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Stable boundary element domain decomposition methods for the Helmholtz equation

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Abstract

In this paper we present a stable boundary element domain decomposition method to solve boundary value problems of the Helmholtz equation via a tearing and interconnecting approach. A possible non–uniqueness of the solution of local boundary value problems due to the appearance of local eigensolutions is resolved by using modified interface conditions of Robin type, which results in a Galerkin boundary element discretization which is robust for all local wave numbers. Numerical examples confirm the stability of the proposed approach.

1 Introduction

The time–harmonic modeling of acoustic waves in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ results in the complex valued boundary value problem

$$\Delta u(x) + [\kappa(x)]^2 u(x) = 0 \quad \text{for } x \in \Omega, \quad \frac{\partial}{\partial n_x} u(x) = g(x) \quad \text{for } \Gamma = \partial\Omega, \tag{1.1}$$

where $\kappa \in \mathbb{R}_+$ is the wave number, and n_x is the exterior normal vector which is defined for almost all $x \in \Gamma$. For simplicity we only consider Neumann boundary conditions in this paper, but Dirichlet or boundary conditions of mixed type can be treated in a similar way. We assume that the boundary value problem (1.1) admits a unique solution, i.e., κ^2 is not an eigenvalue of the associated Neumann eigenvalue problem of the Laplace operator.
The aim of this paper is to formulate and to analyze boundary element domain decomposition methods for an efficient and parallel solution of the boundary value problem (1.1). For this we consider a non–overlapping domain decomposition

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j, \quad \Gamma_i = \partial\Omega_i, \tag{1.2}$$

where the subdomains Ω_i are assumed to be Lipschitz and simply connected. The local interface Γ_{ij} between two neighbored subdomains Ω_i and Ω_j, and the global interface Γ_I are given by

$$\Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j, \quad \Gamma_I = \bigcup_{i,j} \Gamma_{ij}.$$

In addition, $\Gamma_S := \Gamma_I \cup \Gamma$ defines the skeleton of the domain decomposition (1.2). Note that in the case of composite materials, see, e.g., [1, 21], we may also consider piecewise constant wave numbers, i.e.

$$\kappa(x) \;=\; \kappa_i \quad \text{for } x \in \Omega_i, \quad i = 1, \ldots, p. \tag{1.3}$$

Moreover, we may also include an exterior subdomain $\Omega_0 := \mathbb{R}^3 \backslash \overline{\Omega}$ when modeling also the surrounding area. Instead of the global boundary value problem (1.1) we now consider local boundary value problems

$$\Delta u_i(x) + \kappa_i^2 u_i(x) \;=\; 0 \quad \text{for } x \in \Omega_i, \quad \frac{\partial}{\partial n_i} u_i(x) = g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma, \tag{1.4}$$

together with the transmission or interface conditions

$$u_i(x) = u_j(x), \quad \frac{\partial}{\partial n_i} u_i(x) + \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}\,. \tag{1.5}$$

Standard domain decomposition methods, see, e.g., [16, 18, 20], are based on the use of local Dirichlet to Neumann maps

$$t_i(x) := \frac{\partial}{\partial n_i} u_i(x) = (S_{\kappa_i} \widetilde{u}_i)(x) \quad \text{for } x \in \Gamma_i, \tag{1.6}$$

where u_i is the solution of the local Dirichlet boundary value problem

$$\Delta u_i(x) + \kappa_i^2 u_i(x) = 0 \quad \text{for } x \in \Omega_i, \quad u_i(x) = \widetilde{u}_i(x) \quad \text{for } x \in \Gamma_i. \tag{1.7}$$

While in the case of the Laplace equation the solution of a local Dirichlet boundary value problem and therefore the Dirichlet to Neumann map (1.6) is well defined, this is not always the case for the local Helmholtz equation (1.7). In particular, when $\lambda = \kappa_i^2$ is an eigenvalue of the Dirichlet eigenvalue problem

$$-\Delta u_i(x) = \lambda u_i(x) \quad \text{for } x \in \Omega_i, \quad u_i(x) = 0 \quad \text{for } x \in \Gamma_i, \tag{1.8}$$

the local Dirichlet to Neumann map (1.6), i.e. the local Steklev–Poincaré operator S_{κ_i} is not well defined. Note that the problem of non–uniqueness of local Dirichlet boundary value problems can be avoided just by using sufficiently small subdomains Ω_i to ensure $\kappa_i^2 < \lambda_{\min}(\Omega_i)$. But such an approach is not very practicable in applications.

The boundary element discretization of the local Dirichlet to Neumann maps (1.6) leads to a system of linear algebraic equations to be solved in parallel. For an efficient iterative solution we will apply tearing and interconnecting methods [6, 11] which require the solution of local Neumann boundary value problems

$$\Delta u_i(x) + \kappa_i^2 u_i(x) \;=\; 0 \quad \text{for } x \in \Omega_i, \quad \frac{\partial}{\partial n_i} u_i(x) \;=\; \widetilde{t}_i(x) \quad \text{for } x \in \Gamma_i\,. \tag{1.9}$$

Note that the local Neumann boundary value problem (1.9) is not uniquely solvable when $\mu = \kappa_i^2$ is an eigenvalue of the Neumann eigenvalue problem

$$-\Delta u_i(x) \;=\; \mu u_i(x) \quad \text{for } x \in \Omega_i, \quad \frac{\partial}{\partial n_i} u_i(x) \;=\; 0 \quad \text{for } x \in \Gamma_i\,. \tag{1.10}$$

Hence, instead of the Neumann transmission condition in (1.5) we will consider interface conditions of Robin type which allow a unique solution of the related subproblems [5]. Moreover, the use of the symmetric boundary integral formulation to include Robin type interface conditions will also resolve the non–unique definition of the local Dirichlet to Neumann maps in the case when the local wave number κ_i^2 is an eigenvalue of the Dirichlet eigenvalue problem (1.8).

The finite element tearing and interconnecting (FETI) method was introduced in [6] as a dual version of classical iterative substructuring methods. FETI methods are well established as powerful and robust parallel solvers for large–scale finite element equations in different fields of applications, see, e.g., [7, 15]. For a rigorous theoretical analysis, see, for example, [2, 9, 13, 20]. In particular, for second–order self–adjoint elliptic problems such as the potential equation or the linear elasticity problem, the FETI approach is based on an equivalent minimization problem with constraints, which is reformulated by using discrete Lagrange multipliers to enforce the continuity of the primal unknowns. The resulting FETI algorithm then requires the solution of local Dirichlet and Neumann boundary value problems which can be done in parallel. For floating subdomains, appropriate subspace methods can be used for the solution of the local Neumann problem, when the kernel, e.g. the rigid body motions in elasticity, is known. Recently, the boundary element tearing and interconnecting (BETI) methods [11] have been introduced, for an inexact data–sparse BETI algorithm, see [10]. In particular in electromagnetic and acoustic scattering problems a boundary element approach has some advantages over a finite element approach in the treatment of unbounded regions, or when considering composites with piecewise constant material parameters.

In this paper we aim to formulate and to analyze stable boundary element domain decomposition methods for an efficient and parallel solution of an interior boundary value problem of the Helmholtz equation where the related bilinear form is not elliptic anymore. Although we cannot consider an equivalent minimization problem to derive the dual formulation, the latter can be deduced by using algebraic arguments only. But due to the possible appearance of local eigensolutions of related Dirichlet or Neumann eigenvalue problems of the Laplace operator, the local Dirichlet or Neumann boundary value problems may not

be solvable, i.e., the local Steklov–Poincaré operator may not be well defined, or it is not invertible. In [5], a FETI approach was considered to solve Helmholtz problems where the Neumann transmission conditions were replaced by interface conditions of Robin type to ensure a unique elimination of the primal unknowns. Indeed, by using modified Robin type interface conditions it is possible to derive a boundary element domain decomposition method which is stable for all local wave numbers.
This paper is organized as follows: In Sect. 2 we discuss the solution of local Dirichlet and Robin type boundary value problems by using boundary integral equations. This leads to a boundary integral representation of local Steklov–Poincaré operators which are valid for all wave numbers. In Sect. 3 we formulate related domain decomposition methods and analyze their stable discretization by using Galerkin boundary element methods. Numerical results are given in Sect. 4 to demonstrate the stability of the proposed approach.

2 Dirichlet and Robin type boundary value problems

Within this section we consider boundary value problems of the Helmholtz equation with respect to some bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\Gamma = \partial\Omega$. Later we will apply these results to the subdomains Ω_i as introduced in the domain decomposition (1.2).

2.1 Boundary integral operators

Any solution of the Helmholtz equation

$$\Delta u(x) + \kappa^2 u(x) \; = \; 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^3$$

can be described by using the representation formula, see, e.g., [8, 12, 17, 19],

$$u(x) = \int\limits_\Gamma U^*_\kappa(x,y)t(y)ds_y - \int\limits_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y)u(y)ds_y \quad \text{for } x \in \Omega \tag{2.1}$$

where

$$U^*_\kappa(x,y) \; = \; \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}, \quad t(y) := \frac{\partial}{\partial n_y}u(y), \; y \in \Gamma$$

are the fundamental solution of the Helmholtz equation and the associated normal derivative of the solution u, respectively. By taking the Dirichlet and Neumann traces of the representation formula (2.1) we obtain a system of boundary integral equations which can be written by means of the Calderon projector $\mathcal{C}$ on Γ as

$$\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_\kappa & V_\kappa \\ D_\kappa & \frac{1}{2}I + K'_\kappa \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} =: \mathcal{C} \begin{pmatrix} u \\ t \end{pmatrix} \tag{2.2}$$

where

$$(V_\kappa t)(x) = \int\limits_\Gamma U^*_\kappa(x,y)t(y)ds_y, \quad (K_\kappa u)(x) = \int\limits_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y)u(y)ds_y$$

are the single and double layer potentials, and

$$(K'_\kappa t)(x) = \int\limits_\Gamma \frac{\partial}{\partial n_x} U^*_\kappa(x,y) t(y) ds_y, \quad (D_\kappa u)(x) = -\frac{\partial}{\partial n_x} \int\limits_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y) u(y) ds_y$$

are the adjoint double layer potential and the hypersingular boundary integral operator, respectively. The mapping properties of all boundary integral operators as introduced above are well known, see, e.g., [4, 8, 12, 14, 17, 19]. In particular, from the projection property $\mathcal{C}^2 = \mathcal{C}$ of the Calderon projector $\mathcal{C}$ as defined in (2.2) we conclude as in the case of the Laplace operator [19] the following relations:

$$K_\kappa V_\kappa = V_\kappa K'_\kappa, \quad K'_\kappa D_\kappa = D_\kappa V_\kappa, \quad D_\kappa V_\kappa = (\frac{1}{2}I - K_\kappa)(\frac{1}{2}I + K_\kappa)\,. \tag{2.3}$$

Moreover, by using the duality pairing

$$\langle v, \tau \rangle_\Gamma \;=\; \int\limits_\Gamma v(x)\overline{\tau(x)} ds_x$$

for all $v \in H^{1/2}(\Gamma)$ and $\tau \in H^{-1/2}(\Gamma)$ we obtain

$$\langle V_\kappa w, \tau \rangle_\Gamma = \langle w, V_{-\kappa}\tau \rangle_\Gamma, \quad \langle K_\kappa v, \tau \rangle_\Gamma = \langle v, K'_{-\kappa}\tau \rangle_\Gamma, \quad \langle D_\kappa u, v \rangle_\Gamma = \langle u, D_{-\kappa} v \rangle_\Gamma\,.$$

2.2 Dirichlet boundary value problems

The Dirichlet to Neumann map is defined via the solution of the Dirichlet boundary value problem

$$\Delta u(x) + \kappa^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega, \quad u(x) \;=\; g(x) \quad \text{for } x \in \Gamma. \tag{2.4}$$

Lemma 2.1. *If κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem* (1.8)*, then for any $g \in H^{1/2}(\Gamma)$ there exists a unique solution $u \in H^1(\Omega)$ of the Dirichlet boundary value problem* (2.4)*, i.e. $u = \mathcal{S}g$, where the solution operator $\mathcal{S} : H^{1/2}(\Gamma) \to H^1(\Omega)$ is well defined and bounded. If $\lambda = \kappa^2$ is an eigenvalue of the Dirichlet eigenvalue problem* (1.8) *with an associated eigenfunction u_λ, then the Dirichlet datum g has to satisfy the solvability condition*

$$\langle g, t_\lambda \rangle_\Gamma = 0, \quad t_\lambda(x) = \frac{\partial}{\partial n_x} u_\lambda(x) \quad \textit{for } x \in \Gamma. \tag{2.5}$$

Moreover, the solution of the Dirichlet boundary value problem (2.4) *is only unique up to the eigensolution u_λ,*

$$u \;=\; \widetilde{u} + \alpha\, u_\lambda, \quad \alpha \in \mathbb{R}, \quad \langle \widetilde{u}, u_\lambda \rangle_\Omega = 0.$$

Proof. If κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.8), the assertion follows as in the case of the Laplace equation, see, e.g, [12, 18].

It remains to consider the case when $\lambda = \kappa^2$ is an eigenvalue of the Dirichlet eigenvalue problem (1.8). From Green's first formula

$$\int\limits_\Omega \Big[\nabla u(x)\cdot\nabla v(x) - \kappa^2 u(x)v(x)\Big]dx = \int\limits_\Omega \Big[-\Delta u(x) - \kappa^2 u(x)\Big]v(x)dx + \int\limits_\Gamma \frac{\partial}{\partial n_x}u(x)v(x)ds_x$$

we first obtain Green's second formula

$$\int\limits_\Omega \Big[-\Delta u(x) - \kappa^2 u(x)\Big]v(x)dx + \int\limits_\Gamma \frac{\partial}{\partial n_x}u(x)v(x)ds_x$$
$$= \int\limits_\Omega \Big[-\Delta v(x) - \kappa^2 v(x)\Big]u(x)dx + \int\limits_\Gamma \frac{\partial}{\partial n_x}v(x)u(x)ds_x\,.$$

For the solution u of the Dirichlet problem (2.4) and for $v = \overline{u}_\lambda$ we then conclude the solvability condition

$$\int\limits_\Gamma \frac{\partial}{\partial n_x}\overline{u_\lambda(x)}g(x)ds_x \;=\; \overline{\langle g, t_\lambda\rangle_\Gamma} \;=\; 0.$$

If we introduce the Sobolev space

$$H^1_\lambda(\Omega) \;:=\; \Big\{v\in H^1(\Omega) : \langle v, u_\lambda\rangle_\Omega = 0\Big\}$$

we can find $\widetilde{u} \in H^1_\lambda(\Omega)$ with $\widetilde{u} = g$ on Γ as the unique solution of the variational problem

$$\int\limits_\Omega \Big[\nabla\widetilde{u}(x)\cdot\nabla v(x) - \kappa^2\widetilde{u}(x)v(x)\Big]dx \;=\; 0 \quad \text{for all } v\in H^1_\lambda(\Omega)$$

and the general solution of (2.4) is given by $u = \widetilde{u} + \alpha\, u_\lambda$, $\alpha \in \mathbb{R}$. $\square$

Note that the solution operator $\mathcal{S} : H^{1/2}(\Gamma) \to H^1(\Omega)$ of the Dirichlet boundary value problem (2.4) admits an adjoint operator $\mathcal{S}^* : \widetilde{H}^{-1}(\Omega) \to H^{-1/2}(\Gamma)$ which is defined by

$$\langle g, \mathcal{S}^* f\rangle_\Gamma \;:=\; \langle \mathcal{S}g, f\rangle_\Omega \quad \text{for all } f\in\widetilde{H}^{-1}(\Omega), g\in H^{1/2}(\Gamma). \tag{2.6}$$

To solve the Dirichlet boundary value problem (2.4) we may use the first boundary integral equation in (2.2),

$$(V_\kappa t)(x) \;=\; (\frac{1}{2}I + K_\kappa)g(x) \quad \text{for } x\in\Gamma. \tag{2.7}$$

Since $V_\kappa - V_0$ is compact, see, e.g., [8, 17], we conclude that $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible if κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.8). If $\lambda = \kappa^2$ is an eigenvalue of (1.8), we conclude from (2.2)

$$(V_{\pm\kappa}t_\lambda)(x) \;=\; (\frac{1}{2}I - K'_{\pm\kappa})t_\lambda(x) \;=\; 0 \quad \text{for } x\in\Gamma, \tag{2.8}$$

where $t_\lambda = \dfrac{\partial}{\partial n}u_\lambda$ is the normal derivative of the assocated eigenfunction u_λ.

Lemma 2.2. *If κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem* (1.8)*, then for any $g \in H^{1/2}(\Gamma)$ there exists a unique solution $t \in H^{-1/2}(\Gamma)$ of the boundary integral equation* (2.7)*, i.e. $t = S_\kappa g$, where the Steklov–Poincaré operator $S_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is well defined and bounded. If $\lambda = \kappa^2$ is an eigenvalue of the Dirichlet eigenvalue problem* (1.8)*, the solvability condition* (2.5) *ensures solvability of the boundary integral equation* (2.7)*, but the solution is only unique up to eigensolutions t_λ,*

$$t \;=\; \widetilde{t} + \widetilde{\alpha} t_\lambda, \quad \widetilde{\alpha} \in \mathbb{R}, \quad \langle \widetilde{t}, R^{-1} t_\lambda \rangle_\Gamma = 0,$$

where $R : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is some self–adjoint and $H^{1/2}(\Gamma)$–elliptic operator.

Proof. If κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.8), the single layer potential V_κ is invertible.
If $\lambda = \kappa^2$ is an eigenvalue of the Dirichlet eigenvalue problem (1.8), the single layer potential V_κ is not injective and therefore the boundary integral equation (2.7) is not solvable in general. However, the solvability condition (2.5) implies

$$\langle (\frac{1}{2} I + K_\kappa) g, t_\lambda \rangle_\Gamma \;=\; \langle g, t_\lambda \rangle_\Gamma - \langle g, (\frac{1}{2} I - K'_{-\kappa}) t_\lambda \rangle_\Gamma \;=\; 0,$$

i.e., $(\frac{1}{2} I + K_\kappa) g \in \operatorname{Im} V_\kappa$ and the boundary integral equation (2.7) is solvable, but the solution is only unique up to eigensolutions t_λ. $\square$

2.3 Steklov–Poincaré operators

Within this section we only consider the case when κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.8). In this case, the Steklov–Poincaré operator S_κ is well defined, and a boundary integral representation is given by the solution of the boundary integral equation (2.7),

$$t(x) \;=\; V_\kappa^{-1} (\frac{1}{2} I + K_\kappa) g(x) \quad \text{for } x \in \Gamma.$$

When inserting this expression into the second equation of (2.2) this gives a second representation

$$t(x) \;=\; \Big[D_\kappa + (\frac{1}{2} I + K'_\kappa) V_\kappa^{-1} (\frac{1}{2} I + K_\kappa) \Big] g(x) \;=:\; (S_\kappa g)(x) \quad \text{for } x \in \Gamma. \tag{2.9}$$

To investigate the invertibility of S_κ we will derive a Gårding inequality and we will consider the injectivity of S_κ.

Lemma 2.3. *Let κ^2 be not an eigenvalue of the Dirichlet eigenvalue problem* (1.8)*. The Steklov–Poincaré operator S_κ as given in* (2.9) *is then well defined, and satisfies a Gårding inequality,*

$$\langle S_\kappa v, v \rangle_\Gamma + c(v,v) \;\ge\; c_1^S \, \|v\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } v \in H^{1/2}(\Gamma), \tag{2.10}$$

where $c(v,v)$ is a compact perturbation.

Proof. For an arbitrary but fixed $v \in H^{1/2}(\Gamma)$ the related Neumann datum $\tau = S_\kappa v$ is well defined. Hence we can introduce

$$\phi(x) \;=\; \int_\Gamma U_\kappa^*(x,y)\tau(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x,y)v(y)ds_y \quad \text{for } x \in \Omega$$

satisfying

$$\Delta\phi(x) + \kappa^2\phi(x) = 0 \quad \text{for } x \in \Omega, \quad \phi(x) = v(x), \quad \frac{\partial}{\partial n_x}\phi(x) = \tau(x) \quad \text{for } x \in \Gamma.$$

Hence we obtain by Green's first formula

$$\begin{aligned} \langle S_\kappa v, v\rangle_\Gamma \;&=\; \int_\Gamma \frac{\partial}{\partial n_x}\phi(x)\overline{\phi(x)}ds_x \;=\; \int_\Omega |\nabla\phi(x)|^2 dx - \kappa^2 \int_\Gamma |\phi(x)|^2 dx \\ &=\; \|\phi\|^2_{H^1(\Omega)} - (\kappa^2+1)\|\phi\|^2_{L_2(\Omega)}\,. \end{aligned}$$

By using the adjoint solution operator as defined in (2.6) we further conclude

$$\|\phi\|^2_{L_2(\Omega)} \;=\; \|\mathcal{S}v\|^2_{L_2(\Omega)} \;=\; \langle \mathcal{S}v, \mathcal{S}v\rangle_\Omega \;=\; \langle v, \mathcal{S}^*\mathcal{S}v\rangle_\Gamma$$

where $\mathcal{S}^*\mathcal{S} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, since the imbedding $\mathcal{S}g \in H^1(\Omega) \subset \widetilde{H}^{-1}(\Omega)$ is compact. Hence we can use the compact bilinear form

$$c(v,v) \;:=\; (\kappa^2+1)\,\langle v, \mathcal{S}^*\mathcal{S}v\rangle_\Gamma$$

to obtain

$$\langle S_\kappa v, v\rangle_\Gamma + c(v,v) \;=\; \|\phi\|^2_{H^1(\Omega)} \;\geq\; c_1^S\,\|v\|^2_{H^{1/2}(\Gamma)}$$

by the trace theorem. □

Since the Steklov–Poincaré operator $S_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ realizes the Dirichlet to Neumann map it is obvious that S_κ is not injective if $\mu = \kappa^2$ is an eigenvalue of the Neumann eigenvalue problem (1.10).

In the case that κ^2 is neither an eigenvalue of the Dirichlet eigenvalue problem (1.8) nor of the Neumann eigenvalue problem (1.10), the Steklov–Poincaré operator S_κ as given in (2.9) is well defined, injective, and invertible. But to end up with a boundary integral domain decomposition formulation which is stable independent of the wave number κ, instead of a Dirichlet to Neumann map we will use a Dirichlet to Robin map as discussed in the next section.

2.4 Robin type boundary value problems

Instead of the Dirichlet boundary value problem (2.4) we now consider a Helmholtz equation with Robin type boundary conditions,

$$\Delta u(x) + \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega, \quad \frac{\partial}{\partial n_x}u(x) + i\eta(Ru)(x) \;=\; g(x) \quad \text{for } x \in \Gamma, \qquad (2.11)$$

where the regularization operator $R : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is assumed to be self–adjoint and $H^{1/2}$–elliptic, and $\eta \in \mathbb{R}\backslash\{0\}$.

Lemma 2.4. *For any $\eta \in \mathbb{R}\backslash\{0\}$ there exists a unique solution $u \in H^1(\Omega)$ of the Robin type boundary value problem* (2.11)*.*

Proof. The weak formulation of the boundary value problem (2.11) is to find $u \in H^1(\Omega)$ such that

$$\int\limits_\Omega \nabla u(x) \cdot \nabla v(x) dx - \kappa^2 \int\limits_\Omega u(x) v(x) dx + i\eta \int\limits_\Gamma (Ru)(x) v(x) ds_x \;=\; \int\limits_\Gamma g(x) v(x) ds_x$$

is satisfied for all $v \in H^1(\Omega)$. Since the associated bilinear form satisfies a Gårding inequality, i.e. for $v \in H^1(\Omega)$

$$\mathrm{Re}\left\{ \int\limits_\Omega \left[\nabla v(x) \cdot \nabla \overline{v(x)} - \kappa^2 v(x) \overline{v(x)} \right] dx + i\eta \int\limits_\Gamma (Rv)(x) \overline{v(x)} ds_x \right\}$$

$$= \|v\|^2_{H^1(\Omega)} - (\kappa^2 + 1) \|v\|^2_{L_2(\Omega)},$$

it is sufficient to prove injectivity. Let $u \in H^1(\Omega)$ be any solution of the homogeneous variational problem

$$\int\limits_\Omega \nabla u(x) \cdot \nabla v(x) dx - \kappa^2 \int\limits_\Omega u(x) v(x) dx + i\eta \int\limits_\Gamma (Ru)(x) v(x) ds_x \;=\; 0 \tag{2.12}$$

for all $v \in H^1(\Omega)$. By choosing $v = \overline{u}$ this gives

$$\int\limits_\Omega |\nabla u(x)|^2 dx - \kappa^2 \int\limits_\Omega |u(x)|^2 dx + i\eta \int\limits_\Gamma (Ru)(x) \overline{u(x)} ds_x \;=\; 0$$

and therefore, when considering the imaginary part,

$$\int\limits_\Gamma (Ru)(x) \overline{u(x)} ds_x \;=\; 0\,.$$

Since R is self–adjoint and $H^{1/2}(\Gamma)$–elliptic, $u(x) = 0$ for $x \in \Gamma$ follows. Then, since $u \in H^1(\Omega)$ is a solution of the variational problem (2.12), we have

$$\int\limits_\Omega \nabla u(x) \cdot \nabla v(x) dx - \kappa^2 \int\limits_\Omega u(x) v(x) dx \;=\; 0 \quad \text{for all } v \in H^1(\Omega)\,.$$

From this we conclude, by applying integration by parts,

$$\Delta u(x) + \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega, \quad \frac{\partial}{\partial n_x} u(x) = 0 \quad \text{for } x \in \Gamma.$$

Hence, u is a solution of the Helmholtz equation with vanishing Cauchy data. Then, by applying the representation formula (2.1), $u(x) = 0$ for $x \in \Omega$ follows. $\square$

To solve the Robin type boundary value problem (2.11) we aim to derive a boundary integral formulation which is stable for all wave numbers κ. By using the second equation in (2.2) we can rewrite the Robin boundary condition in (2.11) as

$$(D_\kappa u)(x) + i\eta(Ru)(x) + (\frac{1}{2}I + K'_\kappa)t(x) \;=\; g(x) \quad \text{for } x \in \Gamma, \tag{2.13}$$

and where we use in addition the first equation in (2.2),

$$(V_\kappa t)(x) - (\frac{1}{2}I + K_\kappa)u(x) \;=\; 0 \quad \text{for } x \in \Gamma. \tag{2.14}$$

When κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.8) we obtain from (2.14)

$$t(x) \;=\; V_\kappa^{-1}(\frac{1}{2}I + K_\kappa)u(x) \quad \text{for } x \in \Gamma$$

to be inserted into (2.13),

$$\Big[D_\kappa + (\frac{1}{2}I + K'_\kappa)V_\kappa^{-1}(\frac{1}{2}I + K_\kappa)\Big]u(x) + i\eta(Ru)(x) \;=\; g(x) \quad \text{for } x \in \Gamma. \tag{2.15}$$

Since the Steklov–Poincaré operator $S_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is coercive, see Lemma 2.3, and since the injectivity of the operator $S_\kappa + i\eta R$ follows as in the proof of Lemma 2.4, unique solvability of the boundary integral equation (2.15) follows. In particular, the case when $\mu = \kappa^2$ is an eigenvalue of the Neumann eigenvalue problem (1.10), is covered. Obviously, then also the system of boundary integral equations (2.13) and (2.14) admits a unique solution. It remains to prove that the latter holds true also in the case when $\lambda = \kappa^2$ is an eigenvalue of the Dirichlet eigenvalue problem (1.8).

Lemma 2.5. *Let $\lambda = \kappa^2$ be an eigenvalue of the Dirichlet eigenvalue problem* (1.8)*. Then the system* (2.13) *and* (2.14) *of boundary integral equations related to the Robin type boundary value problem* (2.11) *admits a unique solution $(u, t) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.*

Proof. Associated to the system of boundary integral equations (2.13) and (2.14) is the bilinear form

$$a(u,t;v,\tau) = \langle V_\kappa t, \tau\rangle_\Gamma - \langle(\frac{1}{2}I + K_\kappa)u, \tau\rangle_\Gamma + \langle(\frac{1}{2}I + K'_\kappa)t, v\rangle_\Gamma + \langle D_\kappa u, v\rangle_\Gamma + i\eta\langle Ru, v\rangle_\Gamma\,.$$

Since the boundary integral operators $V_\kappa - V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $K_\kappa - K_0 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $D_\kappa - D_0 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, and $K'_\kappa - K'_0 : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are compact, see, e.g. [8, 17], the bilinear form $a(\cdot,\cdot;\cdot,\cdot)$ satisfies a Gårding inequality

$$\mathrm{Re}\Big\{a(v,\tau;v,\tau) + c(v,\tau;v,\tau)\Big\} \;\geq\; c_1^A\left[\|v\|^2_{H^{1/2}(\Gamma)} + \|\tau\|^2_{H^{-1/2}(\Gamma)}\right]$$

for all $(v,\tau)\in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ where the bilinear form $c(\cdot,\cdot;\cdot,\cdot)$ is compact. Hence it remains to prove injectivity.
Let (u,t) be a solution of the homogeneous boundary integral equations

$$V_\kappa t-(\frac{1}{2}I+K_\kappa)u=0,\quad (\frac{1}{2}I+K'_\kappa)t+D_\kappa u+i\eta Ru=0\,. \tag{2.16}$$

Since $\lambda=\kappa^2$ is an eigenvalue of the Dirichlet eigenvalue problem (1.8), i.e.

$$V_\kappa t_\lambda=V_{-\kappa}t_\lambda=(\frac{1}{2}I-K'_\kappa)t_\lambda=(\frac{1}{2}I-K'_{-\kappa})t_\lambda=0,$$

we obtain with

$$0=\langle t,V_{-\kappa}t_\lambda\rangle_\Gamma=\langle V_\kappa t,t_\lambda\rangle_\Gamma=\langle(\frac{1}{2}I+K_\kappa)u,t_\lambda\rangle_\Gamma=\langle u,t_\lambda\rangle_\Gamma-\langle u,(\frac{1}{2}I-K'_{-\kappa})t_\lambda\rangle_\Gamma=\langle u,t_\lambda\rangle_\Gamma$$

the solvability condition

$$\langle u,t_\lambda\rangle_\Gamma\;=\;0\,.$$

The general solution of the first boundary integral equation is then given by

$$t\;=\;\widetilde{t}+\widetilde{\alpha}t_\lambda,\quad \widetilde{\alpha}\in\mathbb{R},\quad \langle\widetilde{t},R^{-1}t_\lambda\rangle_\Gamma=0.$$

Applying V_κ to the second equation in (2.16) this gives, by using (2.3),

$$\begin{aligned}
0 &= V_\kappa\Big[(\frac{1}{2}I+K'_\kappa)t+D_\kappa u+i\eta Ru\Big]\\
&= (\frac{1}{2}I+K_\kappa)V_\kappa t+(\frac{1}{2}I+K_\kappa)(\frac{1}{2}I-K_\kappa)u+i\eta V_\kappa Ru\\
&= (\frac{1}{2}I+K_\kappa)\Big[V_\kappa t+(\frac{1}{2}I-K_\kappa)u\Big]+i\eta V_\kappa Ru\\
&= (\frac{1}{2}I+K_\kappa)u+i\eta V_\kappa Ru\\
&= V_\kappa\Big[t+i\eta Ru\Big],
\end{aligned}$$

i.e.

$$t+i\eta Ru\;=\;\alpha t_\lambda\quad\text{for some }\alpha\in\mathbb{R}.$$

Moreover, when applying D_κ to the first equation in (2.16) we obtain, by inserting the second equation in (2.16),

$$\begin{aligned}
0 &= D_\kappa\Big[V_\kappa t-(\frac{1}{2}I+K_\kappa)u\Big]=(\frac{1}{2}I+K'_\kappa)(\frac{1}{2}I-K'_\kappa)t-(\frac{1}{2}I+K'_\kappa)D_\kappa u\\
&= (\frac{1}{2}I+K'_\kappa)\Big[(\frac{1}{2}I-K'_\kappa)t-D_\kappa u\Big]=(\frac{1}{2}I+K'_\kappa)\Big[t+i\eta Ru\Big]\\
&= \alpha(\frac{1}{2}I+K'_\kappa)t_\lambda=\alpha t_\lambda-\alpha(\frac{1}{2}I-K'_\kappa)t_\lambda=\alpha t_\lambda
\end{aligned}$$

and therefore $\alpha = 0$. Hence we conclude

$$t + i\eta R u \;=\; 0$$

as well as

$$\widetilde{t} + \widetilde{\alpha} t_\lambda + i\eta R u = 0\,.$$

But then we conclude

$$0 = \langle \widetilde{t} + \widetilde{\alpha} t_\lambda + i\eta R u, R^{-1} t_\lambda \rangle_\Gamma = \langle \widetilde{t}, R^{-1} t_\lambda \rangle_\Gamma + \widetilde{\alpha} \| t_\lambda \|^2_{R^{-1}} + i\eta \langle u, t_\lambda \rangle_\Gamma,$$

which implies $\widetilde{\alpha} = 0$, i.e. $t = \widetilde{t}$.
Next we define

$$\phi(x) \;=\; \int\limits_\Gamma U^*_\kappa(x,y)\widetilde{t}(y) ds_y - \int\limits_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y) u(y) ds_y \quad \text{for } x \in \Omega$$

satisfying

$$\Delta \phi(x) + \kappa^2 \phi(x) \;=\; 0 \quad \text{for } x \in \Omega$$

and, due to the first equation in (2.16),

$$\phi(x) \;=\; (V_\kappa \widetilde{t})(x) + \frac{1}{2} u(x) - (K_\kappa u)(x) \;=\; u(x) \quad \text{for } x \in \Gamma.$$

Moreover, we have

$$\frac{\partial}{\partial n_x} \phi(x) \;=\; (\frac{1}{2} I + K'_\kappa)\widetilde{t}(x) + (D_\kappa u)(x) \quad \text{for } x \in \Gamma.$$

From the second equation in (2.16) we therefore conclude, by using Green's first formula,

$$\begin{aligned} 0 \;&=\; \langle (\frac{1}{2} I + K'_\kappa)\widetilde{t} + D_\kappa u + i\eta R u, u \rangle_\Gamma \;=\; \int\limits_\Gamma \frac{\partial}{\partial n_x} \phi(x) \overline{\phi(x)} ds_x + i\eta \langle R u, u \rangle_\Gamma \\ &=\; \int\limits_\Omega \Big[|\nabla \phi(x)|^2 - \kappa^2 |\phi(x)|^2 \Big] dx + i\eta \langle R u, u \rangle_\Gamma\,. \end{aligned}$$

When considering the imaginary part we obtain for $\eta \neq 0$

$$\langle R u, u \rangle_\Gamma \;=\; 0,$$

and since R is self–adjoint and $H^{1/2}(\Gamma)$–elliptic, $u = 0$ follows. Finally, $\widetilde{t} = -i\eta R u = 0$. This proves, that the homogeneous system (2.16) only admits the trivial solution. $\square$

Remark 2.6. *Note that we only assume $\eta \neq 0$ in the Robin type boundary condition* (2.11)*. Moreover it is sufficient to consider Robin type boundary conditions only on a part $\Gamma_R \subset \Gamma$ of the boundary $\Gamma = \partial\Omega$ to ensure unique solvability of the related boundary value problem with boundary conditions of mixed type (Dirichlet, Neumann, Robin), and of a related boundary integral formulation.*

3 Domain decomposition methods

3.1 Variational formulations

Let us consider the local boundary value problems (1.4),

$$\Delta u_i(x) + \kappa_i^2 u_i(x) \;=\; 0 \quad \text{for } x \in \Omega_i, \quad \frac{\partial}{\partial n_i} u_i(x) = g(x) \quad \text{for } x \in \Gamma_i = \partial\Omega_i \cap \Gamma, \tag{3.1}$$

together with the transmission or interface boundary conditions (1.5),

$$u_i(x) = u_j(x), \quad \frac{\partial}{\partial n_i} u_i(x) + \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}\,. \tag{3.2}$$

To avoid non–uniqueness in the solution of either local Dirichlet or Neumann boundary value problems, instead of the Neumann transmission boundary condition in (3.2) we consider a Robin type interface condition given as

$$\frac{\partial}{\partial n_i} u_i(x) + \frac{\partial}{\partial n_j} u_j(x) + i\eta_{ij} R_{ij}[u_i(x) - u_j(x)] \;=\; 0 \quad \text{for } x \in \Gamma_{ij}, i < j, \tag{3.3}$$

together with the Dirichlet transmission condition

$$u_i(x) = u_j(x) \quad \text{for } x \in \Gamma_{ij}. \tag{3.4}$$

Note that $R_{ij} : H^{1/2}(\Gamma_{ij}) \to \widetilde{H}^{-1/2}(\Gamma_{ij})$ is assumed to be self–adjoint and $H^{1/2}(\Gamma_{ij})$–elliptic, and $\eta_{ij} \in \mathbb{R}\backslash\{0\}$. In this case, the equivalence of the interface transmission conditions (3.3) and (3.4) with (3.2) follows immediately.
The local subdomain boundary $\Gamma_i = \partial\Omega_i$ of a subdomain Ω_i is considered as the union

$$\Gamma_i = (\Gamma_i \cap \Gamma) \cup \bigcup_{\Gamma_{ij}} \Gamma_{ij}$$

where $\Gamma_i \cap \Gamma$ corresponds to the original boundary where Neumann boundary conditions are given, while Γ_{ij} denotes the coupling boundary with an adjacent subdomain. We define

$$(R_i u_{|\Gamma_i})(x) \;:=\; (R_{ij} u_{|\Gamma_{ij}})(x) \quad \text{for } x \in \Gamma_{ij} \tag{3.5}$$

and

$$\eta_i(x) \;:=\; \begin{cases} \eta_{ij} & \text{for } x \in \Gamma_{ij},\ i < j, \\ -\eta_{ij} & \text{for } x \in \Gamma_{ij},\ i > j, \\ 0 & \text{for } x \in \Gamma_i \cap \Gamma\,. \end{cases} \tag{3.6}$$

We assume, that $\eta_i(x)$ for $x \in \Gamma_i$ does not change its sign. This can be guaranteed either when considering a checker board domain decomposition [5], or when enforcing Robin type boundary conditions only on a part of the local boundary Γ_i, i.e. setting $\eta_{ij} = 0$ on some coupling boundaries Γ_{ij}.

Let $H^{1/2}(\Gamma_S) = H^1(\Omega)_{|\Gamma_S}$ be the skeleton trace space which is related to the domain decomposition (1.2). For $u \in H^{1/2}(\Gamma_S)$ we set $u_i = u_{|\Gamma_i} \in H^{1/2}(\Gamma_i)$ to ensure the Dirichlet transmission boundary condition (3.4).
For $v \in H^{1/2}(\Gamma_S)$ we then obtain the variational formulation to find $u \in H^{1/2}(\Gamma_S)$ such that, due to (3.3),

$$\begin{aligned}
&\sum_{i=1}^{p} \int\limits_{\Gamma_i} \left[\frac{\partial}{\partial n_i} u_i(x) + i\eta_i(x)(R_i u_{|\Gamma_i})(x) \right] v_{|\Gamma_i}(x) ds_x \\
&\quad = \int\limits_{\Gamma} \frac{\partial}{\partial n_x} u(x) v(x) ds_x \\
&\qquad + \sum_{i<j} \int\limits_{\Gamma_{ij}} \left[\frac{\partial}{\partial n_i} u_i(x) + i\eta_{ij}(R_{ij} u_{|\Gamma_{ij}})(x) + \frac{\partial}{\partial n_i} u_i(x) - i\eta_{ij}(R_{ij} u_{|\Gamma_{ij}})(x) \right] v_{|\Gamma_{ij}}(x) ds_x \\
&\quad = \int\limits_{\Gamma} g(x) v(x) ds_x
\end{aligned}$$

for all $v \in H^{1/2}(\Gamma)$ subject to the local Helmholtz equations

$$\Delta u_i(x) + \kappa_i^2 u_i(x) = 0 \quad \text{for } x \in \Omega_i.$$

By using the local representation formulae

$$u_i(x) = \int\limits_{\Gamma_i} U^*_{\kappa_i}(x,y) t_i(y) ds_y - \int\limits_{\Gamma_i} \frac{\partial}{\partial n_y} U^*_{\kappa_i}(x,y) u_i(y) ds_y \quad \text{for } x \in \Omega_i, i = 1, \ldots, p$$

we obtain a boundary integral formulation to find $u \in H^{1/2}(\Gamma_S)$ and $t_i \in H^{-1/2}(\Gamma_i)$ for $i = 1, \ldots, p$ such that

$$\sum_{i=1}^{p} \left[\langle D_{\kappa_i} u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K'_{\kappa_i}) t_i, v_{|\Gamma_i} \rangle_{\Gamma_i} + \langle i\eta R_i u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i} \right] = \int\limits_{\Gamma} g(x) v(x) ds_x \quad (3.7)$$

for all $v \in H^{1/2}(\Gamma)$ and

$$\langle V_{\kappa_i} t_i, \tau_i \rangle_{\Gamma_i} - \langle (\frac{1}{2} I + K_{\kappa_i}) u_{|\Gamma_i}, \tau_i \rangle_{\Gamma_i} = 0 \quad \text{for all } \tau_i \in H^{-1/2}(\Gamma_i), i = 1, \ldots, p. \quad (3.8)$$

Theorem 3.1. *The coupled variational problem* (3.7) *and* (3.8) *admits a unique solution* $u \in H^{1/2}(\Gamma_S)$. *In particular, the associated bilinear form is coercive. Moreover, if the Neumann boundary value problem* (1.1) *is uniquely solvable, the associated bilinear form is also injective.*

Proof. Coercivity follows as in the proof of Lemma 2.3. It remains to proof injectivity. Let $u \in H^{1/2}(\Gamma_S)$ and $t_i \in H^{-1/2}(\Gamma_i)$ for $i = 1, \ldots, p$ be any solution of the homogeneous system

$$\sum_{i=1}^{p} \Big[\langle D_{\kappa_i} u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K'_{\kappa_i}) t_i, v_{|\Gamma_i} \rangle_{\Gamma_i} + \langle i\eta R_i u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i} \Big] = 0$$

for all $v \in H^{1/2}(\Gamma)$ and

$$\langle V_{\kappa_i} t_i, \tau_i \rangle_{\Gamma_i} - \langle (\frac{1}{2} I + K_{\kappa_i}) u_{|\Gamma_i}, \tau_i \rangle_{\Gamma_i} = 0 \quad \text{for all } \tau_i \in H^{-1/2}(\Gamma_i), i = 1, \ldots, p.$$

With the definition of R_i and η_i we also have

$$\sum_{i=1}^{p} \Big[\langle D_{\kappa_i} u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K'_{\kappa_i}) t_i, v_{|\Gamma_i} \rangle_{\Gamma_i} \Big] = 0 \quad \text{for all } v \in H^{1/2}(\Gamma_S).$$

Let us define

$$\phi_i(x) = \int_{\Gamma_i} U^*_{\kappa_i}(x,y) t_i(y) ds_y - \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*_{\kappa_i}(x,y) u_{|\Gamma_i}(y) ds_y \quad \text{for } x \in \Gamma_i,$$

which satisfies

$$\Delta \phi_i(x) + \kappa^2 \phi_i(x) = 0 \quad \text{for } x \in \Omega_i.$$

and

$$\frac{\partial}{\partial n_x} \phi_i(x) = (\frac{1}{2} I + K'_{\kappa_i}) t_i(x) + (D_{\kappa_i} u_{|\Gamma_i})(x) \quad \text{for } x \in \Gamma_i$$

as well as

$$\phi_i(x) = (V_{\kappa_i} t_i)(x) + \frac{1}{2} u_{|\Gamma_i}(x) - (K_{\kappa_i} u_{|\Gamma_i})(x) = u_{|\Gamma_i} \quad \text{for } x \in \Gamma_i.$$

Hence we may consider $\phi_i = \phi_{|\Omega_i} \in H^1(\Omega_i)$ as the restriction of a function $\phi \in H^1(\Omega)$. Then we obtain, by using Green's first formula,

$$\begin{aligned}
0 &= \sum_{i=1}^{p} \Big[\langle D_{\kappa_i} u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K'_{\kappa_i}) t_i, v_{|\Gamma_i} \rangle_{\Gamma_i} \Big] \\
&= \sum_{i=1}^{p} \int_{\Gamma_i} \frac{\partial}{\partial n_y} \phi_i(x) v_{|\Gamma_i}(x) ds_x \\
&= \sum_{i=1}^{p} \int_{\Omega_i} \Big[\nabla \phi_i(x) \cdot \nabla v_{|\Omega_i}(x) dx - \kappa_i^2 \phi_i(x) v_{|\Omega_i}(x) \Big] dx \\
&= \int_{\Omega} \Big[\nabla \phi(x) \cdot \nabla v(x) - [\kappa(x)]^2 \phi(x) v(x) \Big] dx \quad \text{for all } v \in H^1(\Omega).
\end{aligned}$$

Since this is the weak formulation of the Neumann boundary value problem

$$\Delta\phi(x) + [\kappa(x)]^2\phi(x) \;=\; 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x}\phi(x) = 0 \quad \text{for } x \in \Gamma,$$

$\phi(x) = 0$ for $x \in \Omega$ follows. Recall that the Neumann boundary value problem (1.1) was assumed to be unique solvable. From $\phi_i(x) = 0$ for $x \in \Omega_i$ we conclude $u_{|\Gamma_i}(x) = 0$ for $x \in \Gamma_i$ as well as $n_i \cdot \nabla_x \varphi_i(x) = 0$ for $x \in \Gamma_i$. Hence we conclude

$$(\frac{1}{2}I + K'_{\kappa_i})t_i(x) \;=\; 0, \quad (V_{\kappa_i}t_i)(x) \;=\; 0 \quad \text{for } x \in \Gamma_i.$$

If κ_i^2 is not an eigenvalue of the Dirichlet eigenvalue problem (1.8), the single layer potential V_{κ_i} is injective and $t_i = 0$ follows. On the other hand, if $\lambda = \kappa_i^2$ is an eigenvalue of the Dirichlet eigenvalue problem (1.8), we also have

$$(\frac{1}{2}I - K'_{\kappa_i})t_i(x) \;=\; 0 \quad \text{for } x \in \Gamma_i.$$

Again, $t_i(x) = 0$ follows. $\square$

It remains to define the regularization operator $R_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ which is assumed to be self–adjoint, and $H^{1/2}(\Gamma_i)$–elliptic. In particular, we will also use the restrictions $R_{ij} := R_{i|\Gamma_{ij}} : H^{1/2}(\Gamma_{ij}) \to \widetilde{H}^{-1/2}(\Gamma_{ij})$. A particular choice is the use of the hypersingular integral operator which is related to the Yukawa partial differential equation

$$-\Delta u_i(x) + \kappa_i^2 u_i(x) \;=\; 0 \quad \text{for } x \in \Omega_i.$$

Hence we define

$$(R_i u_i)(x) \;=\; -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int\limits_{\Gamma_i}\frac{\partial}{\partial n_y}\frac{e^{-\kappa_i|x-y|}}{|x-y|}u_i(y)ds_y \quad \text{for } x \in \Gamma_i. \tag{3.9}$$

3.2 Boundary element discretizations

For the Galerkin discretization of the coupled variational formulation (3.7) and (3.8) let

$$W_h = \text{span}\{\varphi_k\}_{k=1}^{M_S} \subset H^{1/2}(\Gamma_S)$$

be a boundary element space on the skeleton of, e.g., piecewise linear basis functions φ_k, with respect to a quasi regular boundary mesh with mesh size h_S. We also define local restrictions of W_h onto Γ_i, in particular

$$W_{i,h} = W_{h|\Gamma_i} = \text{span}\{\varphi_k^i\}_{k=1}^{M_i} \subset H^{1/2}(\Gamma_i).$$

By using the isomorphisms

$$\underline{v}_i \in \mathbb{R}^{M_i} \leftrightarrow v_{i,h} = \sum_{k=1}^{M_i} v_{i,k}\varphi_k^i \in W_{i,h}, \quad \underline{v} \in \mathbb{R}^{M_S} \leftrightarrow v_h = \sum_{k=1}^{M_S} v_k\varphi_k \in W_h$$

there exist Boolian connectivity matrices $A_i \in \mathbb{R}^{M_i \times M_S}$ mapping some $\underline{v} \in \mathbb{R}^{M_S}$ of global nodal values onto the vector $\underline{v}_i = A_i \underline{v} \in \mathbb{R}^{M_i}$ of the local subdomain boundary nodal values. In addition, let

$$Z_{i,h} = \operatorname{span}\{\psi_k^i\}_{k=1}^{N_i} \subset H^{-1/2}(\Gamma_i)$$

be some local boundary element space, e.g., of piecewise constant basis functions ψ_k^i, with respect to a local quasi regular boundary mesh with average mesh size h_i. The Galerkin boundary element discretization of the variational formulation (3.7) and (3.8) now reads: find $u_h \in W_h$ and $t_{i,h} \in Z_{i,h}$ such that

$$\sum_{i=1}^{p} \left[\langle D_{\kappa_i} u_{h|\Gamma_i}, v_{h|\Gamma_i} \rangle_{\Gamma_i} + \langle (\frac{1}{2} I + K'_{\kappa_i}) t_{i,h}, v_{h|\Gamma_i} \rangle_{\Gamma_i} + \langle i\eta R_i u_{h|\Gamma_i}, v_{h|\Gamma_i} \rangle_{\Gamma_i} \right] = \int_{\Gamma} g(x) v_h(x) ds_x \tag{3.10}$$

for all all $v_h \in W_h$ and

$$\langle V_{\kappa_i} t_{i,h}, \tau_{i,h} \rangle_{\Gamma_i} - \langle (\frac{1}{2} I + K_{\kappa_i}) u_{h|\Gamma_i}, \tau_{i,h} \rangle_{\Gamma_i} = 0 \quad \text{for all } \tau_{i,h} \in Z_{i,h}, \ i = 1, \ldots, p. \tag{3.11}$$

Since the bilinear form of the coupled variational problem (3.7) and (3.8) is coercive and injective, see Theorem 3.1, the stability of the Galerkin variational formulation (3.10) and (3.11) follows for a sufficient small mesh size h, see, e.g., [17, 19]. In particular, there holds the quasi–optimal error estimate

$$\begin{aligned} \|u - u_h\|^2_{H^{1/2}(\Gamma_S)} &+ \sum_{i=1}^{p} \|t_i - t_{i,h}\|^2_{H^{-1/2}(\Gamma_i)} \\ &\leq c \left\{ \inf_{v_h \in W_h} \|u - v_h\|^2_{H^{1/2}(\Gamma_S)} + \sum_{i=1}^{p} \inf_{\tau_{i,h} \in Z_{i,h}} \|t_i - \tau_{i,h}\|^2_{H^{-1/2}(\Gamma_i)} \right\}. \end{aligned}$$

When assuming optimal regularity $u_{|\Gamma_S} \in H^2_{\text{pw}}(\Gamma_S)$, i.e. $u \in H^{5/2}(\Omega)$, and when using the Aubin–Nitsche trick, see, e.g., [19], we finally obtain the error estimate

$$\|u - u_h\|_{L_2(\Gamma_S)} \leq c(u, t_i)\, h^2 . \tag{3.12}$$

3.3 Tearing and interconnecting

The Galerkin variational formulation (3.10) and (3.11) is equivalent to a linear system of algebraic equations

$$\begin{pmatrix} V_{\kappa_1,h} & & & -\widetilde{K}_{\kappa_1,h} A_i \\ & \ldots & & \vdots \\ & & V_{\kappa_p,h} & -\widetilde{K}_{\kappa_p,h} A_p \\ A_1^\top \widetilde{K}'_{\kappa_1,h} & \ldots & A_p^\top \widetilde{K}'_{\kappa_p,h} & \sum\limits_{i=1}^{p} A_i^\top [D_{\kappa_i,h} + i\eta_i R_{i,h}] A_i \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \sum\limits_{i=1}^{p} A_i^\top \underline{g}_i \end{pmatrix} \tag{3.13}$$

where the block matrices are defined by

$$\begin{array}{rcl} V_{\kappa_i,h}[\ell,k] &=& \langle V_{\kappa_i}\psi^i_k, \psi^i_\ell\rangle_{\Gamma_i}, \\ \widetilde{K}_{\kappa_i,h}[\ell,n] &=& \langle (\frac{1}{2}I + K_{\kappa_i})\varphi^i_n, \psi^i_\ell\rangle_{\Gamma_i}, \\ \widetilde{K}'_{\kappa_i,h}[m,k] &=& \langle (\frac{1}{2}I + K'_{\kappa_i})\psi^i_k, \varphi^i_m\rangle_{\Gamma_i}, \\ D_{\kappa_i,h}[m,n] &=& \langle D_{\kappa_i}\varphi^i_n, \varphi^i_m\rangle_{\Gamma_i}, \\ R_{i,h}[m,n] &=& \langle R_i\varphi^i_n, \varphi^i_m\rangle_{\Gamma_i} \end{array}$$

for $k,\ell = 1,\ldots,N_i$, $m,n = 1,\ldots,M_i$, and $i = 1,\ldots,p$. In addition,

$$\underline{g}_i[m] \;=\; \langle g, \varphi^i_m\rangle_{\Gamma_i\cap\Gamma} \quad \text{for } m = 1,\ldots,M_i.$$

To tear the global vector $\underline{u} \in \mathbb{R}^{M_S}$ we introduce the local unknowns $\underline{u}_i = A_i\underline{u} \in \mathbb{R}^{M_i}$. To ensure the global continuity, in addition we have to require the interconnecting condition in the form

$$\sum_{i=1}^{p} B_i\underline{u}_i \;=\; \underline{0}\,. \tag{3.14}$$

In particular, for $x_k \in \Gamma_{ij}$ the interconnecting condition (3.14) states the continuity condition

$$u_{i,h}(x_k) = u_{j,h}(x_k).$$

For $i < j$ let k_i and k_j denote the local indices of the global index k. Then we can set

$$B_i[k,k_i] = 1, \quad B_j[k,k_j] = -1\,. \tag{3.15}$$

Moreover, the global equation in (3.13) can be rewritten as

$$\sum_{i=1}^{p} A_i^\top \Big[(D_{\kappa_i,h} + i\eta_i R_{i,h})\underline{u}_i + \widetilde{K}'_{\kappa_i,h}\underline{t}_i - \underline{g}_i\Big] \;=\; \underline{0}.$$

For $x_k \in \Gamma_{ij}$ with local indices k_i and k_j we can rewrite this equation as

$$\Big[(D_{\kappa_i,h} + i\eta_i R_{i,h})\underline{u}_i + \widetilde{K}'_{\kappa_i,h}\underline{t}_i - \underline{g}_i\Big]_{k_i} + \Big[(D_{\kappa_j,h} + i\eta_j R_{j,h})\underline{u}_j + \widetilde{K}'_{\kappa_j,h}\underline{t}_j - \underline{g}_j\Big]_{k_j} = 0\,.$$

Hence, for $i < j$ we may introduce a discrete Lagrange multiplier λ_k to define

$$\Big[(D_{\kappa_i,h} + i\eta_i R_{i,h})\underline{u}_i + \widetilde{K}'_{\kappa_i,h}\underline{t}_i - \underline{g}_i\Big]_{k_i} = \lambda_k, \quad \Big[(D_{\kappa_j,h} + i\eta_j R_{j,h})\underline{u}_j + \widetilde{K}'_{\kappa_j,h}\underline{t}_j - \underline{g}_j\Big]_{k_j} = -\lambda_k.$$

By using (3.15) we therefore end up with the local systems

$$(D_{\kappa_i,h} + i\eta_i R_{i,h})\underline{u}_i + \widetilde{K}'_{\kappa_i,h}\underline{t}_i - \underline{g}_i \;=\; B_i^\top\underline{\lambda} \quad \text{for } i = 1,\ldots,p.$$

Hence, the linear system (3.13) is equivalent to

$$\begin{pmatrix} V_{\kappa_1,h} & -\widetilde{K}_{\kappa_1,h} & & & & \\ \widetilde{K}'_{\kappa_1,h} & D_{\kappa_1,h}+i\eta R_{1,h} & & & & -B_1^\top \\ & & \ddots & & & \vdots \\ & & & V_{\kappa_p,h} & -\widetilde{K}_{\kappa_p,h} & \\ & & & \widetilde{K}'_{\kappa_p,h} & D_{\kappa_p,h}+i\eta R_{p,h} & -B_p^\top \\ & B_1 & \dots & & B_p & \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \underline{u}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{g}_1 \\ \vdots \\ \underline{0} \\ \underline{g}_p \\ \underline{0} \end{pmatrix}. \tag{3.16}$$

Remark 3.2. *In the tearing and interconnecting approach for a second order partial differential equation with an elliptic bilinear form the equivalence of the variational problem with a related minimization problem is used. After localization, Lagrange multipliers are used to enforce global continunity of the primal unknowns. Since in the case of the Helmholtz equation the bilinear forms are only coercive satisfying a Gårding inequality, algebraic arguments have to be used to derive* (3.16)*.*

Let us consider in (3.16) the local systems

$$\begin{pmatrix} V_{\kappa_i,h} & -\widetilde{K}_{\kappa_i,h} \\ \widetilde{K}'_{\kappa_i,h} & D_{\kappa_i,h}+i\eta_i R_{i,h} \end{pmatrix} \begin{pmatrix} \underline{t}_i \\ \underline{u}_i \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{g}_i + B_i^\top \underline{\lambda} \end{pmatrix} \tag{3.17}$$

which correspond to the Galerkin discretization of the boundary integral equations (2.13) and (2.14). Since the associated bilinear form is coercive and injective, stability of the local Galerkin scheme (3.17) follows for a sufficiently small mesh size $h_i < h_0$. Hence we obtain the Schur complement system of (3.16)

$$\begin{aligned} &\sum_{i=1}^p \begin{pmatrix} 0 & B_i \end{pmatrix} \begin{pmatrix} V_{\kappa_i,h} & -\widetilde{K}_{\kappa_i,h} \\ \widetilde{K}'_{\kappa_i,h} & D_{\kappa_i,h}+i\eta_i R_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{0} \\ B_i^\top \underline{\lambda} \end{pmatrix} \\ &\qquad = -\sum_{i=1}^p \begin{pmatrix} 0 & B_i \end{pmatrix} \begin{pmatrix} V_{\kappa_i,h} & -\widetilde{K}_{\kappa_i,h} \\ \widetilde{K}'_{\kappa_i,h} & D_{\kappa_i,h}+i\eta_i R_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{0} \\ \underline{g}_i \end{pmatrix}, \end{aligned}$$

which can be written as

$$F_h \underline{\lambda} = \underline{f}\,. \tag{3.18}$$

Note that the linear system (3.18) corresponds to the standard dual system in tearing and interconnecting domain decomposition methods.

4 Numerical results

As a numerical example we consider the Neumann boundary value problem (1.1) with respect to several computational domains $\Omega \subset \mathbb{R}^3$, and by using different domain decomposition strategies. In all examples, the exact solution is given by

$$u(x) = \frac{e^{i\kappa|x-\widehat{x}|}}{|x-\widehat{x}|}\,. \tag{4.1}$$

First we consider the domain

$$\Omega = \big\{x \in \mathbb{R}^3 \;:\; x_1 \in (-1.0, 1.5),\; x_2 \in (0.0, 1.0),\; x_3 \in (0.0, 1.0)\big\}$$

which is divided into two subdomains, see Fig. 1,

$$\Omega_1 = \{x \in \Omega : x_1 < 0\}\,, \quad \Omega_2 = \{x \in \Omega : x_1 > 0\}\,,$$

and $\widehat{x} = (2, 0, 2)^\top$.

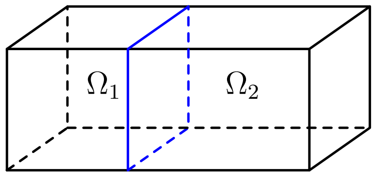

Figure 1: Domain decomposition with two subdomains.

The boundary element discretization of the coupled variational formulation (3.7) and (3.8) is done with respect to a globally uniform boundary mesh of N_i plane triangular elements with M_i nodes per subdomain and by using piecewise constant basis functions ψ_k^i and piecewise linear continuous basis functions φ_n^i. The linear system (3.18) is solved by a GMRES method with a relative error reduction of $\varepsilon = 10^{-8}$.
First we consider the global wave number $\kappa = 2$, which corresponds neither to a Dirichlet nor to a Neumann eigenvalue of both subproblems. The results, which confirm the error estimate (3.12), are given in Table 1.

N_i	M_i	GMRES	$\|u_1 - u_{1,h}\|_{L_2(\Gamma_1)}$
12	8	3	1.759 –1
48	26	8	3.359 –2
192	98	11	7.635 –3
768	386	13	1.853 –3
3072	1538	15	4.586 –4
12288	6146	18	1.142 –4

Table 1: Numerical results for two subdomains, $\kappa = 2$.

In a second example we consider the global wave number $\kappa = \sqrt{3}\pi \approx 5.4414$ which corresponds to the first Dirichlet and Neumann eigenvalue of the unit cube Ω_1. The results given in Table 2 confirm the stability of the proposed approach.

N_i	M_i	GMRES	$\|u_1 - u_{1,h}\|_{L_2(\Gamma_1)}$
12	8	3	4.056 –1
48	26	8	1.716 –1
192	98	15	2.569 –2
768	386	17	5.419 –3
3072	1538	19	1.266 –3
12288	6146	22	3.077 –4

Table 2: Numerical results for two subdomains, $\kappa = \sqrt{3}\pi$.

p	N_i	M_i	GMRES	$\|u_1 - u_{1,h}\|_{L_2(\Gamma_1)}$
8	24	14	23	3.195 –1
8	96	50	29	9.827 –2
8	384	194	31	2.528 –2
8	1536	770	35	6.006 –3
8	6144	3074	41	1.379 –3
27	24	14	54	1.600 –1
27	96	50	58	4.506 –2
27	384	194	59	1.090 –2
27	1536	770	62	2.498 –3
64	24	14	100	9.622 –2
64	96	50	105	2.527 –2
64	384	194	104	5.922 –3
64	1536	770	105	1.474 –3
125	24	14	167	6.347 –2
125	96	50	156	1.567 –2
125	384	194	137	3.817 –3
125	1536	770	137	7.976 –4

Table 3: Domain decomposition with $p = n^3$ subdomains.

In the last example we consider a sequence of domain decompositions where the unit cube $\Omega = (0,1)^3$ is subdivided into $p = n^3$ subdomains, $n = 2, 3, 4, 5$. Again, the global wave number is $\kappa = 2$, and the exact solution is given as in (4.1) with $\widehat{x} = (-0.1, 0, 0)^\top$. The related results are given in Table 3.

5 Conclusions

In this work we have presented a stable boundary element domain decomposition approach to solve interior boundary value problems for the Helmholtz equation via tearing and interconnecting methods. All numerical results confirm the stability of the proposed approach.

But in particular the results given in Table 3 indicate the requirement of an efficient preconditioner for the BETI Schur complement F_h. For this we may apply ideas from finite element tearing and interconnecting methods as described in [5]. But this has to be combined with preconditioned solution strategies for the coupled linear system (3.16), which also require the use of preconditioners for the discrete single layer potential $V_{\kappa_i,h}$ and for the discrete hypersingular boundary integral operator $D_{\kappa_i,h}$. Moreover, fast boundary element methods such as the fast multipole method have to be incorporated to end up with an efficient simulation tool, see, e.g., [10], for the case of data sparse boundary element domain decomposition methods in the case of the potential equation.

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