \in \Omega^c \tag{2.3}$$

where $w \in H^{-1/2}(\Gamma)$ is some unknown density function, or we can describe the solution of (2.1) by means of the double layer potential

$$u(x) = (W_\kappa v)(x) = \int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x,y) v(y) ds_y \quad \text{for } x \in \Omega^c \tag{2.4}$$

where $v \in H^{1/2}(\Gamma)$ is again some unknown density function. When using a direct approach, the solution of (2.1) is given by the representation formula

$$u(x) = -\int_\Gamma U_\kappa^*(x,y) \frac{\partial}{\partial n_y} u(y) ds_y + \int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x,y) u(y) ds_y \quad \text{for } x \in \Omega^c. \tag{2.5}$$

When applying the exterior trace operator γ_0^{ext} to the single and the double layer potentials $\widetilde{V}_\kappa$ and W_κ we obtain certain boundary integral equations for the yet unknown density functions $w \in H^{-1/2}(\Gamma)$, $v \in H^{1/2}(\Gamma)$, and $t = \frac{\partial}{\partial n} u \in H^{-1/2}(\Gamma)$, respectively. In particular, when using the single layer potential (2.3) we have to solve the first kind boundary integral equation

$$(V_\kappa w)(x) := \gamma_0^{\text{ext}}(\widetilde{V}_\kappa w)(x) = \int_\Gamma U_\kappa^*(x,y) w(y) ds_y = g(x) \quad \text{for } x \in \Gamma, \tag{2.6}$$

for the double layer potential (2.4) we obtain the second kind boundary integral equation

$$(\frac{1}{2} I + K_\kappa) v(x) := \gamma_0^{\text{ext}}(W_\kappa v)(x) = \frac{1}{2} v(x) + \int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x,y) v(y) ds_y = g(x) \quad \text{for } x \in \Gamma, \tag{2.7}$$

and for the direct formulation (2.5) we finally get the first kind boundary integral equation

$$(V_\kappa t)(x) = (-\frac{1}{2} I + K_\kappa) g(x) \quad \text{for } x \in \Gamma. \tag{2.8}$$

It is well known that if $\kappa^2 = \lambda$ corresponds to an eigenvalue of the interior Dirichlet eigenvalue problem

$$-\Delta u_\lambda(x) = \lambda u_\lambda(x) \quad \text{for } x \in \Omega, \quad u_\lambda(x) = 0 \quad \text{for } x \in \Gamma, \tag{2.9}$$

the boundary integral equations (2.6) and (2.8) are not uniquely solvable, and if $\kappa^2 = \mu$ corresponds to an eigenvalue of the interior Neumann eigenvalue problem

$$-\Delta u_\mu(x) = \mu u_\mu(x) \quad \text{for } x \in \Omega, \quad \frac{\partial}{\partial n_x} u_\mu(x) = 0 \quad \text{for } x \in \Gamma, \tag{2.10}$$

the boundary integral equation (2.7) is not uniquely solvable.

2.1 Modified Boundary Integral Equations

The idea of using combined boundary integral equations to overcome the problem of non–unique solvability goes back to Brakhage and Werner in 1965 [5]. They used the following representation formula

$$u(x) = (W_k w)(x) - i\eta(\widetilde{V}_k w)(x) \quad \text{for } x \in \Omega^c \tag{2.11}$$

with an unknown density function $w \in L_2(\Gamma)$ and $\eta > 0$, which leads to the boundary integral equation

$$(\frac{1}{2}I + K_\kappa)w(x) - i\eta(V_\kappa w)(x) = g(x) \quad \text{for } x \in \Gamma. \tag{2.12}$$

For domains Ω with a smooth boundary $\Gamma = \partial\Omega$ it can be shown [5] that the boundary integral operator corresponding to (2.12)

$$\frac{1}{2}I + K_\kappa - i\eta V_\kappa : L_2(\Gamma) \to L_2(\Gamma) \tag{2.13}$$

is coercive and injective, i.e. (2.12) admits a unique solution $w \in L_2(\Gamma)$.

In addition to the weakly singular boundary integral equation (2.8) we also consider the hypersingular boundary integral equation

$$(\frac{1}{2}I + K'_\kappa)t(x) \;=\; -(D_\kappa g)(x) \quad \text{for } x \in \Gamma. \tag{2.14}$$

Combining the direct boundary integral equations (2.8) and (2.14), this gives the Burton-Miller formulation [8]

$$\left[V_\kappa - i\eta\left(\frac{1}{2}I + K'_\kappa\right)\right]t(x) \;=\; \left[i\eta D_\kappa + \left(-\frac{1}{2}I + K_\kappa\right)\right]g(x) \quad \text{for } x \in \Gamma. \tag{2.15}$$

Note that the unique solvability of (2.15) in $L_2(\Gamma)$ follows as for the Brakhage–Werner formulation (2.12). But the problem is that the coercivity of the operator (2.13) in $L_2(\Gamma)$ does not carry over to domains with a non–smooth boundary. Therefore there is a need to consider the density function w as an element of $H^{-1/2}(\Gamma)$, and to introduce a regularisation operator $B : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, which lifts w in the right energy space, see, e.g. [6, 7]. Hence we obtain the representation formula

$$u(x) = (\widetilde{V}_\kappa w)(x) + i\eta(W_\kappa Bw)(x) \quad \text{for } x \in \Omega^c$$

and thus the boundary integral equation to be solved

$$(V_\kappa w)(x) + i\eta(\frac{1}{2}I + K_\kappa)Bw(x) \;=\; g(x) \quad \text{for } x \in \Gamma.$$

In [6, 7] several choices of regularization operators B were discussed, mostly the proofs were based on some compactness arguments. Instead, in [10] we have proposed an alternative regularization operator,

$$B = \widetilde{D}_0^{-1}(\frac{1}{2}I + K'_{-\kappa}) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \tag{2.16}$$

which leads to the modified boundary integral equation

$$(A_\kappa w)(x) := (V_\kappa w)(x) + i\eta(\frac{1}{2}I + K_\kappa)\widetilde{D}_0^{-1}(\frac{1}{2}I + K'_{-\kappa})w(x) \quad \text{for } x \in \Gamma. \tag{2.17}$$

Note that $\widetilde{D}_0 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a modified hypersingular boundary integral operator for the Laplace operator [17] defined as

$$\langle \widetilde{D}_0 u, v\rangle_\Gamma \;:=\; \langle Du, v\rangle_\Gamma + \langle u, 1\rangle_\Gamma \langle v, 1\rangle_\Gamma \quad \text{for } u, v \in H^{1/2}(\Gamma).$$

Theorem 2.1 [10] *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Then, for all wave numbers $\kappa \in \mathbb{R}$ and for all positive regularization parameters $\eta > 0$ the boundary integral operator A_κ as defined in* (2.17) *is coercive and injective, i.e. the modified boundary integral equation* (2.17) *admits a unique solution $w \in H^{-1/2}(\Gamma)$.*

Due to the composite structure of the boundary integral operator A_κ in (2.17) a numerical analysis of a Galerkin discretisation of (2.17) will require the use of some Strang lemma, since the operator A_κ has to be approximated in an appropriate manner. Instead, we will first consider a coupled variational problem which is equivalent to the boundary integral equation (2.17) but which itself admits a Gårdings inequality.

2.2 Variational Formulations

The boundary integral equation (2.17) is equivalent to find $w \in H^{-1/2}(\Gamma)$ such that

$$\langle A_\kappa w, \tau\rangle_\Gamma \;=\; \langle V_\kappa w, \tau\rangle_\Gamma + i\eta\langle(\frac{1}{2}I + K_\kappa)\widetilde{D}_0^{-1}(\frac{1}{2}I + K'_{-\kappa})w, \tau\rangle_\Gamma = \langle g, \tau\rangle$$

is satisfied for all $\tau \in H^{-1/2}(\Gamma)$. By introducing $\varphi = \widetilde{D}_0^{-1}(\frac{1}{2}I + K'_{-\kappa})w \in H^{1/2}(\Gamma)$ we obtain

$$\langle A_\kappa w, \tau\rangle_\Gamma \;=\; \langle V_\kappa w, \tau\rangle_\Gamma + i\eta\langle(\frac{1}{2}I + K_\kappa)\varphi, \tau\rangle_\Gamma = \langle g, \tau\rangle$$

where $\varphi \in H^{1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle \widetilde{D}_0\varphi, \phi\rangle_\Gamma \;=\; (\frac{1}{2}I + K'_{-\kappa})w, \phi\rangle_\Gamma \quad \text{for all } \phi \in H^{1/2}(\Gamma).$$

Hence we have to find $(w, \varphi) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that

$$\begin{array}{rcrcl} \langle V_\kappa w, \tau\rangle_\Gamma & + & i\eta\langle(\frac{1}{2}I + K_\kappa)\varphi, \tau\rangle_\Gamma & = & \langle g, \tau\rangle_\Gamma, \\ \langle(\frac{1}{2}I + K'_{-\kappa})w, \phi\rangle_\Gamma & - & \langle \widetilde{D}_0\varphi, \phi\rangle_\Gamma & = & 0 \end{array} \tag{2.18}$$

is satisfied for all $(\tau, \phi) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. According to this variational system we can define a coercive bilinear form as follows.

Lemma 2.2 *The bilinear form*

$$a(w, \varphi; \tau, \phi) = \overline{\langle \widetilde{D}_0 \varphi, \phi \rangle_\Gamma} - \overline{\langle (\frac{1}{2} I + K'_{-\kappa}) w, \phi \rangle_\Gamma} + \langle (\frac{1}{2} I + K_\kappa) \varphi, \tau \rangle_\Gamma - \frac{i}{\eta} \langle V_\kappa w, \tau \rangle_\Gamma$$

where $(w, \varphi), (\tau, \phi) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *is coercive.*

Proof: By identifying $(\tau, \phi) = (w, \varphi) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ we first have

$$\begin{aligned}
|a(w, \varphi; w, \varphi)| &= \\
&= |\langle \widetilde{D}_0 \varphi, \varphi \rangle_\Gamma - \overline{\langle (\frac{1}{2} I + K'_{-\kappa}) w, \varphi \rangle_\Gamma} + \langle (\frac{1}{2} I + K_\kappa) \varphi, w \rangle_\Gamma - \frac{i}{\eta} \langle V_\kappa w, w \rangle_\Gamma| \\
&= |\langle \widetilde{D}_0 \varphi, \varphi \rangle_\Gamma - \overline{\langle w, (\frac{1}{2} I + K_\kappa) \varphi \rangle_\Gamma} + \langle (\frac{1}{2} I + K_\kappa) \varphi, w \rangle_\Gamma - \frac{i}{\eta} \langle V_\kappa w, w \rangle_\Gamma| \\
&= |\langle \widetilde{D}_0 \varphi, \varphi \rangle_\Gamma - \frac{i}{\eta} \langle V_\kappa w, w \rangle_\Gamma|.
\end{aligned}$$

Since $V_\kappa - V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact, and by using

$$\Im \langle V_0 w, w \rangle_\Gamma = \Im \langle \widetilde{D}_0 u, u \rangle_\Gamma = 0$$

we finally conclude the coercivity estimate

$$\begin{aligned}
\left| a(w, \varphi; w, \varphi) + \frac{i}{\eta} \langle (V_\kappa - V_0) w, w \rangle_\Gamma \right| &= |\langle \widetilde{D}_0 \varphi, \varphi \rangle_\Gamma - \frac{i}{\eta} \langle V_0 w, w \rangle_\Gamma| \\
&= \sqrt{\langle \widetilde{D}_0 \varphi, \varphi \rangle_\Gamma^2 + \frac{1}{\eta^2} \langle V_0 w, w \rangle_\Gamma^2} \\
&\geq \frac{1}{\sqrt{2}} \left(\langle \widetilde{D}_0 \varphi, \varphi \rangle_\Gamma + \frac{1}{\eta} \langle V_0 w, w \rangle_\Gamma \right) \\
&\geq \frac{1}{\sqrt{2}} \left(c_1^{\widetilde{D}_0} \|v\|^2_{H^{1/2}(\Gamma)} + \frac{c_1^V}{\eta} \|w\|^2_{H^{-1/2}(\Gamma)} \right) \\
&\geq \frac{1}{\sqrt{2}} \min\{c_1^{\widetilde{D}_0}, \frac{c_1^V}{\eta}\} \left[\|v\|^2_{H^{1/2}(\Gamma)} + \|w\|^2_{H^{-1/2}(\Gamma)} \right]. \qquad \Box
\end{aligned}$$

Combining the coercivity of the bilinear form in the variational problem (2.18) with the injectivity of the modified boundary integral operator A_κ we obtain the unique solvability of the variational problem (2.18).

Theorem 2.3 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with Lipschitz boundary* $\Gamma = \partial\Omega$. *Then, for all wave numbers* $\kappa \in \mathbb{R}_+$ *and for all positive regularization parameters* $\eta > 0$ *there exists a unique solution* $(w, \varphi) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *of the variational problem* (2.18).

Proof: Since the bilinear form of the variational problem (2.18) is coercive, it remains to prove the injectivity. But since the variational problem (2.18) is equivalent to the modified boundary integral equation (2.17), the injectivity of the bilinear form in (2.18) follows from the injectivity of A_κ. $\square$

3 Boundary Element Methods

Let

$$S_h^0(\Gamma) = \mathrm{span}\{\psi_k\}_{k=1}^N \subset H^{-1/2}(\Gamma)$$

be some conforming boundary element space, e.g. of piecewise constant basis functions ψ_k which are defined with respect to some shape regular und globally quasi–uniform boundary element mesh on Γ, where h is the related mesh size. The Galerkin discretization of the modified boundary integral equation (2.17) is then to find $w_h \in S_h^0(\Gamma)$ such that

$$\langle V_\kappa w_h, \tau_h\rangle_\Gamma + i\eta\langle(\frac{1}{2}I + K_\kappa)\widetilde{D}_0^{-1}(\frac{1}{2}I + K'_{-\kappa})w_h, \tau_h\rangle_\Gamma = \langle g, \tau_h\rangle_\Gamma \tag{3.1}$$

is satisfied for all $\tau_h \in S_h^0$.

Proposition 3.1 [22] *Since the modified operator $A_\kappa = V_\kappa + i\eta(\frac{1}{2}I + K_\kappa)\widetilde{D}_0^{-1}(\frac{1}{2}I + K'_{-\kappa})$ is injective and coercive, an associated stability (LBB) condition is satisfied, i.e. there exists a mesh size $h_0 > 0$ such that for all $h < h_0$*

$$c_S\|w_N\|_{H^{-1/2}(\Gamma)} \le \sup_{\tau_h \in S_h^0(\Gamma), \|\tau_h\|_{H^{-1/2}(\Gamma)} > 0} \frac{|\langle A_\kappa w_h, \tau_h\rangle_\Gamma|}{\|\tau_h\|_{H^{-1/2}(\Gamma)}} \tag{3.2}$$

is satisfied for all $w_h \in S_h^0(\Gamma)$.

Using Proposition 3.1 and applying Cea's lemma we can conclude the unique solvability of the discrete variational problem (3.1) as well as the a priori error estimate

$$\|w - w_h\|_{H^{-1/2}(\Gamma)} \le c \inf_{\tau_h \in S_h^0(\Gamma)} \|w - \tau_h\|_{H^{-1/2}(\Gamma)}.$$

But since the operator A_κ is a composition of three boundary integral operators involving the inverse $\widetilde{D}_0^{-1}$ it is in general not possible to compute the Galerkin weights $\langle A_\kappa\psi_\kappa, \psi_\ell\rangle_\Gamma$ exactly. Hence we have to define a suitable approximation of A_κ which can be done by considering the variational problem (2.18).

Let

$$S_h^1(\Gamma) = \mathrm{span}\{\varphi_i\}_{i=1}^M \subset H^{1/2}(\Gamma)$$

be another boundary element space, e.g. of piecewise linear and continuous basis functions φ_i. For simplicity we may assume that $S_h^1(\Gamma)$ is defined with respect to the same boundary element mesh as $S_h^0(\Gamma)$. The Galerkin discretization of the variational problem (2.18) is then to find $(w_h, \varphi_h) \in S_h^0(\Gamma) \times S_h^1(\Gamma)$ such that

$$\begin{array}{rcccl} \langle V_\kappa w_h, \tau_h \rangle_\Gamma & + & i\eta \langle (\frac{1}{2} I + K_\kappa)\varphi_h, \tau_h \rangle_\Gamma & = & \langle g, \tau_h \rangle_\Gamma, \\ \langle (\frac{1}{2} I + K'_{-\kappa}) w_h, \phi_h \rangle_\Gamma & - & \langle \widetilde{D}_0 \varphi_h, \phi_h \rangle_\Gamma & = & 0 \end{array} \tag{3.3}$$

is satisfied for all $(\tau_h, \phi_h) \in S_h^0(\Gamma) \times S_h^1(\Gamma)$. Since the related bilinear form is coercive, see Lemma 2.2, and injective, see Theorem 2.3, an associated stability (LBB) condition follows as in Proposition 2.2 for

$$\begin{pmatrix} V_\kappa & i\eta(\frac{1}{2}I + K_\kappa) \\ (\frac{1}{2}I + K'_{-\kappa}) & -\widetilde{D}_0 \end{pmatrix} : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma).$$

This ensures, when assuming $h < h_0$, the unique solvability of the Galerkin system (3.3) as well as the a priori error estimate

$$\|w - w_h\|_{H^{-1/2}(\Gamma)} + \|\varphi - \varphi_h\|_{H^{1/2}(\Gamma)} \leq c \left[\inf_{\tau_h \in S_h^0(\Gamma)} \|w - \tau_h\|_{H^{-1/2}(\Gamma)} + \inf_{\phi_h \in S_h^1(\Gamma)} \|\varphi - \phi_h\|_{H^{1/2}(\Gamma)} \right].$$

From the approximation properties of the boundary element trial spaces $S_h^0(\Gamma)$ and $S_h^1(\Gamma)$ we further conclude the error estimate

$$\|w - w_h\|_{H^{-1/2}(\Gamma)} + \|\varphi - \varphi_h\|_{H^{1/2}(\Gamma)} \leq c\, h^{3/2} \left[|w|_{H^1_{pw}(\Gamma)} + |\varphi|_{H^2(\Gamma)} \right]$$

when assuming $w \in H^1_{\text{pw}}(\Gamma)$ and $\varphi \in H^2(\Gamma)$. When applying the Aubin–Nitsche trick, see e.g. [22], and when using an inverse inequality for a globally quasi–uniform mesh, we can also derive the general error estimate

$$\|w - w_h\|_{H^s(\Gamma)} + \|\varphi - \varphi_h\|_{H^{s+1}(\Gamma)} \leq c\, h^{1-s} \left[|w|_{H^1_{pw}(\Gamma)} + |\varphi|_{H^2(\Gamma)} \right] \tag{3.4}$$

for all $s \in [-2, 0]$ when assuming $w \in H^1_{\text{pw}}(\Gamma)$ and $\varphi \in H^2(\Gamma)$. In particular for $s = 0$ we obtain

$$\|w - w_h\|_{L_2(\Gamma)} + \|\varphi - \varphi_h\|_{H^1(\Gamma)} \leq c\, h \left[|w|_{H^1_{pw}(\Gamma)} + |\varphi|_{H^2(\Gamma)} \right] \tag{3.5}$$

while for $s = -2$ we get

$$\|w - w_h\|_{H^{-2}(\Gamma)} + \|\varphi - \varphi_h\|_{H^{-1}(\Gamma)} \leq c\, h^3 \left[|w|_{H^1_{pw}(\Gamma)} + |\varphi|_{H^2(\Gamma)} \right]. \tag{3.6}$$

Inserting w_h and φ_h into the representation formula

$$u(x^*) = (\widetilde{V}_\kappa w)(x^*) + i\eta (W_\kappa \varphi)(x^*) \quad \text{for } x^* \in \Omega^c,$$

this defines an approximate solution

$$u_h(x^*) = (\widetilde{V}_\kappa w_h)(x^*) + i\eta (W_\kappa \varphi_h)(x^*) \quad \text{for } x^* \in \Omega^c.$$

To estimate the error, we compute

$$\begin{aligned}
|u(x^*) - u_h(x^*)| &= \left|(\widetilde{V}_\kappa w)(x^*) - (\widetilde{V}_\kappa w_h)(x^*) + i\eta (W_\kappa \varphi)(x^*) - i\eta (W_\kappa \varphi_h)(x^*)\right| \\
&\leq \left|(\widetilde{V}_\kappa w)(x^*) - (\widetilde{V}_\kappa w_h)(x^*)\right| + \eta \left|(W_\kappa \varphi)(x^*) - (W_\kappa \varphi_h)(x^*)\right| \\
&= |\langle U^*_\kappa(x^*,\cdot), w - w_h\rangle_\Gamma| + \eta \left|\langle \frac{\partial}{\partial n} U^*_\kappa(x^*,\cdot), \varphi - \varphi_h\rangle_\Gamma\right| \\
&\leq \|U^*_\kappa(x^*,\cdot)\|_{H^{-s}(\Gamma)} \|w - w_h\|_{H^s(\Gamma)} + \eta \|\frac{\partial}{\partial n} U^*_\kappa(x^*,\cdot)\|_{H^{-s-1}(\Gamma)} \|\varphi - \varphi_h\|_{H^{s+1}(\Gamma)}
\end{aligned}$$

by using duality for some $s \in [-2, 0]$ to obtain, in particular for $s = -2$

$$|u(x^*) - u_h(x^*)| \leq c\, h^3 \left[|w|_{H^1_{pw}(\Gamma)} + |\varphi|_{H^2(\Gamma)}\right] \tag{3.7}$$

when assuming $w \in H^1_{\mathrm{pw}}(\Gamma)$ and $\varphi \in H^2(\Gamma)$.

The Galerkin variational formulation (3.3) is equivalent the following algebraic system of linear equations

$$\begin{pmatrix} V_{\kappa,h} & i\eta(\frac{1}{2}M_h + K_{\kappa,h}) \\ (\frac{1}{2}M_h + K_{\kappa,h})^* & -\widetilde{D}_{0,h} \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{\varphi} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{0} \end{pmatrix} \tag{3.8}$$

where

$$\begin{aligned}
V_{\kappa,h}[\ell,k] &= \langle V_\kappa \psi_k, \psi_\ell\rangle_\Gamma, & K_{\kappa,h}[\ell,i] &= \langle K_\kappa \varphi_i, \psi_\ell\rangle_\Gamma \\
\widetilde{D}_{0,h}[j,i] &= \langle \widetilde{D}_0 \varphi_i, \varphi_j\rangle_\Gamma, & M_h[\ell,i] &= \langle \varphi_i, \psi_\ell\rangle_\Gamma
\end{aligned}$$

for $k, \ell = 1, \ldots, N$ and $i, j = 1, \ldots, M$. In addition,

$$g_\ell = \langle g, \psi_\ell\rangle_\Gamma = \int_\Gamma g(x)\psi_\ell(x)ds_x \quad \text{for } \ell = 1, \ldots, N. \tag{3.9}$$

Since $\widetilde{D}_{0,h}$ is a positve definite and symmetric matrix, we can reformulate the linear system (3.8) as the following Schur complement system

$$[V_{\kappa,h} + i\eta(\frac{1}{2}M_h + K_{\kappa,h})\widetilde{D}^{-1}_{0,h}(\frac{1}{2}M_h + K_{\kappa,h})^*]\underline{w} = \underline{g}. \tag{3.10}$$

Note that the stiffness matrix in (3.10) defines an approximation of the Galerkin matrix $A_{\kappa,h}$ of the composed operator A_κ.

Instead of computing the coefficients (3.9) of the right hand side exactly, one may use an approximation $g_h \in S^1_h(\Gamma)$ of the given Dirichlet data g, which can be computed either by interpolation,

$$g_h(x) = \sum_{i=1}^{M} g(x_i)\varphi_i(x),$$

or by using a L_2 projection satisfying the variational problem

$$\langle g_h, \varphi_j \rangle_\Gamma \;=\; \langle g, \varphi_j \rangle_\Gamma \quad \text{for all } j = 1, \ldots, M.$$

When assuming $g \in H^2(\Gamma)$ we obtain the following error estimate

$$\| g - g_h \|_{H^\sigma(\Gamma)} \;\leq\; c\, h^{2-\sigma}\, |g|_{H^2(\Gamma)}$$

where we have $\sigma \in [0,1]$ in the case of interpolation, and $\sigma \in [-1,1]$ in the case of the L_2 projection due to an Aubin–Nitsche argument.

Now, instead of (3.8) we have to solve the perturbed linear system

$$\begin{pmatrix} V_{\kappa,h} & i\eta(\frac{1}{2}M_h + K_{\kappa,h}) \\ (\frac{1}{2}M_h + K_{\kappa,h})^* & -\widetilde{D}_{0,h} \end{pmatrix} \begin{pmatrix} \underline{\widetilde{w}} \\ \underline{\widetilde{\varphi}} \end{pmatrix} \;=\; \begin{pmatrix} M_h \underline{g} \\ \underline{0} \end{pmatrix} \tag{3.11}$$

implying perturbed solutions $\widetilde{w}_h \in S_h^0(\Gamma)$ and $\widetilde{\varphi}_h \in S_h^1(\Gamma)$, respectively. Moreover, by inserting these perturbed solutions into the representation formula, this defines an approximate solution

$$\widetilde{u}_h(x^*) \;=\; (\widetilde{V}_\kappa \widetilde{w}_h)(x^*) + i\eta(W_\kappa \widetilde{\varphi}_h)(x^*) \quad \text{for } x^* \in \Omega^c.$$

This perturbed solution satisfies the pointwise error estimate

$$|u(x^*) - \widetilde{u}_h(x^*)| \;\leq\; c\, h^3 \left[|w|_{H^1_{pw}(\Gamma)} + |\varphi|_{H^2(\Gamma)} + |g|_{H^2(\Gamma)} \right] \tag{3.12}$$

in the case of a L_2 projection of g, and

$$|u(x^*) - \widetilde{u}_h(x^*)| \;\leq\; c\, h^2 \left[|w|_{H^1_{pw}(\Gamma)} + |\varphi|_{H^2(\Gamma)} + |g|_{H^2(\Gamma)} \right] \tag{3.13}$$

when considering only an interpolation of g.

Note that on a first glance it does not make any difference to compute the right hand side (3.9) directly, or to compute the right hand side of the L_2 projection. However, this question becomes important when considering a direct approach such as the Burton–Miller formulation (2.15).

4 Neumann Boundary Value Problem

In addition to the Dirichlet boundary value problem (2.1) we also consider the exterior Neumann boundary value problem

$$\Delta u(x) + \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega^c = \mathbb{R}^3 \backslash \overline{\Omega}, \quad -\frac{\partial}{\partial n_x} u(x) = f(x) \quad \text{for } x \in \Gamma = \partial\Omega \tag{4.1}$$

together with the Sommerfeld radiation condition (2.2).

The formulation and analysis of stabilized boundary integral equations, as well as the analysis of related boundary element methods is analogous as in the case of the Dirchlet boundary value problem. Hence, here we only provide the formulations, the linear systems resulting from the Galerkin discretization, and comment on the related error estimates.

Resulting from the single layer potential ansatz (2.3) we get the boundary integral equation

$$(-\frac{1}{2}I + K'_\kappa)w(x) = f(x) \quad \text{for } x \in \Gamma, \tag{4.2}$$

while from (2.4) we obtain

$$(D_\kappa v)(x) = f(x) \quad \text{for } x \in \Gamma. \tag{4.3}$$

The direct approach for the Neumann boundary value problem gives the boundary integral equation

$$(D_\kappa \overline{u})(x) = -(\frac{1}{2}I + K'_\kappa)f(x) \quad \text{for } x \in \Gamma. \tag{4.4}$$

As in the case of the Dirichlet boundary value problem it is known that the boundary integral equations (4.2) and (4.4) are not uniquely solvable if $\kappa^2 = \mu$ corresponds to an eigenvalue of the interior Neumann eigenvalue problem (2.10). Moreover, (4.2) does not have a unique solution if $\kappa^2 = \lambda$ corresponds to an eigenvalue of the interior Dirichlet eigenvalue problem (2.9).

Similar to (2.12) we can find a combined boundary integral equation for the Neumann boundary value problem as

$$(D_\kappa v)(x) + i\eta(-\frac{1}{2}I + K'_\kappa)v(x) = f(x) \quad \text{for } x \in \Gamma \tag{4.5}$$

where v is considered as a function in $L_2(\Gamma)$. Introducing the regularisation operator

$$R = V_0^{-1}(-\frac{1}{2}I + K_{-\kappa}) \,:\, H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

leads to the modified boundary integral equation

$$(D_\kappa v)(x) + i\eta(-\frac{1}{2}I + K'_\kappa)V_0^{-1}(-\frac{1}{2}I + K_{-\kappa})v(x) = f(x) \quad \text{for } x \in \Gamma. \tag{4.6}$$

As for the Dirichlet boundary value problem it can be shown that the corresponding boundary integral operator

$$D_\kappa + i\eta(-\frac{1}{2}I + K'_\kappa)V_0^{-1}(-\frac{1}{2}I + K_{-\kappa}) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is coercive and injective, and therefore invertible. Hence, the modified boundary integral equation (4.6) admits a unique solution $v \in H^{1/2}(\Gamma)$ for any wave number $k \in \mathbb{R}_+$.

The related variational formulation of (4.6) can be reformulated as a saddle point formulation to find $(v, \psi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\begin{array}{rcccl} \langle D_\kappa v, \mu\rangle_\Gamma & + & i\eta\langle(-\frac{1}{2}I + K'_\kappa)\psi, \mu\rangle_\Gamma & = & \langle f, \mu\rangle_\Gamma, \\ \langle(\frac{1}{2}I + K_{-\kappa})v, \zeta\rangle_\Gamma & - & \langle V_0\psi, \zeta\rangle_\Gamma & = & 0 \end{array} \tag{4.7}$$

is satisfied for all $(\mu, \zeta) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. As in Lemma 2.2 we can define a coercive bilinear form which is related to the variational problem (4.7). From this the unique solvability of the variational problem (4.7) follows as in Theorem 2.3.

As in Section 3 we now introduce the boundary element spaces

$$S_1^h(\Gamma) \times S_0^h(\Gamma) \subset H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$$

of piecewise linear and piecewise constant basis functions φ_i and ψ_k, respectively. Since the associated bilinear form is coercive there also holds the corresponding stability (LBB) condition if the mesh size h is small enough. This ensures the unique solvability of the discrete Galerkin equations which result from a Galerkin variational formulation of (4.7). The discrete Galerkin system is equivalent to a linear system of algebraic equations,

$$\begin{pmatrix} D_{\kappa,h} & i\eta(-\frac{1}{2}M_h + K_{-\kappa,h})^* \\ (-\frac{1}{2}M_h + K_{-\kappa,h}) & -V_{0,h} \end{pmatrix} \begin{pmatrix} \underline{v} \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix} \tag{4.8}$$

where

$$\begin{array}{rclrcl} V_{0,h}[\ell,k] & = & \langle V_0\psi_k, \psi_\ell\rangle_\Gamma, & K_{-\kappa,h}[\ell,i] & = & \langle K_{-\kappa}\varphi_i, \psi_\ell\rangle_\Gamma \\ D_{\kappa,h}[j,i] & = & \langle D_\kappa\varphi_i, \varphi_j\rangle_\Gamma, & M_h[\ell,i] & = & \langle \varphi_i, \psi_\ell\rangle_\Gamma \end{array}$$

for $k, \ell = 1, \ldots, N$ and $i, j = 1, \ldots, M$. In addition,

$$f_j \,=\, \langle f, \varphi_j\rangle_\Gamma \,=\, \int_\Gamma f(x)\varphi_j(x)ds_x \quad \text{for } j = 1, \ldots, M. \tag{4.9}$$

Since the matrix $V_{0,h}$ is positive definite we can reformulate (4.8) as the Schur complement system

$$[D_{\kappa,h} + i\eta(-\frac{1}{2}M_h + K_{-\kappa,h})^* V_{0,h}^{-1}(-\frac{1}{2}M_h + K_{-\kappa,h})]\underline{v} \,=\, \underline{f}. \tag{4.10}$$

As for the exterior Dirichlet boundary value problem we can derive related error estimates, in particular there holds

$$\|v - v_h\|_{H^{1/2}(\Gamma)} + \|w - w_h\|_{H^{-1/2}(\Gamma)} \,\leq\, c\, h^{3/2} \left[\|v\|_{H^2(\Gamma)} + \|w\|_{H^1_{\text{pw}}(\Gamma)} \right]$$

when assuming $v \in H^2(\Gamma)$ and $w \in H^1_{\text{pw}}(\Gamma)$. Moreover, when applying the Aubin–Nitsche trick the optimal error estimate is given by

$$\|v - v_h\|_{H^{-1}(\Gamma)} + \|w - w_h\|_{H^{-2}(\Gamma)} \,\leq\, c\, h^3 \left[\|v\|_{H^2(\Gamma)} + \|w\|_{H^1_{\text{pw}}(\Gamma)} \right].$$

Inserting the approximate solutions (v_h, w_h) into the representation formula

$$u(x^*) \,=\, (W_\kappa v)(x^*) + i\eta(\widetilde{V}_\kappa w)(x^*) \quad \text{for } x \in \Omega^c,$$

this defines an approximate solution

$$u_h(x^*) \,=\, (W_\kappa v_h)(x^*) + i\eta(\widetilde{V}_\kappa w_h)(x^*) \quad \text{for } x \in \Omega^c$$

where we can conclude the pointwise error estimate

$$|u(x^*) - u_h(x^*)| \,\leq\, c\, h^3 \left[\|v\|_{H^2(\Gamma)} + \|w\|_{H^1_{\text{pw}}(\Gamma)} \right]. \tag{4.11}$$

5 Numerical Results

In this chapter we consider several numerical examples for the Dirichlet problem (2.1) as well as for the Neumann problem (4.1). In particular we will compare different boundary integral formulations as discussed in this paper.

5.1 Exterior Dirichlet Boundary Value Problem

We first consider the exterior Dirichlet boundary value problem (2.1) where the given Dirichlet datum g is choosen such that the solution of (2.1) is defined by the monopole function

$$u(x) = \frac{e^{i\kappa|x-x_s|}}{|x-x_s|} \quad \text{for } x \in \Omega^c, x_s \in \Omega, \tag{5.1}$$

which is a radiating solution of the Helmholtz equation. For the discretiation we consider the Galerkin variational problem (3.3) which leads to the Schur complement system (3.10)

$$[V_{\kappa,h} + i\eta(\frac{1}{2}M_h + K_{\kappa,h})\widetilde{D}_{0,h}^{-1}(\frac{1}{2}M_h + K_{\kappa,h})^*]\underline{w} \;=\; \underline{g}.$$

This system is solved by using a GMRES algorithm with complex arithmetics, and with a relative error reduction of $\varepsilon = 10^{-8}$. The k–th matrix by vector multiplication reads as follows

$$A_{\kappa,h}\underline{w}^k = V_{\kappa,h}\underline{w}^k + i\eta(\frac{1}{2}M_h + K_{\kappa,h})\underline{p}^k, \tag{5.2}$$

where

$$\underline{p}^k = \widetilde{D}_{0,h}^{-1}(\frac{1}{2}M_h + K_{\kappa,h})^*\underline{w}^k.$$

Thus we have to determine the solution $\underline{p}^k$ of the linear system

$$\widetilde{D}_{0,h}\underline{p}^k = (\frac{1}{2}M_h + K_{\kappa,h})^*\underline{w}^k \tag{5.3}$$

at any step of the GMRES iteration. Since the real valued matrix $\widetilde{D}_{0,h}$ is symmetric and positive definite we can compute the solution of (5.3) by applying a conjugate gradient method. Once the approximate solution (w_h, φ_h) is found, we check the error behaviour by computing an approximate solution

$$u_h(x^*) = (\widetilde{V}_\kappa w_h)(x^*) + i\eta(W_\kappa\varphi_h)(x^*) \quad \text{for } x^* \in \Omega^c.$$

As computational domain we first consider the unit sphere

$$\Omega = B_1(0) = \{x \in \mathbb{R}^3 : |x| < 1\},$$

with the boundary $\Gamma = \partial B_1(0)$ and where $x_s = (0, 0, 0.9)^\top$ in (5.1).

		(2.17)		(2.6)	
M	N	Iter	error	Iter	error
42	80	20	7.26 –3	20	1.20 –2
162	320	41	1.14 –3	38	5.99 –4
642	1280	55	2.20 –4	49	4.51 –5
2562	5120	70	8.94 –5	61	4.31 –6

Table 1: Exterior Dirichlet Boundary Value Poblem for a Sphere, $\kappa = 4$.

In Table 1 we present the results for the modified boundary integral equation (2.17) and for the single layer potential ansatz (2.6) when the wave number is $\kappa = 4$, i.e. no spurious modes appear.
Note that Iter stands for the number of GMRES iterations, and error stands for the pointwise error

$$\text{error} = |u_h(x^*) - u(x^*)|$$

which is evaluated in the point $x^* = (1.5, 0, 0)^\top$. At a first glance we can see that both methods lead to a good accuracy at the finest level but the convergence rate for the modified boundary integral equation is worse than the rate for the single layer potential. This might be due to the additional approximation error which is due to the composed boundary integral operator as used in the modified formulation.

Next we consider the wave number $\kappa = 13$ which is close to the eigenvalue $\lambda_4 = 4\pi$ of the interior Dirichlet eigenvalue problem (2.9). The results are given in Table 2. Note that the standard single layer potential approach fails while the proposed modified formulation behaves as predicted.

		(2.17)		(2.6)	
M	N	Iter	error	Iter	error
42	80	26	3.77 –1	28	2.90 –1
162	320	53	2.21 –2	84	9.38 –2
642	1280	79	2.53 –3	240	4.32 –1
2562	5120	105	2.39 –4	137	1.19 –3

Table 2: Exterior Dirichlet Boundary Value Poblem for a Sphere, $\kappa = 13$.

Finally we consider the exterior Dirichlet boundary value problem where the computational domain is the cube $\Omega = (0, \frac{1}{2})^3$ with the monopole (5.1) for $x_s = (0.2, 0.1, 0.4)^\top$. The first critical wave number is $k = 2\sqrt{3}\pi \approx 10.883$ for which the results are given in Table 3 where the evaluation point is $x^* = (1, 0.5, 0.5)^\top \in \Omega^c$. Again the standard approach fails while the modified formulation behaves as predicted.

		(2.17)		(2.6)	
M	N	Iter	error	Iter	error
14	24	9	1.57 –1	9	1.94 –1
50	96	27	3.05 –3	27	2.82 –1
184	384	42	1.05 –3	42	4.93 –1
770	1536	59	3.12 –4	59	2.32
3074	6144	73	1.38 –4	76	9.05 –1

Table 3: Exterior Dirichlet Boundary Value Poblem for a Cube, $\kappa = 2\sqrt{3}\pi$.

5.2 Exterior Neumann Boundary Value Problem

Now we consider the exterior Neumann boundary value problem (4.1). As for the Dirichlet problem the solution is defined by using the monopole function (5.1). The Galerkin discretisation of the modified boundary integral eqation (4.6) leads to the Schur complement system (4.10)

$$[D_{\kappa,h} + i\eta(-\frac{1}{2}M_h + K_{-\kappa,h})^* V_{0,h}^{-1}(-\frac{1}{2}M_h + K_{-\kappa,h})]\underline{v} \;=\; \underline{f}. \tag{5.4}$$

As for the Dirichlet problem we solve the system (5.4) using the GMRES algorithm for the outer iteration and the CG algorithm for the inner iteration.

First we consider the unit sphere $\Omega = B_1(0)$, $x_s = (0, 0, 0.9)^\top$, and $\kappa = 4$, i.e. the case of no spurious modes. In Table 4 we give the results for the modified boundary integral formulation (4.6), for the Brakhage–Werner formulation (4.5), for the double layer potential integral equation (4.2), and for the hypersingular boundary integral equation (4.3).

		(4.6)		(4.5)		(4.2)		(4.3)	
M	N	Iter	error	Iter	error	Iter	error	Iter	error
42	80	14	2.00 –2	14	4.06 –2	14	6.94 –2	14	5.18 –2
162	320	23	4.89 –3	18	1.69 –2	22	2.91 –3	19	3.97 –3
642	1280	28	1.29 –4	25	7.88 –4	25	1.11 –4	26	9.64 –4
2562	5120	36	9.57 –6	35	1.13 –4	26	3.92 –5	36	1.45 –4

Table 4: Exterior Neumann Boundary Value Problem for a Sphere, $\kappa = 4$.

Next we consider the cube $\Omega = (0, \frac{1}{2})^3$ with the monopole (5.1) for $x_s = (0.2, 0.1, 0.4)^\top$. For the critical wave number $\kappa = 2\sqrt{3}\pi \approx 10.883$ the hypersingular boundary integral equation (4.3) fails, see Table 5, since the wave number corresponds to an eigenvalue of the interior Neumann eigenvalue problem.
In addition to Table 5 we also consider the direct boundary integral equation (4.4) and the Burton–Miller formulation (2.15), see Table 6.

		(4.6)		(4.5)		(4.2)		(4.3)	
M	N	Iter	error	Iter	error	Iter	error	Iter	error
14	24	7	1.46 –1	7	3.13 –1	7	8.11 –1	7	2.90 –1
50	96	22	7.67 –3	19	9.61 –2	20	2.22 –1	19	8.10 –2
194	384	29	2.43 –3	26	1.25 –2	30	1.45 –1	29	6.01 –2
770	1536	36	3.80 –4	39	1.44 –3	35	7.05 –2	44	5.83 –2
3074	6144	50	5.56 –5	54	2.25 –4	37	3.56 –2	64	6.19 –2

Table 5: Exterior Neumann Boundary Value Problem for a Cube, $\kappa = 2\sqrt{3}\pi$.

		(4.4)		(2.15)	
M	N	Iter	error	Iter	error
14	24	6	2.65 –1	6	2.85 –1
50	96	19	7.33 –2	19	7.95 –2
194	384	29	1.11 –2	25	1.07 –2
770	1536	44	2.58 –3	38	2.47 –3
3074	6144	64	6.35 –4	53	6.09 –4

Table 6: Neumann Boundary Value Problem for a Cube, $\kappa = 2\sqrt{3}\pi$.

6 Conclusions

In this paper we have formulated and analyzed modified boundary element methods for the Dirichlet and Neumann boundary value problems of the Helmholtz equation which are stable for all wave numbers. In contrast to other existing approaches the proposed method relies on the use boundary integral operators which are already available in standard boundary element methods. Therefore, the additional effort in the implementation is neglectible.

Open questions concern the construction of efficient preconditioned iterative solution strategies where also an optimal choice of the scaling parameter $\eta \in \mathbb{R}_+$ is crucial, see [1, 14, 15]. Note that the use of fast boundary element methods [11, 12, 18] will enable us to handle challenging applications.

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