Technische Universität Graz

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Is the one–equation coupling of finite and boundary element methods always stable?

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Dedicated to Wolfgang L. Wendland on the occasion of his 75th birthday

Abstract

In this paper we present a sufficient and necessary condition to ensure the ellipticity of the bilinear form which is related to the one–equation coupling of finite and boundary element methods to solve a scalar free space transmission problem for a second order uniform elliptic partial differential equation in the case of general Lipschitz interfaces. This condition relates the minimal eigenvalue of the coefficient matrix in the bounded interior domain to the contraction constant of the shifted double layer integral operator which reflects the shape of the interface. Numerical examples confirm the theoretical results on the sharpeness of the presented estimates.

1 Introduction

The coupling of finite and boundary element methods is well established in many areas of applications, see, e.g., [5], in particular when considering the coupling of rather general partial differential equations in an interior domain with a partial differential equation with constant coefficients in an unbounded exterior domain. While on the continuous level the exterior boundary value problem can be reduced to the use of the Steklov–Poincaré operator describing the Dirichlet to Neumann map [9], the numerical analysis of related boundary element discretizations is more involved. From a mathematical point of view, the symmetric coupling [2] of finite and boundary element methods provides a sound stability and error analysis for a rather general choice of finite and boundary elements. Although there are efficient implementations available, the use of the symmetric formulation is still not very popular in engineering or for more advanced applications.

Hence, the one–equation coupling of finite and boundary element methods [1, 4], using single and double layer boundary integral operators only, is an attractive alternative. However, the mathematical analysis requires either the compactness of the double layer integral

operator, and therefore a smooth interface, or a sufficiently fine boundary element mesh for the approximation of the Neumann data [13]. While for the general case a rigorous mathematical analysis was not available for some time, numerical examples indicated the stability of this coupling scheme for more general situations [3]. In a recent paper [8], the stability of the standard finite and boundary element coupling scheme was proved for the first time. In [11] this approach was extended to prove the ellipticity of the bilinear form for the coupled formulation. An essential ingredient of this approach is the use of different variational and boundary integral formulations of the Steklov–Poincaré operator, or equivalently, of the energies which are related to the interior and exterior boundary value problems.
The result presented in [11, Theorem 2.2] is based on the assumption that the minimal eigenvalue of the coefficient matrix of the interior boundary value problem is uniformly bounded below by $\frac{1}{4}$. It was not clear whether this sufficient condition is also necessary. This paper presents a refined analysis and provides a sufficient and necessary condition to ensure ellipticity of the coupled bilinear form. While the main part of the proof is rather similar to the proof as given in [11], we present more precise estimates for the involved boundary integral operators. In particular, it is possible to derive an improved ellipticity result by using the contraction property [12] of the shifted double layer integral operator. It turns out, that this estimate and the underlying condition are sharp, i.e. there are situations where the coupled bilinear form fails to be elliptic.
This paper is structured as follows: In Sect. 2 we introduce the standard boundary integral operators, and we prove some norm equivalence inequalities for the interior Steklov–Poincaré operator which are based on the contraction property of the shifted double layer integral operator. The variational formulation for the one–equation coupling of finite and boundary elements is given in Sect. 3, where the main result on the ellipticity of the related bilinear form is stated in Theorem 3.1. In addition we describe a situation where the bilinear form fails to be elliptic. In Sect. 4 we present some numerical examples to investigate the ellipticity of the coupled bilinear form in the case of a two–dimensional free space transmission problem for several interfaces. For a sequence of coefficients of the interior partial differential equation we compute minimal eigenvalues of the generalized eigenvalue problem representing the ellipticity estimate as given in Theorem 3.1. The numerical results are in good agreement with the theoretical estimates on critical values for the coefficient.

2 Boundary integral equations

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$, and let $\Omega^c := \mathbb{R}^n \backslash \overline{\Omega}$. The solution of the exterior Dirichlet boundary value problem

$$-\Delta u_e(x) = 0 \quad \text{for } x \in \Omega^c, \quad u_e(x) = g(x) \quad \text{for } x \in \Gamma, \quad u_e(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty \tag{2.1}$$

is given by the representation formula

$$u_e(x) = -\int_\Gamma U^*(x,y)\frac{\partial}{\partial n_y}u_e(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)g(y)ds_y \quad \text{for } x \in \Omega^c, \tag{2.2}$$

where

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{for } n=2, \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{for } n=3 \end{cases}$$

is the fundamental solution of the Laplace operator. From (2.2) we obtain the boundary integral equation

$$(Vt_e)(x) = -\frac{1}{2}g(x) + (Kg)(x) \quad \text{for almost all } x \in \Gamma \tag{2.3}$$

where

$$t_e(x) = \frac{\partial}{\partial n_x}u_e(x) \quad \text{for almost all } x \in \Gamma$$

is the yet unknown normal derivative and

$$(Vt_e)(x) = \int_\Gamma U^*(x,y)t_e(y)ds_y, \quad (Kg)(x) = \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)g(y)ds_y \quad \text{for } x \in \Gamma$$

denote the single and double layer integral operators, respectively. Recall that the single layer integral operator V symmetrizes the double layer integral operator K [7, 10], i.e. $KV = VK'$. Moreover, $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is $H^{-1/2}(\Gamma)$–elliptic [6] satisfying

$$\langle V\tau, \tau\rangle_\Gamma \geq c_1^V \, \|\tau\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } \tau \in H^{-1/2}(\Gamma), \tag{2.4}$$

where we assume the scaling condition diam $\Omega < 1$ for $n = 2$. To ensure that the solution u_e as given by the representation formula (2.2) fulfils the radiation condition in (2.1) in the two–dimensional case, we need to assume that the normal derivative t_e satisfies the scaling condition

$$\int_\Gamma t_e(x)ds_x = 0. \tag{2.5}$$

Since the single layer integral operator V is invertible, and since we have $(\frac{1}{2}I + K)1 = 0$, by using (2.3) this is equivalent to

$$\begin{aligned} 0 &= \langle t_e, 1\rangle_\Gamma = \langle Vt_e, V^{-1}1\rangle_\Gamma = \langle(-\frac{1}{2}I + K)g, V^{-1}1\rangle_\Gamma \\ &= \langle g, V^{-1}(\frac{1}{2}I + K)1\rangle_\Gamma - \langle g, V^{-1}1\rangle_\Gamma = -\langle g, V^{-1}1\rangle_\Gamma, \end{aligned}$$

which implies for $n = 2$ the solvability condition

$$\langle g, t_{\text{eq}}\rangle_\Gamma = 0 \quad \text{with} \quad t_{\text{eq}} = V^{-1}1.$$

For the analysis of the coupled boundary and finite element variational formulation of a free space transmission problem, we need to state several results on boundary integral operators and related norm equivalences. These results are mainly based on the contraction property of the double layer integral operator [12]

$$\|(\frac{1}{2}I + K)v\|_{V^{-1}} \leq c_K \, \|v\|_{V^{-1}} \quad \text{for all } v \in H^{1/2}(\Gamma) \tag{2.6}$$

with

$$c_K = \frac{1}{2} + \sqrt{\frac{1}{4} - c_1^V c_1^D} < 1, \quad \|v\|^2_{V^{-1}} = \langle V^{-1}v, v\rangle_\Gamma. \tag{2.7}$$

To derive the contraction estimate (2.6) we used (2.4) and the ellipticity estimate

$$\langle Dv, v\rangle_\Gamma \geq c_1^D \, \|v\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}_*(\Gamma) := \left\{v \in H^{1/2}(\Gamma) : \langle v, 1\rangle_\Gamma = 0\right\} \tag{2.8}$$

for the hypersingular boundary integral operator

$$(Dv)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) v(y) ds_y \quad \text{for } x \in \Gamma.$$

The solution of the exterior Dirichlet boundary value problem (2.1) defines, by the solution of the boundary integral equation (2.3), the exterior Dirichlet to Neumann map $t_e = -S^{\text{ext}} u_e$, where the exterior Steklov–Poincaré operator is given by

$$S^{\text{ext}} = V^{-1}(\frac{1}{2}I - K).$$

For the analysis of the coupled finite and boundary element formulation we will also make use of the interior Steklov–Poincaré operator which is given by

$$S^{\text{int}} := V^{-1}(\frac{1}{2}I + K) = D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\,. \tag{2.9}$$

Lemma 2.1 *For all $v \in H^{1/2}(\Gamma)$ there hold the equivalence inequalities*

$$\frac{1}{c_K} \, \|(\frac{1}{2}I + K)v\|^2_{V^{-1}} \leq \langle S^{int} v, v\rangle_\Gamma \leq \frac{1}{1 - c_K} \, \|(\frac{1}{2}I + K)v\|^2_{V^{-1}} \tag{2.10}$$

where $c_K < 1$ is the contraction constant as given in (2.6).

Proof. Let us first consider $v \in H^{1/2}_*(\Gamma)$. By using the ellipticity estimates (2.8) for the hypersingular integral operator D and (2.4) for the single layer integral operator V, we have, see also [12, Proposition 5.2],

$$\langle Dv, v\rangle_\Gamma \geq c_1^D \, \|v\|^2_{H^{1/2}(\Gamma)} \geq c_1^D c_1^V \, \langle V^{-1}v, v\rangle_\Gamma = c_K(1 - c_K) \, \|v\|^2_{V^{-1}}.$$

Hence, the lower estimate follows from (2.6) by using the symmetric representation of $S^{\rm int}$. On the other hand, the non–symmetric representation of $S^{\rm int}$ gives

$$\langle S^{\rm int} v, v\rangle_\Gamma = \langle V^{-1}(\frac{1}{2}I+K)v, v\rangle_\Gamma \leq \|(\frac{1}{2}I+K)v\|_{V^{-1}} \|v\|_{V^{-1}},$$

and therefore the upper estimate follows from, see [12, Theorem 5.1],

$$(1-c_K)\,\|v\|_{V^{-1}} \leq \|(\frac{1}{2}I+K)v\|_{V^{-1}} \quad \text{for all } v \in H^{1/2}_*(\Gamma).$$

Due to $\ker S^{\rm int} = \ker(\frac{1}{2}I+K) = \operatorname{span}\{1\}$, the assertion holds for all $v \in H^{1/2}(\Gamma)$. ■

Remark 2.1 *While the particular values of the ellipticity constants c_1^V and c_1^D of the single layer integral operator V and of the hypersingular boundary integral operator D may depend on the definition of the underlying Sobolev norms $\|\cdot\|_{H^{-1/2}(\Gamma)}$ and $\|\cdot\|_{H^{1/2}(\Gamma)}$, respectively, the product of both constants can be characterized by*

$$c_1^V c_1^D = \min_{0\neq v\in H^{1/2}_*(\Gamma)} \frac{\langle Dv, v\rangle_\Gamma}{\langle V^{-1}v, v\rangle_\Gamma}\,.$$

This characterization implies that the lower estimate in (2.10) *is sharp, i.e. there exists a $\widehat{v} \in H^{1/2}_*(\Gamma)$ such that*

$$\frac{1}{c_K}\,\|(\frac{1}{2}I+K)\widehat{v}\|^2_{V^{-1}} = \langle S^{int}\widehat{v}, \widehat{v}\rangle_\Gamma.$$

3 Non–symmetric BEM/FEM coupling

Next we consider the free space transmission problem

$$-\mathrm{div}[A(x)\nabla u_i(x)] = f(x) \quad \text{for } x\in\Omega, \quad -\Delta u_e(x) = 0 \quad \text{for } x \in \Omega^c := \mathbb{R}^n\backslash\overline{\Omega} \tag{3.1}$$

with the interface transmission conditions

$$u_i(x) = u_e(x), \quad n_x \cdot [A(x)\nabla u_i(x)] = \frac{\partial}{\partial n_x} u_e(x) = t_e(x) \quad \text{for almost all } x\in\Gamma, \tag{3.2}$$

and with the radiation boundary condition

$$u_e(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x|\to\infty. \tag{3.3}$$

We assume that the coefficient matrix $A(x) \in \mathbb{R}^{n\times n}$ is symmetric and uniformly positive definite, i.e.

$$\lambda_{\min}(A) := \inf_{x\in\Omega} \min_{i=1,\ldots,n} \lambda_i(A(x)) \,>\, 0.$$

Moreover, $f \in L_2(\Omega)$ is a given function, and n_x is the exterior normal vector which is defined for almost all $x \in \Gamma$.
By considering the Neumann transmission condition in (3.2), the variational formulation of the interior Poisson equation in (3.1) is to find $u_i \in H^1(\Omega)$ such that

$$\int_\Omega [A(x)\nabla u_i(x)] \cdot \nabla v(x) dx - \int_\Gamma t_e(x) v(x) ds_x = \int_\Omega f(x) v(x) dx$$

is satisfied for all $v \in H^1(\Omega)$ while $t_e \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle V t_e, \tau \rangle_\Gamma + \langle (\frac{1}{2} I - K) u_i, \tau \rangle_\Gamma = 0 \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

Since the domain bilinear form defines only a semi–norm in $H^1(\Omega)$, we introduce the splitting [11]

$$u_i(x) = u_0 + \widetilde{u}_i(x) \quad \text{for } x \in \Omega, \quad u_0 = \frac{1}{\langle 1, t_{\rm eq} \rangle_\Gamma} \int_\Omega f(x) dx, \quad \langle \widetilde{u}_i, t_{\rm eq} \rangle_\Gamma = 0.$$

Recall that in the two–dimensional case the scaling condition (2.5) implies the solvability condition

$$\int_\Omega f(x) dx = 0,$$

and therefore $u_0 = 0$ follows when considering the case $n = 2$. In general, the coupled finite and boundary element variational formulation [11] of the transmission problem (3.1)–(3.3) reads, when using the constraint $\langle \widetilde{u}_i, t_{\rm eq} \rangle_\Gamma = 0$ for stabilization, to find $(\widetilde{u}_i, t_e) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \int_\Omega [A(x)\nabla \widetilde{u}_i(x)] \cdot \nabla v(x)\, dx + \alpha \langle \widetilde{u}_i, t_{\rm eq} \rangle_\Gamma \langle v, t_{\rm eq} \rangle_\Gamma - \langle t_e, v \rangle_\Gamma &= \int_\Omega f(x) v(x) dx, && (3.4) \\ \langle V t_e, \tau \rangle_\Gamma + \langle (\frac{1}{2} I - K) \widetilde{u}_i, \tau \rangle_\Gamma &= -\langle u_0, \tau \rangle_\Gamma && (3.5) \end{aligned}$$

is satisfied for $(v, \tau) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ where $\alpha \in \mathbb{R}_+$ is a suitable stabilization parameter. The unique solvability of the coupled variational formulation (3.4)–(3.5) and the stability and error analysis of related Galerkin boundary element methods is based on the following ellipticity result, which improves the results of [11, Theorem 2.2]. Recall that

$$\|v\|^2_{H^1(\Omega),\Gamma} := \int_\Omega |\nabla v(x)|^2 dx + [\langle v, t_{\rm eq} \rangle_\Gamma]^2, \quad \|\tau\|^2_V = \langle V\tau, \tau \rangle_\Gamma$$

define equivalent norms in $H^1(\Omega)$ and $H^{-1/2}(\Gamma)$, respectively.

Theorem 3.1 *Let $\alpha \geq \lambda_{\min}(A) > \frac{1}{4} c_K$ be satisfied, where $c_K < 1$ is the contraction constant as given in* (2.7). *Then the bilinear form*

$$\begin{aligned} a(u, t; v, \tau) \;:=\; & \int_\Omega [A(x)\nabla u(x)] \cdot \nabla v(x)\, dx + \alpha \langle u, t_{eq} \rangle_\Gamma \langle v, t_{eq} \rangle_\Gamma - \langle t, v \rangle_\Gamma && (3.6) \\ & + \langle V t, \tau \rangle_\Gamma + \langle (\frac{1}{2} I - K) u, \tau \rangle_\Gamma \end{aligned}$$

is $H^1(\Omega) \times H^{-1/2}(\Gamma)$*–elliptic satisfying*

$$a(v,\tau;v,\tau) \geq \frac{1}{2}(1-c_K)\left[1+\frac{\lambda_{\min}(A)}{c_K} - \sqrt{\left(1-\frac{\lambda_{\min}(A)}{c_K}\right)^2+1}\,\right]\left[\|v\|^2_{H^1(\Omega),\Gamma} + \|\tau\|^2_V\right] \tag{3.7}$$

for all $(v,\tau) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$.

Proof. As in the proof given in [11, Section 4], we have, by using $\alpha \geq \lambda_{\min}(A)$,

$$\begin{aligned}
a(v,\tau;v,\tau) &= \int_\Omega [A(x)\nabla v(x)]\cdot\nabla v(x)dx + \alpha\,[\langle v,t_{\mathrm{eq}}\rangle_\Gamma]^2 + \langle V\tau,\tau\rangle_\Gamma - \langle(\frac{1}{2}I+K)v,\tau\rangle_\Gamma \\
&\geq \lambda_{\min}(A)\left[\int_\Omega |\nabla v(x)|^2dx + [\langle v,t_{\mathrm{eq}}\rangle_\Gamma]^2\right] + \langle V\tau,\tau\rangle_\Gamma - \langle(\frac{1}{2}I+K)v,\tau\rangle_\Gamma \\
&= \lambda_{\min}(A)\left[\int_\Omega |\nabla \overline{v}(x)|^2dx + \langle S^{\mathrm{int}}v,v\rangle_\Gamma + [\langle v,t_{\mathrm{eq}}\rangle_\Gamma]^2\right] + \langle V\tau,\tau\rangle_\Gamma - \langle(\frac{1}{2}I+K)v,\tau\rangle_\Gamma,
\end{aligned}$$

where we have used the splitting $v = v_\Gamma + \overline{v}$ with v_Γ being the harmonic extension of $v_{|\Gamma}$, and $\widetilde{v} \in H^1_0(\Omega)$. Note that then we have

$$\int_\Omega \nabla v_\Gamma(x)\cdot\nabla\overline{v}(x)\,dx = 0, \quad \int_\Omega \nabla v_\Gamma(x)\cdot\nabla v_\Gamma(x)\,dx = \langle S^{\mathrm{int}}v,v\rangle_\Gamma.$$

From the equivalence inequalities (2.10) we now conclude

$$\begin{aligned}
&\lambda_{\min}(A)\langle S^{\mathrm{int}}v,v\rangle_\Gamma + \langle V\tau,\tau\rangle_\Gamma - \langle(\frac{1}{2}I+K)v,\tau\rangle_\Gamma \\
&\quad\geq \frac{\lambda_{\min}(A)}{c_K}\|(\frac{1}{2}I+K)v\|^2_{V^{-1}} + \|\tau\|^2_V - \|(\frac{1}{2}I+K)v\|_{V^{-1}}\|\tau\|_V \\
&\quad= \left(\frac{\lambda_{\min}(A)}{c_K} - \frac{1}{2}\frac{1}{\gamma^2}\right)\|(\frac{1}{2}I+K)v\|^2_{V^{-1}} + \left(1-\frac{1}{2}\gamma^2\right)\|\tau\|^2_V \\
&\qquad\qquad + \frac{1}{2}\left(\gamma\|\tau\|_V - \frac{1}{\gamma}\|(\frac{1}{2}I+K)v\|_{V^{-1}}\right)^2 \\
&\quad\geq \left(1-\frac{1}{2}\gamma_*^2\right)\left[\|(\frac{1}{2}I+K)v\|^2_{V^{-1}} + \|\tau\|^2_V\right]
\end{aligned}$$

if

$$\frac{\lambda_{\min}(A)}{c_K} - \frac{1}{2}\frac{1}{\gamma_*^2} = 1 - \frac{1}{2}\gamma_*^2$$

is satisfied. Hence we find

$$\gamma_*^2 = 1 - \frac{\lambda_{\min}(A)}{c_K} + \sqrt{\left(1-\frac{\lambda_{\min}(A)}{c_K}\right)^2+1},$$

and therefore

$$1 - \frac{1}{2}\gamma_*^2 = \frac{1}{2}\left[1 + \frac{\lambda_{\min}(A)}{c_K} - \sqrt{\left(1 - \frac{\lambda_{\min}(A)}{c_K}\right)^2 + 1}\right] \, > \, 0$$

if we assume

$$\lambda_{\min}(A) > \frac{1}{4}\, c_K \, .$$

As in [11, Section 4] we now obtain

$$\begin{aligned} a(v,\tau;v,\tau) \quad &\geq \quad \lambda_{\min}(A)\left[\int_\Omega |\nabla\overline{v}(x)|^2 dx + [\langle v, t_{\text{eq}}\rangle_\Gamma]^2\right] \\ &\quad + \frac{1}{2}\left[1 + \frac{\lambda_{\min}(A)}{c_K} - \sqrt{\left(1 - \frac{\lambda_{\min}(A)}{c_K}\right)^2 + 1}\right] \left[\|(\frac{1}{2}I + K)v\|_{V^{-1}}^2 + \|\tau\|_V^2\right] \end{aligned}$$

and the assertion follows from (2.10). ■

Lemma 3.2 *The ellipticity estimate* (3.7) *as given in Theorem* 3.1 *is sharp. In particular for $A(x) = \mu I$, $x \in \Omega$, and $0 < \mu \leq \frac{1}{4}c_K$, there exist nontrivial $(v^*, \tau^*) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that*

$$a(v^*, \tau^*; v^*, \tau^*) = 0\, .$$

Proof. As introduced in Remark 2.1, let $\widehat{v} \in H_*^{1/2}(\Gamma)$ be given such that

$$\frac{1}{c_K}\, \|(\frac{1}{2}I + K)\widehat{v}\|_{V^{-1}}^2 = \langle S^{\text{int}}\widehat{v}, \widehat{v}\rangle_\Gamma.$$

Let $\widehat{u} \in H^1(\Omega)$ be the unique weak solution of the Dirichlet boundary value problem

$$-\Delta\widehat{u}(x) = 0 \quad \text{for } x \in \Omega, \quad \widehat{u}(x) = \widehat{v}(x) \quad \text{for } x \in \Gamma.$$

Moreover, let $\widehat{t} \in H^{-1/2}(\Gamma)$ be the unique solution of the boundary integral equation

$$(V\widehat{t})(x) = (\frac{1}{2}I + K)\widehat{v}(x) \quad \text{for } x \in \Gamma.$$

In particular for $(v^*, \tau^*) = (\widehat{u}, \beta\widehat{t}) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$, $\beta \in \mathbb{R}_+$, we then obtain

$$\begin{aligned} a(\widehat{u}, \beta\widehat{t}; \widehat{u}, \beta\widehat{t}) \quad &= \quad \lambda \int_\Omega |\nabla\widehat{u}(x)|^2\, dx - \beta\langle(\frac{1}{2}I + K)\widehat{v}, \widehat{t}\rangle_\Gamma + \beta^2\langle V\widehat{t}, \widehat{t}\rangle_\Gamma \\ &= \quad \lambda\langle S^{\text{int}}\widehat{v}, \widehat{v}\rangle_\Gamma + \beta(\beta - 1)\langle V\widehat{t}, \widehat{t}\rangle_\Gamma \\ &= \quad \frac{\lambda}{c_K}\|(\frac{1}{2}I + K)\widehat{v}\|_{V^{-1}}^2 + \beta(\beta - 1)\langle V\widehat{t}, \widehat{t}\rangle_\Gamma \\ &= \quad \left[\frac{\lambda}{c_K} + \beta(\beta - 1)\right] \langle V\widehat{t}, \widehat{t}\rangle_\Gamma = 0 \end{aligned}$$

if we chose

$$\beta = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\lambda}{c_K}} \in \mathbb{R} \quad \text{for } \lambda \le \frac{1}{4} c_K \, .$$

∎

Remark 3.1 *The ellipticity estimate as given in Theorem* 3.1 *implies the stability and related error estimates for any admissible choice of finite and boundary elements. However, by using the exterior Steklov–Poincaré operator*

$$t_e = -S^{ext} \widetilde{u}_i = -V^{-1}(\frac{1}{2}I - K)\widetilde{u}_i$$

we may consider the reduced bilinear form

$$\widetilde{a}(u,v) := \int_\Omega [A(x)\nabla u(x)] \cdot \nabla v(x)\, dx + \alpha\, \langle u, t_{eq}\rangle_\Gamma \langle v, t_{eq}\rangle_\Gamma + \langle V^{-1}(\frac{1}{2}I - K)u, v\rangle_\Gamma$$

which is elliptic without any further restriction. But since the coupled finite and boundary element approximation corresponds to a mixed discretization scheme, the Galerkin discretization of the exterior Steklov–Poincaré operator requires a related stability condition, which restricts the choice of boundary elements, see, e.g., [9, 13].

4 Numerical examples

In this section, we will test the theoretical estimate (3.7) and the sharpness statement of Lemma 3.2 with some numerical examples. We use a two–dimensional discretization of the coupled variational formulation (3.4)–(3.5) with piecewise linear and continuous basis functions φ_i and piecewise constant basis functions ψ_k for the approximation of $\widetilde{u}_i$ and t_e, respectively. In particular, we consider the case of $A(x) = \mu I$ for $\mu \in (0,1]$.
The best possible constant in the ellipticity estimate (3.7) is characterized by the Rayleigh quotient

$$\begin{aligned} c_1^A &= \inf_{(0,0)\neq(v,\tau)\in H^1(\Omega)\times H^{-1/2}(\Gamma)} \frac{a(v,\tau;v,\tau)}{\|v\|^2_{H^1(\Omega),\Gamma} + \langle V\tau,\tau\rangle_\Gamma} \\ &= \inf_{(0,0)\neq(v,\tau)\in H^1(\Omega)\times H^{-1/2}(\Gamma)} \frac{a_S(v,\tau;v,\tau)}{\|v\|^2_{H^1(\Omega),\Gamma} + \langle V\tau,\tau\rangle_\Gamma} \end{aligned}$$

with the symmetrized bilinear form

$$\begin{aligned} a_S(u,t;v,\tau) &:= \mu\left[\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \langle u, t_{\mathrm{eq}}\rangle_\Gamma \langle v, t_{\mathrm{eq}}\rangle_\Gamma\right] - \frac{1}{2}\langle t, v\rangle_\Gamma - \frac{1}{2}\langle u,\tau\rangle_\Gamma \\ &\quad + \langle Vt,\tau\rangle_\Gamma + \frac{1}{2}\langle(\frac{1}{2}I - K)u,\tau\rangle_\Gamma + \frac{1}{2}\langle t, (\frac{1}{2}I-K)v\rangle_\Gamma . \end{aligned}$$

An approximation of the ellipticity constant c_1^A is now given by the minimal eigenvalue of the algebraic eigenvalue problem

$$\begin{pmatrix} \mu\widetilde{A}_h & -\frac{1}{4}M_h^\top - \frac{1}{2}K_h^\top \\ -\frac{1}{4}M_h - \frac{1}{2}K_h & V_h \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix} = \lambda \begin{pmatrix} \widetilde{A}_h & \\ & V_h \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix}, \tag{4.1}$$

where the single blocks are given by

$$\begin{aligned} M_h[\ell,i] &= \langle \varphi_{i|\Gamma}, \psi_\ell \rangle_\Gamma, & \widetilde{A}_h[j,i] &= \int_\Omega \nabla\varphi_i(x)\cdot\nabla\varphi_j(x)dx + \langle \varphi_{i|\Gamma}, t_{eq,h}\rangle_\Gamma \langle \varphi_{j|\Gamma}, t_{eq,h}\rangle_\Gamma, \\ K_h[\ell,i] &= \langle K\varphi_{i|\Gamma}, \psi_\ell \rangle_\Gamma, & V_h[\ell,k] &= \langle V\psi_k, \psi_\ell\rangle_\Gamma \end{aligned}$$

for $i,j = 1,\ldots,M,\ k,\ell = 1,\ldots,N$, where M denotes the number of nodes of the finite element mesh and N is the number of elements on the boundary. In addition we compute an approximation $t_{eq,h}$ of t_{eq} from

$$V_h \underline{t}_{eq} = \underline{b} \qquad \text{where } b[i] = \langle 1, \psi_i\rangle_\Gamma.$$

We compute the minimal eigenvalue of the eigenvalue problem (4.1) for a sequence of coefficients $\mu_i = \frac{i}{100}$, $i = 1,\ldots,100$, by using some appropriate eigenvalue solver, e.g., the inverse power method, and the Lapack routine for generalized symmetric eigenvalue problems with a positive definite matrix on the right hand side.
Since the contraction constant c_K as given in (2.7) is in general unknown, we compute an approximation from the maximal eigenvalue of the algebraic eigenvalue problem

$$(\frac{1}{2}\overline{M}_h^\top + \overline{K}_h^\top)\overline{V}_h^{-1}(\frac{1}{2}\overline{M}_h + \overline{K}_h)\underline{v} = \lambda \overline{M}_h^\top \overline{V}_h^{-1}\overline{M}_h \underline{v}, \tag{4.2}$$

where the boundary element matrices are given by

$$\overline{V}_h[\ell,k] = \langle V\phi_k, \phi_\ell\rangle_\Gamma, \quad \overline{M}_h[\ell,i] = \langle \varphi_{i|\Gamma}, \phi_\ell\rangle_\Gamma, \quad \overline{K}_h[\ell,i] = \langle K\varphi_{i|\Gamma}, \phi_\ell\rangle_\Gamma$$

for $i = 1,\ldots,M,\ k,\ell = 1,\ldots,2N$. Note that in addition to the piecewise linear and continuous basis functions φ_i we use piecewise linear but discontinuous basis functions ϕ_k on the boundary to ensure the stability of the Galerkin discretization.
As a first example we consider the transmission boundary value problem (3.1)–(3.3) in the case when Ω is a circle with radius $\sqrt{1/8}$. For a circular domain the contraction constant is $c_K = \frac{1}{2}$. For all computations we consider polygonal approximations of the circular boundary and a globally uniform boundary element mesh. In Fig. 1, the computed minimal eigenvalues of several refinement levels are compared to the behavior of the constant in the ellipticity estimate (3.7) as a function in μ_i. The approximations of the considered refinement levels are almost the same. Therefore the lines of the selected refinement levels $L = 2$, i.e. $N = 16$ boundary elements, and $L = 5$, i.e. $N = 128$ boundary elements, are on top of each other. For $\mu = 0.125$, we observe two zero eigenvalues for each refinement level. As can be seen in the zoomed plot in Fig. 1, all lines intersect in the point $(0.125, 0)$,

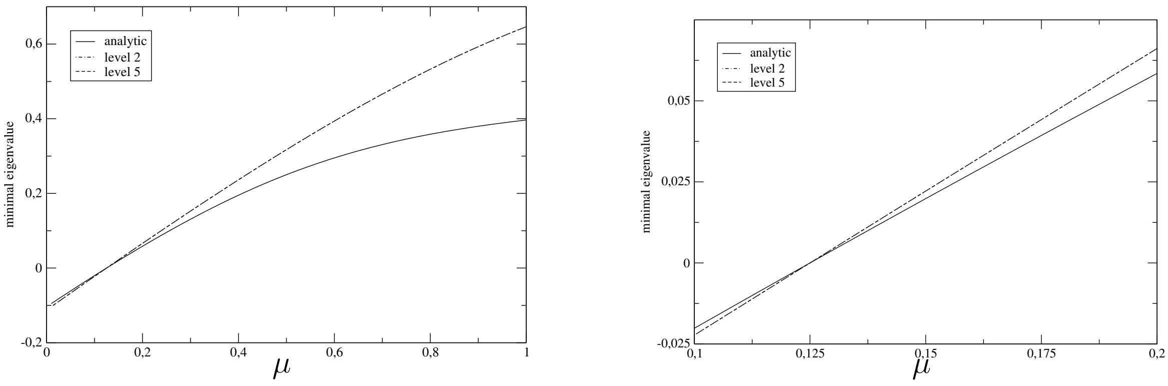

Figure 1: Minimal eigenvalues for the circle and $\mu \in (0, 1]$.

which is in perfect agreement with the requirement $\mu > \frac{1}{4}c_K = \frac{1}{8}$ of Theorem 3.1 and the sharpness statement of Proposition 3.2.
As a second example, we consider the case when $\Omega = (0, \frac{1}{2})^2$ is a square. Since the contraction constant c_K is not explicitely known in this case, we consider an approximation $c_K \approx 0.73$ by computing the maximal eigenvalue of the algebraic eigenvalue problem (4.2) for a refinement level $L = 10$, i.e. $N = 4096$ boundary elements. Note that we observed a slow convergence of the largest eigenvalue while most of the other eigenvalues converge fast. In Fig. 2, the computed minimal eigenvalues of the system (4.1) are compared to the behavior of the constant in the ellipticity estimate (3.7) by using the approximation $c_K \approx 0.73$ for several refinement levels. In this case, we observe distinguishable curves for the considered refinement levels, but the lines of the forth and fifth refinement level are in good agreement. As limit case we estimate $\mu > \frac{1}{4}c_K \approx 0.1825$ which is in good agreement with the observation that $\mu_{critical} \in (0.17, 0.18)$ for the fifth refinement level, as can be seen from the zoom in Fig. 2. On level $L = 8$ we estimate $\frac{1}{4}c_K \approx 0.18066$ and observe small negative eigenvalues for $\mu = 0.1795$.

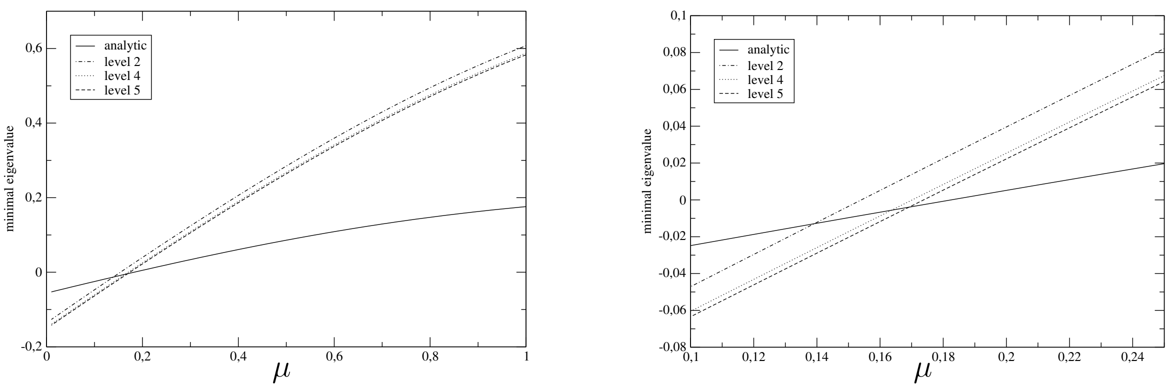

Figure 2: Minimal eigenvalues for the square and $\mu \in (0, 1]$.

As a last example for a non–convex domain we consider the L shaped domain with the corners $(0, 0)$, $(0.25, 0)$, $(0.5, 0)$, $(0.5, 0.25)$, $(0.25, 0.25)$, $(0.25, 0.5)$, $(0, 0.5)$, and $(0, 0.25)$.

On the ninth refinement level, we approximate the contraction constant by $c_K \approx 0.805$ and observe a faster convergence to the largest eigenvalue of (4.2) as for the square. In Fig. 3, the computed minimal eigenvalues of the system (4.1) are compared to the behavior of the constant in the ellipticity estimate (3.7) with the approximation $c_K \approx 0.805$ for the L shaped domain. Again the lines of the considered refinement levels are in good agreement. On the fifth refinement level we observe the critical value of μ close to 0.2 and the theoretical bound is $\mu > \frac{1}{4}c_K \approx 0.20125$.

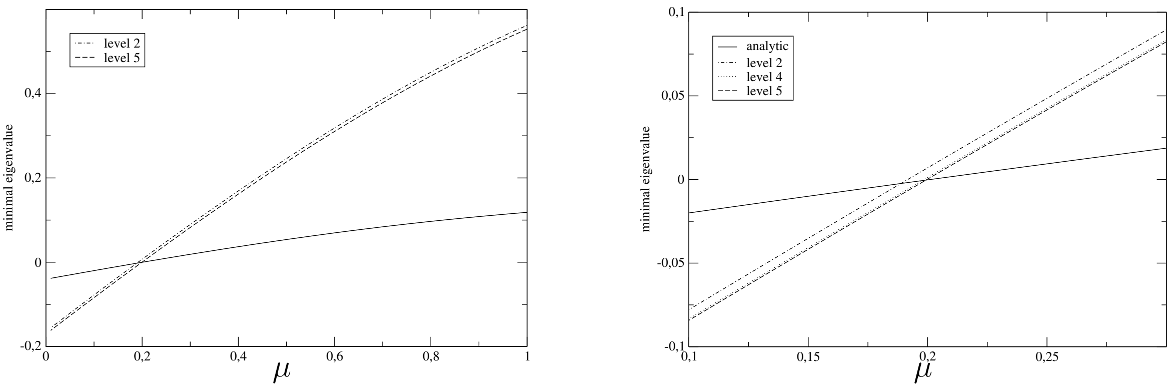

Figure 3: Minimal eigenvalues for the L shaped domain and $\mu \in (0, 1]$.

5 Conclusions

In this paper we have considered the stability of the one–equation coupling of finite and boundary element methods for an almost arbitrary choice of finite and boundary element trial spaces. It turns out that the non–symmetric boundary element approximation of the Steklov–Poincaré operator requires sufficient energy from the interior problem to result in a stable discretization. An essential tool is to rewrite the energy of the interior problem by using the Steklov–Poincaré operator which is related to an interior problem with constant coefficients. Hence, by analyzing related eigenvalue problems, we can generalize this approach to problems with (interior) boundary conditions. While the theoretical results of this paper are independent of the space dimension, for simplicity we just considered numerical examples for two–dimensional model problems. Moreover, this approach can also be extended to coupled problems in linear elasticity.

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