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A Boundary Element Method for the Dirichlet Eigenvalue Problem of the Laplace Operator

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Abstract

The solution of eigenvalue problems for partial differential operators by using boundary integral equation methods usually involves some Newton potentials which may be resolved by using a multiple reciprocity approach. Here we propose an alternative approach which is in some sense equivalent to the above. Instead of a linear eigenvalue problem for the partial differential operator we consider a nonlinear eigenvalue problem for an associated boundary integral operator. This nonlinear eigenvalue problem can be solved by using some approriate iterative scheme, here we will consider a Newton scheme. We will discuss the convergence and the boundary element discretization of this algorithm, and give some numerical results.

1 Introduction

As a model problem we consider the interior Dirichlet eigenvalue problem of the Laplace operator,

$$-\Delta u_\lambda(x) = \lambda u_\lambda(x) \quad \text{for } x \in \Omega, \quad u_\lambda(x) = 0 \quad \text{for } x \in \Gamma = \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. The variational formulation of (1.1) reads to find $u_\lambda \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_\lambda(x) \nabla v(x) dx = \lambda \int_\Omega u_\lambda(x) v(x) dx \tag{1.2}$$

is satisfied for all $v \in H_0^1(\Omega)$. It is well known that the set of eigensolutions $\{(u_{\lambda_k}, \lambda_k)\}$ is countable, and that the eigenfunctions $\{u_{\lambda_k}\}$ form a complete orthonormal system in $L_2(\Omega)$ and in $H_0^1(\Omega)$, respectively. Moreover, all eigenvalues λ_k have a finite multiplicity, and we have $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ as well as $\lambda_k \to \infty$ as $k \to \infty$.

A finite element approximation of the variational formulation (1.2) results in the linear algebraic eigenvalue problem

$$K_h \underline{u}_{\lambda_k} = \lambda_k M_h \underline{u}_{\lambda_k} \tag{1.3}$$

where K_h is the finite element stiffness matrix, and M_h is the related mass matrix. For a numerical analysis of this approach, and for appropriate eigenvalue solvers for (1.3), see, for example, [1, 4, 9, 14].

Instead of a finite element approach, which always requires a discretization of the computational domain Ω, we will use boundary integral formulations and related boundary element methods [15, 16] to solve the eigenvalue problem (1.1). Then, only a discretization of the boundary $\Gamma = \partial\Omega$ is required, and the computational complexity is lower than in a finite element approach when using fast boundary element methods [12].

Considering the eigenvalue problem (1.1) as a Poisson equation with a certain right hand side λu_λ we obtain a boundary–domain integral formulation [8] to be solved. By using the so–called Multiple Reciprocity Method (MRM) [6] it is possible to approximate the volume integrals by some boundary integrals. Then, a polynomial eigenvalue problem has to be solved. Since our approach is in some sense equivalent with the latter we first describe the multiple reciprocity method.

By using the fundamental solution of the Laplace operator,

$$U^*(x,y) \;=\; \frac{1}{4\pi}\,\frac{1}{|x-y|} \quad \text{for } x,y \in \mathbb{R}^3,$$

the solution of the eigenvalue problem (1.1) is given by the representation formula

$$u_\lambda(x) \;=\; \int_\Gamma U^*(x,y)t_\lambda(y)ds_y + \lambda \int_\Omega U^*(x,y)u_\lambda(y)dy \quad \text{for } x \in \Omega \tag{1.4}$$

where $t_\lambda(x) = n_x \cdot \nabla u_\lambda(x)$, $x \in \Gamma$, is the associated normal derivative of the eigensolution u_λ. The basic idea of the multiple reciprocity method is to rewrite the volume integral in the representation formula (1.4) by using integration by parts. For this we first note

$$\frac{\partial}{\partial y_i}|x-y| \;=\; \frac{y_i - x_i}{|x-y|}, \quad \frac{\partial^2}{\partial y_i^2}|x-y| \;=\; \frac{1}{|x-y|} - \frac{(y_i-x_i)^2}{|x-y|^3}.$$

Therefore the fundamental solution of the Laplace operator can be written as

$$U^*(x,y) = \frac{1}{4\pi}\,\frac{1}{|x-y|} = \Delta_y\left(\frac{1}{8\pi}|x-y|\right) = \Delta_y U_1^*(x,y), \quad U_1^*(x,y) = \frac{1}{8\pi}|x-y|.$$

Hence, by using Green's second formula, we obtain

$$\begin{aligned}
\int_\Omega U^*(x,y)u_\lambda(y)dy \;&=\; \int_\Omega \Delta_y U_1^*(x,y)u_\lambda(y)dy \\
&= \int_\Omega U_1^*(x,y)\Delta_y u_\lambda(y)dy - \int_\Gamma U_1^*(x,y)\frac{\partial}{\partial n_y}u_\lambda(y)ds_y + \int_\Gamma u_\lambda(y)\frac{\partial}{\partial n_y}U_1^*(x,y)ds_y \\
&= -\lambda\int_\Omega U_1^*(x,y)u_\lambda(y)dy - \int_\Gamma U_1^*(x,y)t_\lambda(y)ds_y
\end{aligned}$$

when u_λ is a solution of the eigenvalue problem (1.1). The representation formula (1.4) is therefore equivalent to

$$u_\lambda(x) \;=\; \int_\Gamma \left[U^*(x,y) - \lambda U_1^*(x,y)\right] t_\lambda(y) ds_y - \lambda^2 \int_\Omega U_1^*(x,y) u_\lambda(y) dy, \quad x \in \Omega.$$

By doing this recursively, and by setting $U_0^*(x,y) = U^*(x,y)$, we further obtain

$$u_\lambda(x) = \int_\Gamma \left[\sum_{k=0}^{n} (-\lambda)^k U_k^*(x,y)\right] t_\lambda(y) ds_y - (-\lambda)^{n+1} \int_\Omega U_n^*(x,y) u_\lambda(y) dy, \quad x \in \Omega,$$

where

$$U_k^*(x,y) \;=\; \Delta_y U_{k+1}^*(x,y) \quad \text{for } k = 0, \ldots, n-1.$$

For $k \in \mathbb{N}_0$ we define

$$U_k^*(x,y) \;=\; \frac{\alpha_k}{4\pi} |x-y|^{2k-1}, \quad \alpha_0 = 1, \quad \alpha_1 = \frac{1}{2}$$

to obtain, for $k \in \mathbb{N}$,

$$\frac{\partial}{\partial y_j} U_{k+1}^*(x,y) \;=\; \frac{\alpha_{k+1}}{4\pi} \frac{\partial}{\partial y_j} |x-y|^{2k+1} \;=\; \frac{\alpha_{k+1}}{4\pi} (2k+1)(y_j - x_j)|x-y|^{2k-1}$$

and therefore

$$\frac{\partial^2}{\partial y_j^2} U_{k+1}^*(x,y) = \frac{\alpha_{k+1}}{4\pi}(2k+1)|x-y|^{2k-1} + \frac{\alpha_{k+1}}{4\pi}(2k+1)(2k-1)(y_j - x_j)^2 |x-y|^{2k-3}.$$

Hence,

$$\Delta_y U_{k+1}^*(x,y) = \alpha_{k+1}(2k+1)(2k+2)\frac{1}{4\pi}|x-y|^{2k-1} = \frac{\alpha_k}{4\pi}|x-y|^{2k-1} = U_k^*(x,y),$$

and by induction we find

$$\alpha_{k+1} \;=\; \frac{\alpha_k}{(2k+1)(2k+2)} \;=\; \frac{1}{(2k+2)!},$$

i.e.

$$U_k^*(x,y) \;=\; \frac{1}{(2k)!}\frac{1}{4\pi}|x-y|^{2k-1}.$$

For $x \in \Omega$ we therefore have the representation formula

$$u_\lambda(x) \;=\; \frac{1}{4\pi} \int_\Gamma \frac{1}{|x-y|} \left[\sum_{k=0}^{n} (-1)^k \lambda^k \frac{1}{(2k)!} |x-y|^{2k}\right] t_\lambda(y) ds_y + R_n(x, u_\lambda, \lambda) \tag{1.5}$$

where the remainder is given by

$$R_n(x, u_\lambda, \lambda) \;=\; (-1)^n \lambda^{n+1} \frac{1}{(2n)!} \frac{1}{4\pi} \int_\Omega |x-y|^{2n-1} u_\lambda(y) dy.$$

In the approach of the multiple reciprocity method (MRM) the remainder R_n is neglected [5]. Then, due to the homogeneous Dirichlet boundary condition in (1.1), the polynomial eigenvalue problem

$$\frac{1}{4\pi} \int_\Gamma \frac{1}{|x-y|} \left[\sum_{k=0}^{n} (-1)^k \widetilde{\lambda}^k \frac{1}{(2k)!} |x-y|^{2k} \right] \widetilde{t}_\lambda(y) ds_y \;=\; 0 \quad \text{for } x \in \Gamma \tag{1.6}$$

is to be solved. In particular when taking the limit for $n \to \infty$ this results in the nonlinear eigenvalue problem

$$\frac{1}{4\pi} \int_\Gamma \frac{\cos \sqrt{\lambda} |x-y|}{|x-y|} t_\lambda(y) ds_y \;=\; 0 \quad \text{for } x \in \Gamma. \tag{1.7}$$

Let us now describe an alternative approach to derive the nonlinear eigenvalue problem (1.7). For $\lambda = \kappa^2 > 0$ we can write the interior Dirichlet eigenvalue problem (1.1) as a Helmholtz equation with homogeneous Dirichlet boundary conditions,

$$-\Delta u_\kappa(x) - \kappa^2 u_\kappa(x) = 0 \quad \text{for } x \in \Omega, \quad u_\kappa(x) = 0 \quad \text{for } x \in \Gamma. \tag{1.8}$$

Solutions of the boundary value problem (1.8) are given by the representation formula

$$u_\kappa(x) = \int_\Gamma U^*_\kappa(x, y) t(y) ds_y \quad \text{for } x \in \Omega \tag{1.9}$$

where the fundamental solution of the Helmholtz operator is given by

$$U^*_\kappa(x, y) \;=\; \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x-y|} \quad \text{for } x, y \in \mathbb{R}^3. \tag{1.10}$$

Applying the interior trace operator to the representation formula (1.9) we obtain a boundary integral equation to find wave numbers $\kappa \in \mathbb{R}_+$ and related nontrivial eigensolutions $t \in H^{-1/2}(\Gamma)$ such that

$$\int_\Gamma U^*_\kappa(x, y) t(y) ds_y = 0 \quad \text{for } x \in \Gamma. \tag{1.11}$$

Since the eigenfuctions u_λ of the eigenvalue problem (1.1) and therefore the solutions u_κ of the boundary value problem (1.8) are real valued, instead of (1.11) we have to find nontrivial solutions $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ satisfying

$$\frac{1}{4\pi} \int_\Gamma \frac{\cos \kappa |x-y|}{|x-y|} t(y) ds_y \;=\; 0 \quad \text{for } x \in \Gamma \tag{1.12}$$

and

$$\frac{1}{4\pi}\int_\Gamma \frac{\sin\kappa|x-y|}{|x-y|}t(y)ds_y \;=\; 0 \quad \text{for } x\in\Gamma. \tag{1.13}$$

Note that (1.12) corresponds to (1.7).

Summarizing the above we conclude that for any solution $(u_\lambda,\lambda)\in H^1_0(\Omega)\times\mathbb{R}$ of the eigenvalue problem (1.1) the nonlinear eigenvalue problem (1.12) is satisfied where the wave number is $\kappa=\sqrt{\lambda}$ and $t(x)=n_x\cdot\nabla u_\lambda(x)$, $x\in\Gamma$, is the corresponding normal derivative. On the other hand, if $(t,\kappa)\in H^{-1/2}(\Gamma)\times\mathbb{R}$ is a solution of the nonlinear eigenvalue problem (1.12), the function

$$u_\kappa(x) \;=\; \frac{1}{4\pi}\int_\Gamma \frac{\cos\kappa|x-y|}{|x-y|}t(y)ds_y \quad \text{for } x\in\Omega$$

solves the eigenvalue problem (1.1). Hence, the linear eigenvalue problem (1.1) is equivalent to the nonlinear eigenvalue problem (1.12). A boundary element approximation of the eigenvalue problem (1.11) would lead to a polynomial eigenvalue problem, see [7] for a collocation approach. Here we will first consider a Newton scheme to solve (1.12) and then we will apply a Galerkin boundary element discretization afterwards.

In Section 2 we consider an iterative solution approach for the nonlinear eigenvalue problem (2.3) which is an analogon of the inverse iteration for linear and for nonlinear matrix eigenvalue problems, see, e.g., [11, 13, 17]. In fact, we will apply a Newton scheme to solve the nonlinear equation (2.3) where in addition we introduce an appropriate scaling condition in $H^{-1/2}(\Gamma)$. In particular we will prove the invertibility of the related Fréchet derivative. However, our theoretical approach is restricted to simple eigenvalues only. A Galerkin boundary element method to solve the nonlinear eigenvalue problem is formulated and analyzed in Section 3 where we also prove optimal convergence for the approximate solutions. Numerical results given in Section 4 confirm not only the theoretical results, the experiments indicate that our approach also works for multiple eigenvalues.

2 Application of Newton's method

The nonlinear eigenvalue problem (1.12) can be written as

$$(V_\kappa t)(x)=\frac{1}{4\pi}\int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|}t(y)ds_y=0 \quad \text{for } x\in\Gamma \tag{2.1}$$

where for fixed κ the operator $V_\kappa : H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$ is linear and bounded, see, e.g., [10, 15, 16]. To normalize the eigensolutions $t\in H^{-1/2}(\Gamma)$ of (2.1) we introduce a scaling condition by using an equivalent norm in $H^{-1/2}(\Gamma)$,

$$\|t\|_V^2 \;=\; \langle Vt,t\rangle_\Gamma \;=\; \frac{1}{4\pi}\int_\Gamma t(x)\int_\Gamma \frac{1}{|x-y|}t(y)ds_yds_x \;=\; 1, \tag{2.2}$$

where $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the single layer potential of the Laplace operator. Note that we have, see for example [16],

$$\langle Vt, t\rangle_\Gamma \,\geq\, c_1^V \,\|t\|^2_{H^{-1/2}(\Gamma)}, \quad \|Vt\|_{H^{1/2}(\Gamma)} \,\leq\, c_2^V \,\|t\|_{H^{-1/2}(\Gamma)} \quad \text{for all } t \in H^{-1/2}(\Gamma).$$

Now we have to find solutions $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ of the nonlinear eigenvalue problem

$$F_1(t,\kappa) \,=\, (V_\kappa t)(x) \,=\, 0 \quad \text{for } x \in \Gamma, \quad F_2(t,\kappa) \,=\, \langle Vt, t\rangle_\Gamma - 1 \,=\, 0. \tag{2.3}$$

Hence we define the function $\boldsymbol{F} : H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma) \times \mathbb{R}$ as

$$\boldsymbol{F}(t,\kappa) \,=\, \begin{pmatrix} F_1(t,\kappa) \\ F_2(t,\kappa) \end{pmatrix} \,=\, \begin{pmatrix} \dfrac{1}{4\pi}\displaystyle\int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|} t(y) ds_y \\ \langle Vt, t\rangle_\Gamma - 1 \end{pmatrix}. \tag{2.4}$$

Then, to obtain eigensolutions of the scaled eigenvalue problem (2.3) we have to find solutions $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ of the nonlinear equation

$$\boldsymbol{F}(t,\kappa) \,=\, \boldsymbol{0} \tag{2.5}$$

which is to be solved by applying Newton's method. For the Fréchet derivative of $\boldsymbol{F}(t,\kappa)$ we obtain

$$\boldsymbol{F}'(t,\kappa) \,=\, \begin{pmatrix} V_\kappa & -A_\kappa t \\ 2\langle Vt, \cdot\rangle_\Gamma & 0 \end{pmatrix} \,:\, H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma) \times \mathbb{R} \tag{2.6}$$

where

$$(A_\kappa t)(x) \,=\, \frac{1}{4\pi}\int_\Gamma \sin(\kappa|x-y|) t(y) ds_y \quad \text{for } x \in \Gamma.$$

As for the Laplace single layer potential, see for example [10, 16], we conclude that the boundary integral operator $A_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded, i.e. for all $\kappa \in \mathbb{R}$ we have

$$\|A_\kappa t\|_{H^{-1/2}(\Gamma)} \,\leq\, c_{A_\kappa} \,\|t\|_{H^{-1/2}(\Gamma)} \quad \text{for all } t \in H^{-1/2}(\Gamma). \tag{2.7}$$

When applying a Newton scheme to find solutions $(t_*, \kappa_*) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ of the nonlinear equation (2.5) the new iterates $(t_{n+1}, \kappa_{n+1}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ are the unique solutions of the linear operator equation

$$\boldsymbol{F}'(t_n,\kappa_n) \begin{pmatrix} t_{n+1} - t_n \\ \kappa_{n+1} - \kappa_n \end{pmatrix} + \boldsymbol{F}(t_n,\kappa_n) \,=\, \boldsymbol{0} \tag{2.8}$$

where for the previous iterates we assume $(t_n, \kappa_n) \in U_\varrho(t_*, \kappa_*)$ where $\varrho > 0$ is sufficient small, i.e.

$$\|t_* - t_n\|^2_{H^{-1/2}(\Gamma)} + |\kappa_* - \kappa_n|^2 \,\leq\, \varrho^2. \tag{2.9}$$

Note that the linearized equation (2.8) is equivalent to a saddle point problem to find $(t_{n+1}, \kappa_{n+1}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\kappa_n} t_{n+1}, w\rangle_\Gamma - \kappa_{n+1}\langle A_{\kappa_n} t_n, w\rangle_\Gamma &= -\kappa_n \langle A_{\kappa_n} t_n, w\rangle_\Gamma \\ 2\langle V t_n, t_{n+1}\rangle_\Gamma &= \langle V t_n, t_n\rangle_\Gamma + 1. \end{aligned} \tag{2.10}$$

is satisfied for all $w \in H^{-1/2}(\Gamma)$.

The local convergence of the Newton method (2.8) to solve the nonlinear eigenvalue problem (2.3) is guaranteed if the Fréchet derivative $\boldsymbol{F}'(t_*, \kappa_*)$ is invertible for the solution (t_*, κ_*) of $\boldsymbol{F}(t, \kappa) = \mathbf{0}$. For this we first show that the Fréchet derivative $\boldsymbol{F}'(t, \kappa)$ satisfies a Gårdings inequality.

Lemma 2.1. *The Fréchet derivative $\boldsymbol{F}'(t, \kappa) : H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma) \times \mathbb{R}$ is coercive for all $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$, i.e. there exists a compact operator*

$$\boldsymbol{C}(t, \kappa) \,:\, H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma) \times \mathbb{R}$$

such that the following Gårdings inequality is satisfied,

$$\langle (\boldsymbol{F}'(t, \kappa) + \boldsymbol{C}(t, \kappa))(w, \alpha), (w, \alpha)\rangle \,\geq\, c\, \left[\|w\|^2_{H^{-1/2}(\Gamma)} + |\alpha|^2 \right] \tag{2.11}$$

for all $(w, \alpha) \in H^{-1/2}(\Gamma) \times \mathbb{R}$.

Proof. For

$$\boldsymbol{C}(t, \kappa) \begin{pmatrix} w \\ \alpha \end{pmatrix} := \begin{pmatrix} C_1(w, \alpha) \\ C_2(w, \alpha) \end{pmatrix} = \begin{pmatrix} (V - V_\kappa) w + \alpha A_\kappa t \\ -2\langle w, t\rangle_\Gamma + \alpha \end{pmatrix}$$

we obtain

$$\boldsymbol{F}'(t, \kappa) + \boldsymbol{C}(t, \kappa) \;=\; \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}$$

and therefore Gårdings inequality (2.11) is fulfilled since the Laplace single layer potential V is $H^{-1/2}(\Gamma)$–elliptic. It remains to show that $\boldsymbol{C}(t, \kappa)$ is compact.

We first note that the operator $V - V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact, see, e.g., [15]. For a fixed $t \in H^{-1/2}(\Gamma)$ also the operator $\alpha A_\kappa t : \mathbb{R} \to H^{1/2}(\Gamma)$ is compact. Hence,

$$C_1(w, \alpha) \;=\; (V - V_\kappa) w + \alpha A_\kappa t, \quad C_1 : H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma)$$

is compact. Finally, the operator $C_2 : H^{-1/2}(\Gamma) \times \mathbb{R} \to \mathbb{R}$ is compact since every linear and bounded operator which maps into a finite dimensional space is compact. $\square$

Since for all $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ the Fréchet derivative $\boldsymbol{F}'(t, \kappa)$ satisfies a Gårdings inequality, it is sufficient to investigate the injectivity of $\boldsymbol{F}'(t_*, \kappa_*)$.

Lemma 2.2. *Let $(t_*, \kappa_*) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ be a solution of $\boldsymbol{F}(t, \kappa) = \mathbf{0}$. Assume*

(A1) κ_* *is a simple eigenvalue of* $V_\kappa t = 0$,

(A2) $A_{\kappa_*} t_* \notin \mathcal{R}(V_{\kappa_*})$.

Then $\boldsymbol{F}'(t_*, \kappa_*)$ *is injective.*

Proof. Consider

$$\boldsymbol{F}'(t_*, \kappa_*)(w, \alpha) \; = \; \mathbf{0}$$

for some $(w, \alpha) \in H^{-1/2}(\Gamma) \times \mathbb{R}$. In particular we have

$$V_{\kappa_*} w \; = \; \alpha A_{\kappa_*} t_* \, .$$

Since we assume $A_{\kappa_*} t_* \notin \mathcal{R}(V_{\kappa_*})$ it follows that $\alpha = 0$ and $V_{k_*} w = 0$. Hence we obtain $w = \omega t_*$ for some $\omega \in \mathbb{R}$. Then,

$$0 \; = \; 2\langle w, V t_* \rangle_\Gamma \; = \; 2\omega \langle t_*, V t_* \rangle_\Gamma$$

implies $w \equiv 0$, i.e. $\boldsymbol{F}'(t_*, \kappa_*)$ is injective. □

Corollary 2.3. *Let* (t_*, κ_*) *be a solution of* $\boldsymbol{F}(t, \kappa) = \mathbf{0}$ *and* (A1) *and* (A2) *be satisfied. Then* $\boldsymbol{F}'(t_*, \kappa_*)$ *is invertible. Since* $\boldsymbol{F}'(t, \kappa)$ *is continuous in* $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ *it follows that also* $\boldsymbol{F}'(t, \kappa)$ *is invertible for all* $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R} \cap U_\varrho(t_*, \kappa_*)$ *for some* $\varrho > 0$.

Summarizing the above we now can formulate the main result of this section.

Theorem 2.4. *Let* (t_*, κ_*) *be a solution of* $\boldsymbol{F}(t, \kappa) = \mathbf{0}$ *and* (A1) *and* (A2) *be satisfied. Then* $\boldsymbol{F}'(t_*, \kappa_*)$ *is invertible and Newton's method converges for all initial values in a sufficient small neighborhood* $U_\varrho(t_*, \kappa_*)$ *to* (t_*, κ_*).

Remark 2.5. *For multiple eigenvalues* κ_* *the Fréchet derivative* $\boldsymbol{F}'(t_*, \kappa_*)$ *is not invertible, because* $\boldsymbol{F}'(t_*, \kappa_*)$ *is not injective. Nevertheless Newton's method may also converge* [2, 3]. *The convergence rate may then be smaller and the convergence domain is not a small neighborhood of the solution but rather a restricted region which avoids the set on which* $\boldsymbol{F}'$ *is singular. In our case numerical examples show that Newton's method converges also for all multiple eigenvalues, see the numerical example in Section* 4.

3 A boundary element method

Let us recall the variational formulation (2.10) to find $(t_{n+1}, \kappa_{n+1}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\kappa_n} t_{n+1}, w \rangle_\Gamma - \kappa_{n+1} \langle A_{\kappa_n} t_n, w \rangle_\Gamma &= -\kappa_n \langle A_{\kappa_n} t_n, w \rangle_\Gamma \\ 2\langle V t_n, t_{n+1} \rangle_\Gamma &= \langle V t_n, t_n \rangle_\Gamma + 1. \end{aligned} \tag{3.1}$$

is satisfied for all $w \in H^{-1/2}(\Gamma)$. For a Galerkin discretization of (3.1) we first define trial spaces $S_h^0(\Gamma)$ of piecewise constant basis functions ψ_k which are defined with respect to a globally quasi–uniform boundary element mesh of mesh size h. Then the Galerkin discretization of (3.1) reads to find $(t_{n+1,h}, \kappa_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\kappa_n} t_{n+1,h}, w_h \rangle_\Gamma - \kappa_{n+1,h} \langle A_{\kappa_n} t_n, w_h \rangle_\Gamma &= -\kappa_n \langle A_{\kappa_n} t_n, w_h \rangle_\Gamma \\ 2\langle V t_n, t_{n+1,h} \rangle_\Gamma &= \langle V t_n, t_n \rangle_\Gamma + 1. \end{aligned} \tag{3.2}$$

is satisfied for all $w_h \in S_h^0(\Gamma)$.

Theorem 3.1. *Let (t_*, κ_*) be a solution of $\boldsymbol{F}(t,\kappa) = 0$ and let the assumptions* (A1) *and* (A2) *be satisfied. Let $(t_n, \kappa_n) \in U_\varrho(t_*, \kappa_*)$ be satisfied where ϱ is appropriately chosen as discussed in Corollary* 2.3. *Then, for a sufficient small mesh size $h < h_0$, the Galerkin variational problem* (3.2) *has a unique solution $(t_{n+1,h}, \kappa_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ satisfying the error estimate*

$$\|t_{n+1} - t_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \kappa_{n+1,h}|^2 \le c \inf_{w_h \in S_h^0(\Gamma)} \|t_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)}. \tag{3.3}$$

Proof. Since the Fréchet derivative $\boldsymbol{F}'(t_n, \kappa_n)$ is injective and satisfies a Gårdings inequality, the proof follows by applying standard arguments, see, e.g., [16]. □

In practical computations we have to replace in (3.2) $(t_n, \kappa_n) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ by previously computed approximations $(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) \in S_h^0(\Gamma) \times \mathbb{R}$. In particular we have to find $(\hat{t}_{n+1,h}, \hat{\kappa}_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\hat{\kappa}_{n,h}} \hat{t}_{n+1,h}, w_h \rangle_\Gamma - \hat{\kappa}_{n+1,h} \langle A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h}, w_h \rangle_\Gamma &= -\hat{\kappa}_{n,h} \langle A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h}, w_h \rangle_\Gamma \\ 2\langle V \hat{t}_{n,h}, \hat{t}_{n+1,h} \rangle_\Gamma &= \langle V \hat{t}_{n,h}, \hat{t}_{n,h} \rangle_\Gamma + 1 \end{aligned} \tag{3.4}$$

is satisfied for all $w_h \in S_h^0(\Gamma)$. To analyze the perturbed variational problem (3.4) we also need to consider the continuous variational problem to find $(\hat{t}_{n+1}, \hat{\kappa}_{n+1}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\hat{\kappa}_{n,h}} \hat{t}_{n+1}, w \rangle_\Gamma - \hat{\kappa}_{n+1} \langle A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h}, w \rangle_\Gamma &= -\hat{\kappa}_{n,h} \langle A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h}, w \rangle_\Gamma \\ 2\langle V \hat{t}_{n,h}, \hat{t}_{n+1} \rangle_\Gamma &= \langle V \hat{t}_{n,h}, \hat{t}_{n,h} \rangle_\Gamma + 1. \end{aligned} \tag{3.5}$$

is satisfied for all $w \in H^{-1/2}(\Gamma)$. Note that (3.4) is the Galerkin discretization of (3.5).

Theorem 3.2. *Let (t_*, κ_*) be a solution of $\boldsymbol{F}(t,\kappa) = 0$ and let the assumptions* (A1) *and* (A2) *be satisfied. Let $(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) \in S_h^0(\Gamma) \times \mathbb{R} \cap U_\varrho(t_*, \kappa_*)$ be satisfied where ϱ is appropriately chosen as discussed in Corollary* 2.3. *Then, for a sufficient small mesh size $h < h_0$, the Galerkin variational problem* (3.4) *has a unique solution $(\hat{t}_{n+1,h}, \hat{\kappa}_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ satisfying the error estimate*

$$\begin{aligned} &\|t_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1,h}|^2 \\ &\qquad \le c \left[\|t_n - \hat{t}_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat{\kappa}_{n,h}|^2 + \inf_{w_h \in S_h^0(\Gamma)} \|t_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)} \right] \end{aligned} \tag{3.6}$$

where the constant c depends on (t^*, κ^*), and on ϱ.

Proof. Since (3.4) is the Galerkin formulation of the variational problem (3.5), the application of Theorem 3.1 gives the error estimate

$$\|\hat{t}_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\hat{\kappa}_{n+1} - \hat{\kappa}_{n+1,h}|^2 \;\le\; c \inf_{w_h \in S_h^0(\Gamma)} \|\hat{t}_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)}$$

and therefore, by using the triangle inequality twice,

$$\begin{aligned} &\|t_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1,h}|^2 \\ &\le\; 2(1+2c)\left[\|t_{n+1} - \hat{t}_{n+1}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1}|^2 + \inf_{w_h \in S_h^0(\Gamma)} \|t_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)}\right] \end{aligned}$$

where the constant c is given as in (3.3).

Recall that the variational formulation (3.1) is equivalent to the Newton equation (2.8),

$$\boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_{n+1} - t_n \\ \kappa_{n+1} - \kappa_n \end{pmatrix} + \boldsymbol{F}(t_n,\kappa_n) \;=\; \boldsymbol{0}.$$

In the same way we can write the variational problem (3.5) as

$$\boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\begin{pmatrix} \hat{t}_{n+1} - \hat{t}_{n,h} \\ \hat{\kappa}_{n+1} - \hat{\kappa}_{n,h} \end{pmatrix} + \boldsymbol{F}(\hat{t}_{n,h},\hat{\kappa}_{n,h}) \;=\; \boldsymbol{0}.$$

Hence,

$$\begin{aligned} \boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_{n+1} - \hat{t}_{n+1} \\ \kappa_{n+1} - \hat{\kappa}_{n+1} \end{pmatrix} \;&=\; \boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_{n+1} - t_n \\ \kappa_{n+1} - \kappa_n \end{pmatrix} + \boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_n - \hat{t}_{n,h} \\ \kappa_n - \hat{\kappa}_{n,h} \end{pmatrix} \\ &+ \left[\boldsymbol{F}'(t_n,\kappa_n) - \boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\right]\begin{pmatrix} \hat{t}_{n,h} - \hat{t}_{n+1} \\ \hat{\kappa}_{n,h} - \hat{\kappa}_{n+1} \end{pmatrix} + \boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\begin{pmatrix} \hat{t}_{n,h} - \hat{t}_{n+1} \\ \hat{\kappa}_{n,h} - \hat{\kappa}_{n+1} \end{pmatrix} \\ &=\; \boldsymbol{F}(\hat{t}_{n,h},\hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n,\kappa_n) + \boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_n - \hat{t}_{n,h} \\ \kappa_n - \hat{\kappa}_{n,h} \end{pmatrix} \\ &+ \left[\boldsymbol{F}'(t_n,\kappa_n) - \boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\right]\begin{pmatrix} \hat{t}_{n,h} - \hat{t}_{n+1} \\ \hat{\kappa}_{n,h} - \hat{\kappa}_{n+1} \end{pmatrix}. \end{aligned}$$

Since $\boldsymbol{F}'(t_n,\kappa_n) : H^{-1/2}(\Gamma)\times\mathbb{R} \to H^{1/2}(\Gamma)\times\mathbb{R}$ is assumed to be invertible, see Corollary 2.3, we therefore have

$$\begin{aligned} \begin{pmatrix} t_{n+1} - \hat{t}_{n+1} \\ \kappa_{n+1} - \hat{\kappa}_{n+1} \end{pmatrix} \;&=\; [\boldsymbol{F}'(t_n,\kappa_n)]^{-1}\left[\boldsymbol{F}(\hat{t}_{n,h},\hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n,\kappa_n)\right] + \begin{pmatrix} t_n - \hat{t}_{n,h} \\ \kappa_n - \hat{\kappa}_{n,h} \end{pmatrix} \\ &+ [\boldsymbol{F}'(t_n,\kappa_n)]^{-1}\left[\boldsymbol{F}'(t_n,\kappa_n) - \boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\right]\begin{pmatrix} \hat{t}_{n,h} - \hat{t}_{n+1} \\ \hat{\kappa}_{n,h} - \hat{\kappa}_{n+1} \end{pmatrix} \end{aligned}$$

and thus

$$\begin{aligned} \|(t_{n+1} - \hat{t}_{n+1}, \kappa_{n+1} - \hat{\kappa}_{n+1})\|_{H^{-1/2}(\Gamma)\times\mathbb{R}} \;&\le\; \|(t_n - \hat{t}_{n,h}, \kappa_n - \hat{\kappa}_{n,h})\|_{H^{-1/2}(\Gamma)\times\mathbb{R}} \\ &+ c_{F'}\,\|\left[\boldsymbol{F}(\hat{t}_{n,h},\hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n,\kappa_n)\right]\|_{H^{1/2}(\Gamma)\times\mathbb{R}} \\ &+ c_{F'}\,\|\left[\boldsymbol{F}'(t_n,\kappa_n) - \boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\right]\begin{pmatrix} \hat{t}_{n,h} - \hat{t}_{n+1} \\ \hat{\kappa}_{n,h} - \hat{\kappa}_{n+1} \end{pmatrix}\|_{H^{1/2}(\Gamma)\times\mathbb{R}} \end{aligned}$$

where $c_{F'}$ is the boundedness constant of $[\boldsymbol{F}'(t_n, \kappa_n)]^{-1}$. For the second term we now obtain

$$\begin{aligned}
\| \left[\boldsymbol{F}(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n, \kappa_n)\right] \|^2_{H^{1/2}(\Gamma)\times\mathbb{R}} \\
&= \|F_1(\hat{t}_{h,h}, \hat{\kappa}_{n,h}) - F_1(t_n, \kappa_n)\|^2_{H^{1/2}(\Gamma)} + |F_2(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) - F_2(t_n, \kappa_n)|^2 \\
&= \|V_{\hat{\kappa}_{n,h}} \hat{t}_{n,h} - V_{\kappa_n} t_n\|^2_{H^{1/2}(\Gamma)} + |\langle V\hat{t}_{n,h}, \hat{t}_{n,h}\rangle_\Gamma - \langle Vt_n, t_n\rangle_\Gamma|^2.
\end{aligned}$$

By using a Taylor expansion with respect to κ_n we have

$$\begin{aligned}
(V_{\kappa_n} - V_{\hat{\kappa}_{n,h}})t_n(x) &= \frac{1}{4\pi}\int_\Gamma \frac{\cos(\kappa_n|x-y|) - \cos(\hat{\kappa}_{n,h}|x-y|)}{|x-y|} t_n(y) ds_y \\
&= (\hat{\kappa}_{n,h} - \kappa_n)\frac{1}{4\pi}\int_\Gamma \sin(\kappa_n^*|x-y|)\, t_n(y) ds_y \\
&= (\hat{\kappa}_{n,h} - \kappa_n)(A_{\kappa_n^*} t_n)(x)
\end{aligned}$$

and therefore we obtain, by using (2.7),

$$\begin{aligned}
\|V_{\hat{\kappa}_{n,h}} \hat{t}_{n,h} - V_{\kappa_n} t_n\|_{H^{1/2}(\Gamma)} &= \|V_{\hat{\kappa}_{n,h}}(\hat{t}_{n,h} - t_n) - (V_{\kappa_n} - V_{\hat{\kappa}_{n,h}})t_n\|_{H^{1/2}(\Gamma)} \\
&\le \|V_{\hat{\kappa}_{n,h}}(\hat{t}_{n,h} - t_n)\|_{H^{1/2}(\Gamma)} + |\hat{\kappa}_{n,h} - \kappa_n|\, \|A_{\kappa_n^*} t_n\|_{H^{1/2}(\Gamma)} \\
&\le c_{V_{\hat{\kappa}_{n,h}}} \|t_n - \hat{t}_{n,h}\|_{H^{-1/2}(\Gamma)} + c_{A_{\kappa_n^*}}\, |\hat{\kappa}_{n,h} - \kappa_n|\, \|t_n\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\langle V\hat{t}_{n,h}, \hat{t}_{n,h}\rangle_\Gamma - \langle Vt_n, t_n\rangle_\Gamma| &= |\langle V(\hat{t}_{n,h} - t_n), \hat{t}_{n,h}\rangle_\Gamma + \langle Vt_n, \hat{t}_{n,h} - t_n\rangle_\Gamma| \\
&\le c_2^V\, \|t_n - \hat{t}_{n,h}\|_{H^{-1/2}(\Gamma)} \left[\|\hat{t}_{n,h}\|_{H^{-1/2}(\Gamma)} + \|t_n\|_{H^{-1/2}(\Gamma)}\right].
\end{aligned}$$

Hence we conclude

$$\| \left[\boldsymbol{F}(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n, \kappa_n)\right] \|^2_{H^{1/2}(\Gamma)\times\mathbb{R}} \le c\left[\|t_n - \hat{t}_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat{\kappa}_{n,h}|^2\right]$$

where the constant c depends on (t_n, κ_n), and on ϱ.

Finally, for $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ we have

$$\begin{aligned}
\| \left[\boldsymbol{F}'(t_n, \kappa_n) - \boldsymbol{F}'(\hat{t}_{n,h}, \hat{\kappa}_{n,h})\right] \begin{pmatrix} t \\ \kappa \end{pmatrix} \|^2_{H^{1/2}(\Gamma)\times\mathbb{R}} \\
&= \|(V_{\kappa_n} - V_{\hat{\kappa}_{n,h}})t + \kappa(A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h} - A_{\kappa_n} t_n)\|^2_{H^{1/2}(\Gamma)} + 4|\langle V(t_n - \hat{t}_{n,h}), t\rangle_\Gamma|^2
\end{aligned}$$

where we can bound

$$\begin{aligned}
\|(V_{\kappa_n} - V_{\hat{\kappa}_{n,h}})t\|_{H^{1/2}(\Gamma)} &= |\hat{\kappa}_{n,h} - \kappa_n|\, \|A_{\kappa_n^*} t\|_{H^{1/2}(\Gamma)} \le c_{A_{\kappa_n^*}}\, |\hat{\kappa}_{n,h} - \kappa_n|\, \|t\|_{H^{-1/2}(\Gamma)}, \\
|\langle V(t_n - \hat{t}_{n,h}), t\rangle_\Gamma| &\le c_2^V\, \|t\|_{H^{-1/2}(\Gamma)} \|t_n - \hat{t}_{n,h}\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

For the remaining term we first consider

$$\begin{aligned}\|A_{\hat\kappa_{n,h}}\hat t_{n,h} - A_{\kappa_n} t_n\|_{H^{1/2}(\Gamma)} &\le \|A_{\hat\kappa_{n,h}}(\hat t_{n,h} - t_n)\|_{H^{1/2}(\Gamma)} + \|(A_{\hat\kappa_{n,h}} - A_{\kappa_n})t_n\|_{H^{1/2}(\Gamma)} \\ &\le c_{A_{\hat\kappa_{n,h}}} \|\hat t_{n,h} - t_n\|_{H^{-1/2}(\Gamma)} + \|(A_{\hat\kappa_{n,h}} - A_{\kappa_n})t_n\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

Again by using a Taylor expansion with respect to κ_n we have

$$\begin{aligned}(A_{\hat\kappa_{n,h}} - A_{\kappa_n})t_n(x) &= \frac{1}{4\pi}\int_\Gamma \left[\sin(\hat\kappa_{n,h}|x-y|) - \sin(\kappa_n|x-y|)\right] t_n(y)ds_y \\ &= (\hat\kappa_{n,h} - \kappa_n)\frac{1}{4\pi}\int_\Gamma |x-y|\cos(\kappa_n^*|x-y|)t_n(y)ds_y \\ &= (\hat\kappa_{n,h} - \kappa_n)(B_{\kappa_n^*}t_n)(x)\end{aligned}$$

where the operator $B_{\kappa_n^*} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded. This follows as for the Laplace single layer potential, we skip the details. Hence we conclude

$$\begin{aligned}&\| \left[\boldsymbol{F}'(t_n,\kappa_n) - \boldsymbol{F}'(\hat t_{n,h},\hat\kappa_{n,h})\right] \begin{pmatrix} \hat t_{n,h} - \hat t_{n+1} \\ \hat\kappa_{n,h} - \hat\kappa_{n+1}\end{pmatrix} \|^2_{H^{1/2}(\Gamma)\times\mathbb{R}} \\ &\qquad\qquad \le c\left[\|t_n - \hat t_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat\kappa_{n,h}|^2\right]\end{aligned}$$

where the constant c depends on (t_n,κ_n), and on ϱ. Therefore we can conclude

$$\|t_{n+1} - \hat t_{n+1}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat\kappa_{n+1}|^2 \le c\left[\|t_n - \hat t_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat\kappa_{n,h}|^2\right]$$

from which the assertion follows. ☐

Corollary 3.3. *Let (t_*,κ_*) be a solution of $\boldsymbol{F}(t,\kappa) = 0$ and let the assumptions* (A1) *and* (A2) *be satisfied. Let $(t_{0,h},\kappa_{0,h}) \in S_h^0(\Gamma)\times\mathbb{R}\cap U_\varrho(t_*,\kappa_*)$ be satisfied where ϱ is appropriately chosen as discussed in Corollary* 2.3. *Then, for a sufficient small mesh size $h < h_0$, the Galerkin variational problem* (3.4) *has a unique solution $(\hat t_{n+1,h},\hat\kappa_{n+1,h}) \in S_h^0(\Gamma)\times\mathbb{R}$ satisfying the error estimate*

$$\|t_{n+1} - \hat t_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat\kappa_{n+1,h}|^2 \le c\left[\varrho^4 + h^3\right] \tag{3.7}$$

when assuming $t^* \in H^1_{pw}(\Gamma)$. Note that the constant c depends on (t^*,κ^*), n, and on ϱ.

Proof. Let us define the error e_n as

$$e_n := \|t_n - \hat t_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat\kappa_{n,h}|^2.$$

Then, by using the error estimate (3.6) we conclude

$$\begin{aligned}e_{n+1} &\le c\left[e_n + \inf_{w_h\in S_h^0(\Gamma)}\|t_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)}\right] \\ &\le c\left[e_n + \|t_* - t_{n+1}\|^2_{H^{-1/2}(\Gamma)} + \inf_{w_h\in S_h^0(\Gamma)}\|t^* - w_h\|^2_{H^{-1/2}(\Gamma)}\right] \\ &\le c\left[e_n + c_1\varrho^4 + c_2h^3\|t_*\|^2_{H^1_{\rm pw}}\right] \\ &\le \widetilde{c}\left[e_n + \varrho^4 + h^3\right]\end{aligned}$$

when assuming $t^* \in H^1_{\rm pw}(\Gamma)$. Since for $n = 0$ we set $(\hat{t}_0, \hat{\kappa}_0) = (t_0, \kappa_0) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ we conclude

$$e_1 \,\leq\, c\, \left[\varrho^4 + h^3\right]$$

when assuming $(t_0, \kappa_0) \in S^0_h(\Gamma) \times \mathbb{R} \cap U_\varrho(t_*, \kappa_*)$. Now the assertion follows by induction. $\square$

When using the Aubin–Nitsche trick, see for example [16], it is possible to derive error estimates in Sobolev spaces with lower Sobolev index. In particular we obtain the error estimate

$$\|t_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1,h}|^2 \,\leq\, c\, \left[\varrho^4 + h^6\right] \tag{3.8}$$

when assuming $t^* \in H^1_{\rm pw}(\Gamma)$. Hence we can expect a cubic convergence rate for the eigenvalues,

$$|\kappa_{n+1} - \hat{\kappa}_{n+1,h}| \,\leq\, c\, \left[\varrho^4 + h^6\right]^{1/2} \,=\, \mathcal{O}(h^3). \tag{3.9}$$

4 Numerical results

In this section we present some numerical results to investigate the behavior of the nonlinear boundary element approach as presented in this paper. As a model problem we consider the interior Dirichlet eigenvalue problem (1.1) where the domain $\Omega = (0, \frac{1}{2})^3$ is a cube. Hence the eigenvalues are given by

$$\lambda_k \,=\, 4\pi^2 \left[k_1^2 + k_2^2 + k_3^2\right]$$

and the associated eigenfunctions are

$$u_k(x) \,=\, (\sin 2\pi k_1 x_1)(\sin 2\pi k_2 x_2)(\sin 2\pi k_3 x_3).$$

It turns out that the first eigenvalue ($k_1 = k_2 = k_3 = 1$)

$$\lambda_1 \,=\, 12\pi^2, \quad \kappa_1 \,=\, 2\sqrt{3}\pi$$

is simple, while the second eigenvalue ($k_1 = 2, k_2 = k_3 = 1$)

$$\lambda_2 \,=\, 24\pi^2, \quad \kappa_2 \,=\, 2\sqrt{6}\pi$$

is multiple.

For the boundary element discretization the boundary $\Gamma = \partial\Omega$ was decomposed into N uniform triangular boundary elements. The numerical results to approximate the simple eigenvalue $\kappa_1 = \sqrt{\lambda_1}$ are given in Table 1.
Note that the convergence rate of approximately 8 corresponds to the cubic convergence as predicted in (3.9). Next we consider the case of a multiple eigenvalue, the results to approximate $\kappa_2 = \sqrt{\lambda_2}$ are given in Table 2.
As in the multiple reciprocity method the problem of the so–called spurious eigenvalues occours close to multiple eigenvalues. In particular, several distinct discrete eigenvalues are

N	$\kappa_{1,N}$	$\|\kappa_1 - \kappa_{1,N}\|$	rate
384	10.8768	5.986e-03	-
1536	10.8821	6.962e-04	8.6
6144	10.8827	8.619e-05	8.1

Table 1: Approximation of $\kappa_1 = 2\sqrt{3}\pi \approx 10.8828$, simple eigenvalue.

N	$\kappa_{2_1,N}$	$\|\kappa_2 - \kappa_{2_1,N}\|$	rate	$\kappa_{2_2,N}$	$\|\kappa_2 - \kappa_{2_2,N}\|$	$\kappa_{2_3,N}$	$\|\kappa_2 - \kappa_{2_3,N}\|$
384	15.3774	1.680e-02	-	14.7057	0.68	15.8867	0.50
1536	15.3889	1.888e-03	8.9	14.6902	0.70	15.8579	0.47
6144	15.3904	2.280e-04	8.3	14.6839	0.71	15.8499	0.46

Table 2: Approximation of $\kappa = 2\sqrt{6}\pi \approx 15.3906$, multiple eigenvalue.

obtained to approximate a multiple eigenvalue. This phenomena also occours for algebraic eigenvalue problems when an approximation of the matrix is used, see e.g. [14].

The spurious eigenvalues can be filtered out with an a posteriori error control by using the complex valued fundamental solution (1.10) for an eigensolution (t, κ),

$$\frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}t(y)ds_y \;=\; (V_\kappa t)(x) + i\frac{1}{4\pi}\int_\Gamma \frac{\sin\kappa|x-y|}{|x-y|}t(y)ds_y \;=\; 0.$$

Then the residuum

$$\frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa_h|x-y|}}{|x-y|}t_h(y)ds_y$$

for actual approximations of eigensolutions (t_h, κ_h) is significant smaller as for spurious eigensolutions, and the residuum tends to zero for actual approximations of the eigensolutions if h gets smaller. Note that no spurious eigenvalues occur when an analogous algorithm is used which is based on the complex valued fundamental solution (1.10). But then we have to use complex arithmetics so that the computational complexity is twice expensive as for the real valued version.

5 Conclusions

In this paper we have presented and analyzed a boundary element method for the solution of the interior Dirichlet eigenvalue problem for the Laplace operator. Hereby, the linear eigenvalue problem for the partial differential operator is transformed into a nonlinear eigenvalue problem for an associated boundary integral operator which is solved via a Newton iteration. The discretization by using a Galerkin boundary element method gives a cubic convergence of the approximated eigenvalues. When using fast boundary element methods [12] an almost optimal computational complexity can be obtained. For this, also efficient preconditioned iterative solution methods to solve the Galerkin equations (3.4) are mandatory. As already mentioned in Remark 2.5 a further analysis in the case of multiple eigenvalues is needed.

Finally we mention that the proposed approach can be used to solve the interior Neumann eigenvalue problem for the Laplace operator, and to solve related eigenvalue problems in linear elastostatics.

References

[1] Z. Cai, J. Mandel, and S. McCormick. Multigrid methods for nearly singular linear equations and eigenvalue problems. *SIAM J. Numer. Anal.*, 34(1):178–200, 1997.

[2] D. W. Decker and C. T. Kelley. Newton's method at singular points. I. *SIAM J. Numer. Anal.*, 17(1):66–70, 1980.

[3] D. W. Decker and C. T. Kelley. Convergence acceleration for Newton's method at singular points. *SIAM J. Numer. Anal.*, 19(1):219–229, 1982.

[4] W. Hackbusch. *Theorie und Numerik elliptischer Differentialgleichungen.* Teubner Studienbücher Mathematik. [Teubner Mathematical Textbooks]. B. G. Teubner, Stuttgart, second edition, 1996.

[5] N. Kamiya and E. Andoh. Robust boundary element scheme for Helmholtz eigenvalue equation. In *Boundary elements, XIII (Tulsa, OK, 1991)*, pages 839–850. Comput. Mech., Southampton, 1991.

[6] N. Kamiya, E. Andoh, and K. Nogae. Eigenvalue analysis by the boundary element method: new developments. *Engng. Anal. Bound. Elms.*, 12:151–162, 1993.

[7] S. M. Kirkup and S. Amini. Solution of the Helmholtz eigenvalue problem via the boundary element method. *Internat. J. Numer. Methods Engrg.*, 36:321–330, 1993.

[8] M. Kitahara. *Boundary integral equation methods in eigenvalue problems of elastodynamics and thin plates*, volume 10 of *Studies in Applied Mechanics.* Elsevier Scientific Publishing Co., Amsterdam, 1985.

[9] A. V. Knyazev. Preconditioned eigensolvers—an oxymoron? *Electron. Trans. Numer. Anal.*, 7:104–123 (electronic), 1998. Large scale eigenvalue problems (Argonne, IL, 1997).

[10] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[11] V. Mehrmann and H. Voss. Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods. *GAMM Mitt. Ges. Angew. Math. Mech.*, 27(2):121–152 (2005), 2004.

[12] S. Rjasanow and O. Steinbach. *The fast solution of boundary integral equations.* Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2007.

[13] A. Ruhe. Algorithms for the nonlinear eigenvalue problem. *SIAM J. Numer. Anal.*, 10:674–689, 1973.

[14] Y. Saad. *Numerical methods for large eigenvalue problems.* Algorithms and Architectures for Advanced Scientific Computing. Manchester University Press, Manchester, 1992.

[15] S. A. Sauter and C. Schwab. *Randelementmethoden. Analyse, Numerik und Implementierung schneller Algorithmen.* B. G. Teubner, Stuttgart, Leipzig, Wiesbaden, 2004.

[16] O. Steinbach. *Numerical approximation methods for elliptic boundary value problems. Finite and boundary elements.* Springer, New York, 2007.

[17] J. H. Wilkinson. *The algebraic eigenvalue problem.* Clarendon Press, Oxford, 1965.

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