Technische Universität Graz

The All-Floating BETI Method: Numerical Results

G. Of

Berichte aus dem
Institut für Numerische Mathematik

Bericht 2006/7

Technische Universität Graz

The All-Floating BETI Method: Numerical Results

G. Of

Berichte aus dem
Institut für Numerische Mathematik

Bericht 2006/7

Technische Universität Graz
Institut für Numerische Mathematik
Steyrergasse 30
A 8010 Graz

WWW: `http://www.numerik.math.tu-graz.at`

The All-Floating BETI Method: Numerical Results

G. Of

Institute of Computational Mathematics
Graz University of Technology
Steyrergasse 30,
A-8010 Graz, Austria
of@tugraz.at

Abstract

The all-floating BETI method considers all subdomains as floating subdomains and improves the overall asymptotic complexity of the BETI method. This effect and the scalability of the method are shown in numerical examples.

1 Introduction

The boundary element tearing and interconnecting (BETI) method has been derived by Langer and Steinbach [9] as the boundary element counterpart of the well-known FETI methods [4]. In the standard BETI method, floating and non-floating subdomains have to be treated differently. This is rather easy for the potential equation, but in linear elastostatics it gets more involved since the number of rigid body motions which have to be considered may vary from one subdomain to another. The FETI–DP methods [3] introduce some global primal variables to guarantee the invertibility of all local Steklov Poincaré operators. The choice of these primal variables is important for the performance and gets more involved in linear elastostatics [7]. The all-floating BETI method overcomes these difficulties and improves the overall asymptotic complexity. The idea is to consider all subdomains as floating subdomains by tearing off the Dirichlet boundary conditions. This gives a unified treatment for all subdomains and an "optimal" preconditioning of the Steklov Poincaré operators. At the DD17 Conference a similar approach was presented for the FETI method, called TotalFETI [2].

2 The Boundary Element Tearing and Interconnecting Method

As model problem the Dirichlet boundary value problem

$$\begin{aligned} -\mathrm{div}[\alpha(x)\nabla u(x)] &= 0 \qquad && \text{for } x \in \Omega \\ u(x) &= g(x) && \text{for } x \in \Gamma = \partial\Omega \end{aligned}$$

of the potential equation is considered. $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain decomposed into p non–overlapping subdomains Ω_i with Lipschitz boundaries $\Gamma_i = \partial\Omega_i$. Further, the coefficient function $\alpha(\cdot)$ is piecewise constant such that $\alpha(x) = \alpha_i > 0$ for $x \in \Omega_i$, $i = 1, \ldots, p$. Instead of the global boundary value problem, the corresponding local boundary value problems may be considered for local functions u_i with transmission conditions

$$u_i(x) = u_j(x) \qquad \text{and} \qquad \alpha_i \frac{\partial}{\partial n_i} u_i(x) + \alpha_j \frac{\partial}{\partial n_j} u_j(x) = 0$$

for $x \in \Gamma_{ij} = \Gamma_i \cap \Gamma_j$ and $i \leq j$. n_i denotes the outer normal vector of the subdomain Ω_i. The Dirichlet domain decomposition method is based on an strong coupling of the Dirichlet data across the coupling interfaces by introducing a global function $u \in H^{1/2}(\Gamma_s)$ on the skeleton $\Gamma_S := \bigcup_{i=1}^{p} \Gamma_i$. A weak coupling of the Neumann data is applied using a variational formulation. After the discretization, the global system of linear equations

$$\widetilde{S}_h \widetilde{\mathbf{u}} = \sum_{i=1}^{p} A_i^\top \widetilde{S}_{i,h} A_i \widetilde{\mathbf{u}} = \sum_{i=1}^{p} A_i^\top \mathbf{f}_i \tag{1}$$

has to be solved. The connectivity matrices $A_i \in \mathbb{R}^{M_i \times M}$ map the global nodes to the local nodes and the global vector $\widetilde{\mathbf{u}}$ to the local vectors $\widetilde{\mathbf{u}}_i = A_i \widetilde{\mathbf{u}}$. The coefficients of the vectors $\mathbf{f}_i$ of the right-hand side are given by

$$\mathbf{f}_i[k] = -\int_{\Gamma_i} (S_i \widetilde{g})(x) \varphi_k^i(x) ds_x.$$

Here, the potential $u = \widetilde{u} + \widetilde{g}$ is split into an extension $\widetilde{g}$ of the given Dirichlet data g and into the unknown part $\widetilde{u}$. A matching discretization of the boundaries Γ_i into plane triangles is used. The potentials u_i are approximated by piecewise linear and continuous basis functions $\{\psi_n^i\}_{n=1}^{N_i}$ on each subdomain. Piecewise constant basis functions $\{\varphi_k^i\}_{k=1}^{M_i}$ are used for the approximation of the local fluxes t_i. The matrices

$$\widetilde{S}_{i,h} = D_{i,h} + (\frac{1}{2} M_{i,h}^\top + K_{i,h}^\top) V_{i,h}^{-1} (\frac{1}{2} M_{i,h} + K_{i,h})$$

are discrete approximations of the local Steklov Poincaré operators S_i, the so-called Dirichlet to Neumann maps. The boundary element matrices

$$\begin{aligned} V_{i,h}[\ell, k] &= \langle V_i \varphi_k^i, \varphi_\ell^i \rangle_{\Gamma_i}, \qquad & K_{i,h}[\ell, n] &= \langle K_i \psi_n^i, \varphi_\ell^i \rangle_{\Gamma_i}, \\ D_{i,h}[m, n] &= \langle D_i \psi_n^i, \psi_m^i \rangle_{\Gamma_i}, & M_{i,h}[\ell, n] &= \langle \psi_n^i, \varphi_\ell^i \rangle_{\Gamma_i} \end{aligned}$$

are realized by the fast multipole method [5] using integration by parts for the matrix $D_{i,h}$ of the hypersingular operator, see [12]. The use of the fast multipole method reduces the quadratic effort for a matrix times vector multiplication and the quadratic memory requirements of a standard boundary element method to almost linear ones up to some polylogarithmic factor. The involved boundary integral operators are the single layer potential V_i, the double layer potential K_i and the hypersingular operator D_i defined by

$$\begin{aligned}(V_i t_i)(x) &= \int_{\Gamma_i} U^*(x,y) t_i(y) ds_y, \\ (K_i u_i)(x) &= \int_{\Gamma_i} \frac{\partial}{\partial n_{i,y}} U^*(x,y) u_i(y) ds_y, \\ (D_i u_i)(x) &= -\frac{\partial}{\partial n_{i,x}} \int_{\Gamma_i} \frac{\partial}{\partial n_{i,y}} U^*(x,y) u_i(y) ds_y.\end{aligned}$$

The global system of linear equations (1) is preconditioned by

$$C_{\widetilde{S}}^{-1} = \sum_{i=1}^{p} A_i^\top V_{i,lin,h} A_i$$

in the conjugate gradient method. This preconditioner provides a good preconditioning of the local Steklov Poincaré operators S_i [14].

2.1 Standard BETI Method

Instead of the global system (1) of linear equations, the equivalent minimization problem

$$F(\widetilde{\mathbf{u}}) = \min_{\widetilde{\mathbf{v}} \in \mathbb{R}^M} \sum_{i=1}^{p} \left[\frac{1}{2} (\widetilde{S}_{i,h} A_i \widetilde{\mathbf{v}}, A_i \widetilde{\mathbf{v}}) - (\mathbf{f}_i, A_i \widetilde{\mathbf{v}}) \right] \tag{2}$$

is considered in the BETI method. Introducing local vectors $\widetilde{\mathbf{v}}_i := A_i \widetilde{\mathbf{v}}$ tears off the local potentials $\widetilde{v}_i$ at the coupling interfaces. Therefore, only local minimization problems have to be considered. The interconnection is done by introducing the constraints

$$\sum_{i=1}^{p} B_i \mathbf{v}_i = \mathbf{0} \tag{3}$$

to reinforce the continuity of the potentials across the coupling interfaces. Each line of a matrix B_i has at most one non-zero entry. This entry is either 1 or -1. At a global node with r adjacent subdomains $r-1$ constraints are used to guarantee that the corresponding local coefficients of these subdomains are equal. It remains to solve p local minimization problems with the constraints (3). Introducing Lagrangian multipliers λ gives the system of linear equations

$$\begin{pmatrix} \widetilde{S}_{1,h} & & & -B_1^\top \\ & \ddots & & \vdots \\ & & \widetilde{S}_{p,h} & -B_p^\top \\ B_1 & \dots & B_p & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{u}}_1 \\ \vdots \\ \widetilde{\mathbf{u}}_p \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_p \\ \mathbf{0} \end{pmatrix}. \tag{4}$$

The local Steklov Poincaré operators of the subdomains which share a piece with the Dirichlet boundary Γ are invertible, i.e., $\widetilde{\mathbf{u}}_i = \widetilde{S}_{i,h}^{-1}(\mathbf{f}_i + B_i^\top \lambda)$. The local Steklov Poincaré operators of the other subdomains, called floating subdomains, are singular. For suitable compatibility and normalization conditions, the local solutions can be given by $\widetilde{\mathbf{u}}_i = \widehat{S}_{i,h}^{-1}(\mathbf{f}_i + B_i^\top \lambda) + \gamma_i$ with some arbitrary constants γ_i and modified local Steklov Poincaré operators defined by

$$\langle \widehat{S}_i u, v \rangle := \langle S_i u, v \rangle + \beta_i \langle u, 1 \rangle_{\Gamma_i} \langle v, 1 \rangle_{\Gamma_i}.$$

The parameters β_i of this stabilization can be chosen suitably [13]. If the first q subdomains are floating subdomains and the remaining are non-floating, the system (4) can be written as the Schur complement system

$$\left[\sum_{i=1}^{q} B_i \widehat{S}_{i,h}^{-1} B_i^\top + \sum_{i=q+1}^{p} B_i S_{i,h}^{-1} B_i^\top \right] \lambda + G\gamma = \sum_{i=1}^{q} B_i \widehat{S}_{i,h}^{-1} \mathbf{f}_i + \sum_{i=q+1}^{p} B_i S_{i,h}^{-1} \mathbf{f}_i \tag{5}$$

or in the compact form

$$F\lambda + G\gamma = \mathbf{d} \qquad \text{with} \qquad G^\top \lambda = ((\mathbf{f}_i, \mathbf{1}))_{i=1:q}$$

and $G = (B_1\mathbf{1}, \dots, B_q\mathbf{1})$. Like in [6], the Lagrangian multipliers λ and the constants γ are determined by

$$P^\top F \lambda = P^\top \mathbf{d} \qquad \text{and} \qquad \gamma = (G^\top Q G)^{-1} G^\top Q (\mathbf{d} - F\lambda).$$

with the orthogonal projection $P = I - QG(G^\top QG)^{-1}G^\top$. Using the scaled hypersingular BETI preconditioner [9]

$$C^{-1} = (BC_\alpha^{-1}B^\top)^{-1} BC_\alpha^{-1} D_h C_\alpha^{-1} B^\top (BC_\alpha^{-1} B^\top)^{-1}.$$

with appropriate scaling matrices C_α [1, 6] the condition number of the preconditioned BETI system can be estimated by

$$\kappa(C^{-1}F) \leq c\,(1 + \log H/h)^2$$

independent of jumps in the coefficients α_i.

2.2 All-Floating BETI Method

A disadvantage of the BETI formulation (4) is that the condition number for the inversion of the local Steklov Poincaré operator of non-floating subdomains is increasing logarithmically for the used preconditioning by the single layer potential as a boundary integral operator of opposite order, see [10]. The all-floating BETI method considers all subdomains as floating subdomains by tearing off the Dirichlet boundary conditions. This gives a simple unified treatment for all subdomains and a "optimal" preconditioning of the local Steklov Poincaré operators.

As in the case of the standard BETI method, the global minimization problem (2) is split into local minimization problems

$$F(\widetilde{\mathbf{u}}_i) = \min_{\widetilde{\mathbf{v}}_i} \frac{1}{2}(\widetilde{S}_{i,h}\widetilde{\mathbf{v}}_i, \widetilde{\mathbf{v}}_i) + (\widetilde{S}_{i,h}\widetilde{\mathbf{g}}_i, \widetilde{\mathbf{v}}_i)$$

by introducing the local vectors $\widetilde{\mathbf{v}}_i = A_i\widetilde{\mathbf{v}}$. Now, the unknown part $\widetilde{\mathbf{v}}_i$ of the local Dirichlet datum and the known part $\widetilde{\mathbf{g}}_i$ given by the Dirichlet boundary conditions are reunited in the function $v_i = \widetilde{v}_i + \widetilde{g}_i$. The coefficients of these local functions can be determined by equivalent local minimization problems

$$\widetilde{F}(\mathbf{u}_i) = \min_{\mathbf{v}_i \in \mathbb{R}^{M_i}} \frac{1}{2}(\widetilde{S}_{i,h}\mathbf{v}_i, \mathbf{v}_i).$$

Additional local constraints are used to guarantee that the Dirichlet boundary conditions are satisfied, i.e., $\sum_{i=1}^{p} \widetilde{B}_i\mathbf{v}_i = \mathbf{b}$. These constraints include the constraints of the standard BETI method, for which the entries of the right hand side $\mathbf{b}$ are zero. The additional local constraints are of the type $\mathbf{v}_i[k] = g(x_k)$ where k is the local index of a Dirichlet node x_k. Then the corresponding line of the matrix $\widetilde{B}_i$ has one non-zero entry equal to one and the entry of the right hand side $\mathbf{b}$ is given by $g(x_k)$. Again, Lagrangian multipliers λ are introduced to get the system of linear equations

$$\begin{pmatrix} \widetilde{S}_{1,h} & & & -\widetilde{B}_1^\top \\ & \ddots & & \vdots \\ & & \widetilde{S}_{p,h} & -\widetilde{B}_p^\top \\ \widetilde{B}_1 & \dots & \widetilde{B}_p & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_p \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{b} \end{pmatrix}.$$

The corresponding Schur complement system has the simpler structure

$$\sum_{i=1}^{p} \widetilde{B}_i \widehat{S}_{i,h}^{-1} \widetilde{B}_i^\top \lambda + G\gamma = \mathbf{b}.$$

This system can be solved as described for the standard BETI method, but now all subdomains are floating subdomains. The all-floating approach can be extended to mixed boundary value problems and lienar elastostatics, see [11].

The BETI system and the all-floating system are solved as two-fold saddle point problems, which are derived by reintroducing the local fluxes. In these formulations, no inner inversions of the local Steklov Poincaré operators and of the single layer potentials are needed. Therefore, this is the fastest way to solve the BETI systems [8]. If the used algebraic multigrid preconditioners of the local single layer potentials are optimal and the two-fold saddle point formulations are used, the number of iterations of the conjugate gradient method for the standard BETI method is bounded by $\mathcal{O}((1+\log(H/h))^2)$ and that the total complexity is of order $\mathcal{O}(N_i \log^4 N_i)$, as the fast multipole method has a complexity of order $\mathcal{O}(N_i \log^2 N_i)$. In this notation, h is the global mesh-size and H denotes the diameter of the subdomains. For the all-floating BETI method, the number of iterations is reduced to the order of $\mathcal{O}(1+\log(H/h))$ and the number of arithmetic operations is of order $\mathcal{O}(N_i \log^3 N_i)$ correspondingly. This will be proven in an upcoming paper for linear elastostatics.

3 Numerical Results

As a academic test example, the domain decomposition is given by a cube subdivided into eight smaller cubes with boundaries of 24 triangles each. The robustness of the preconditioner with respect to jumping coefficients has been shown in a previous paper [8]. Here, a constant coefficient $\alpha = 1$ for all subdomains is considered. In Table 1, the computational times t_1 and t_2 for setting up the system and for solving in seconds and the numbers of iterations It of the conjugate gradient method with a relative accuracy of 10^{-8} are compared for the standard Dirichlet domain decomposition method (1) and for the twofold saddle point formulations of the standard and of the all-floating BETI methods are compared for six uniformed refinement steps. Note that the problem sizes of the local subproblems with N_i boundary elements are large.

Table 1: Comparison of the BETI methods

		DDD (1)			BETI			AF–BETI		
L	N_i	t_1	t_2	It.	t_1	t_2	It.	t_1	t_2	It.
0	24	1	0	4(1)	2	1	11	2	2	28
1	96	1	1	14(8)	3	2	37	3	3	40
2	384	5	4	20(10)	4	6	40	8	7	40
3	1536	14	25	22(11)	15	38	45	22	38	43
4	6144	77	167	24(11)	76	243	50	93	227	46
5	24576	333	1626	26(11)	333	2036	56	342	1798	48
6	98304	1443	9262	29(12)	1445	10130	62	1477	8693	51

On the first refinement levels, there is an overhead of the iteration numbers and the times for solving the system of the all-floating method in comparison to the standard

Table 2: Scalability of the all-floating BETI method

	8 finer subdomains						64 subdomains					
L	N_i	#duals	t_1	t_2	It.	D-error	N_i	#duals	t_1	t_2	It.	D-error
0	96	221	4	3	29	$1.35e-2$	24	613	5	6	32	$1.10e-2$
1	384	753	8	7	36	$3.88e-3$	96	1865	7	5	37	$3.45e-3$
2	1536	2777	21	34	41	$9.91e-4$	384	6481	11	10	47	$9.09e-4$
3	6144	10665	82	194	46	$2.28e-4$	1536	24161	23	48	52	$2.22e-4$
4	24576	41801	287	1811	53	$6.13e-5$	6144	93313	90	307	60	$4.67e-5$
5	98304	165513	1358	10485	62	$1.61e-5$	24576	366785	312	2658	70	$1.40e-5$

BETI method. This effect is due to the larger number of degrees of freedom of the all-floating method. The improved asymptotic complexity of the all-floating formulation pays off for the last three refinement levels, as the numbers of iterations are reduced. Finally, the all-floating method is faster than the standard domain decomposition method. The Dirichlet problem is the most challenging problem for the all-floating method, since the Dirichlet problem gives more additional constraints than a mixed boundary value problem. Therefore, the speedup by the all-floating method is better for mixed boundary value problems, see [11].

Finally, a test of the scalability of the all-floating method is presented. The results of a domain decomposition of the cube into 64 subcubes are compared to the results of a domain decomposition into eight subcubes. The triangulation of the surfaces of the eight cubes is finer with 96 instead of 24 triangles per subcube on the coarsest level, such that the triangulations of the whole cube are the same for both decompositions. Therefore, the decompositions are comparable. The results for five refinement steps are given in Table 2.

The iteration numbers of the decomposition into 64 subdomains are slightly increased compared to the eight subdomains, but this may be caused by the more complex coupling with up to 26 neighbors instead of seven. Except for the first three levels, the computational times for the 64 subdomains are about four times faster than for the eight subdomains. This is the most one can expect, since the additional coupling interfaces double the numbers of local degrees of freedom. Due to these additional local degrees of freedom, the more accurate approximations of the local Steklov Poincaré operators give a reduced L_2 error for the approximation of the potential. The total numbers of degrees of freedom are 1345177 for the decomposition into eight subdomains and 2726209 for the decomposition into 64 subdomains.

4 Conclusions

The all-floating BETI method simplifies the treatment of floating and non-floating subdomains and improves the asymptotic behavior of the BETI method. In combination with a fast multipole boundary element method, it provides an almost optimal complexity with respect to the number of iterations, the arithmetical complexity and the memory requirements. The all-floating BETI method has already been extended to mixed boundary value problems and linear elastostatics.

Acknowledgement

This work has been supported by the German Research Foundation 'Deutsche Forschungsgemeinschaft (DFG)' under the grant SFB 404 'Multifield Problems in Continuum Mechanics'.

References

[1] S. C. Brenner. An additive Schwarz preconditioner for the FETI method. *Numer. Math.*, 94(1):1–31, 2003.

[2] Z. Dostál, D. Horák, and R. Kučera. Total feti - an easier implementable variant of the feti method for numerical solution of elliptic pde. *Commun. Numer. Methods Eng.*, 2006. published online.

[3] C. Farhat, M. Lesoinne, and K. Pierson. A scalable dual-primal domain decomposition method. *Numer. Linear Algebra Appl.*, 7(7-8):687–714, 2000.

[4] C. Farhat and F.-X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Int. J. Numer. Methods Eng.*, 32(6):1205–1227, 1991.

[5] L. Greengard and V. Rokhlin. A fast algorithm for particle simulations. *J. Comput. Phys.*, 73:325–348, 1987.

[6] A. Klawonn and O. B. Widlund. FETI and Neumann-Neumann iterative substructuring methods: Connections and new results. *Commun. Pure Appl. Math.*, 54(1):57–90, 2001.

[7] A. Klawonn and O. B. Widlund. Dual-Primal FETI methods for linear elasticity. Technical Report TR2004-855, Courant Institute of Mathematical Sciences, Department of Computer Science, September 2004.

[8] U. Langer, G. Of, O. Steinbach, and W. Zulehner. Inexact data–sparse boundary element tearing and interconnecting methods. *SIAM J. Sci. Comput.*, 2006, accepted.

[9] U. Langer and O. Steinbach. Boundary element tearing and interconnecting methods. *Computing*, 71(3):205–228, 2003.

[10] W. McLean and O. Steinbach. Boundary element preconditioners for a hypersingular integral equation on an interval. *Adv. Comput. Math.*, 11(4):271–286, 1999.

[11] G. Of. *BETI–Gebietszerlegungsmethoden mit schnellen Randelementverfahren und Anwendungen*. Doctoral thesis, Universtät Stuttgart, 2006.

[12] G. Of, O. Steinbach, and W. L. Wendland. The fast multipole method for the symmetric boundary integral formulation. *IMA J. Numer. Anal.*, 26:272–296, 2006.

[13] O. Steinbach. *Numerische Näherungsverfahren für elliptische Randwertprobleme. Finite Elemente und Randelemente*. B.G. Teubner, Stuttgart, Leipzig, Wiesbaden, 2003.

[14] O. Steinbach and W. L. Wendland. The construction of some efficient preconditioners in the boundary element method. *Adv. Comput. Math.*, 9(1-2):191–216, 1998.

Erschienene Preprints ab Nummer 2005/1

2005/1	O. Steinbach	Numerische Mathematik 1. Vorlesungsskript.
2005/2	O. Steinbach	Technische Numerik. Vorlesungsskript.
2005/3	U. Langer G. Of O. Steinbach W. Zulehner	Inexact Fast Multipole Boundary Element Tearing and Interconnecting Methods
2005/4	U. Langer G. Of O. Steinbach W. Zulehner	Inexact Data–Sparse Boundary Element Tearing and Interconnecting Methods
2005/5	U. Langer O. Steinbach W. L. Wendland	Fast Boundary Element Methods in Industrial Applications Söllerhaus Workshop, 25.–28.9.2005, Book of Abstracts.
2005/6	U. Langer A. Pohoata O. Steinbach	Dual–Primal Boundary Element Tearing and Interconnecting Methods
2005/7	O. Steinbach (ed.)	Jahresbericht 2004/2005
2006/1	S. Engleder O. Steinbach	Modified Boundary Integral Formulations for the Helmholtz Equation
2006/2	O. Steinbach (ed.)	2nd Austrian Numerical Analysis Day, Book of Abstracts
2006/3	B. Muth G. Of P. Eberhard O. Steinbach	Collision Detection for Complicated Polyhedra Using the Fast Multipole Method or Ray Crossing
2006/4	G. Of, B. Schneider	Numerical Tests for the Recovery of the Gravity Field by Fast Boundary Element Methods
2006/5	U. Langer O. Steinbach W. L. Wendland	4th Workshop on Fast Boundary Element Methods in Industrial Applications, Book of Abstracts
2006/6	O. Steinbach (ed.)	Jahresbericht 2005/2006