Technische Universität Graz

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# Challenges and Applications of Boundary Element Domain Decomposition Methods

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**Abstract**

Boundary integral equation methods are well suited to represent the Dirichlet to Neumann maps which are required in the formulation of domain decomposition methods. Based on the symmetric representation of the local Steklov–Poincaré operators by a symmetric Galerkin boundary element method, we describe a stabilised variational formulation for the local Dirichlet to Neumann map. By a strong coupling of the Neumann data across the interfaces, we obtain a mixed variational formulation. For biorthogonal basis functions the resulting system is equivalent to nonredundant finite and boundary element tearing and interconnecting methods. We will also address several open questions, ideas and challenging tasks in the numerical analysis of boundary element domain decomposition methods, in the implementation of those algorithms, and their applications.

## 1 Introduction

Boundary element methods are well established approximation methods to solve exterior boundary value problems, or to solve partial differential equations with (piecewise) constant coefficients considered in complicated substructures and in domains with moving boundaries. For an state of the art overview on recent advances on mathematical aspects and engineering applications of boundary integral equation methods, see, for example, Schanz and Steinbach [2007]. However, for partial differential equations with nonlinear coefficients the coupling of finite and boundary element methods seems to be an efficient tool to solve complex problems in complicated domains. For the formulation and for an efficient solution of the resulting systems of equations, domain decomposition methods are mandatory.

The classical approach to couple finite and boundary element methods is to use only the weakly singular boundary integral equation with single and double layer potentials, see, e.g., Brezzi and Johnson [1979], Johnson and Nedelec [1980], and Wendland [1988]. In Costabel [1987] a symmetric coupling of finite and boundary elements using the so–called hypersingular boundary integral operator was introduced. This approach was then extended to symmetric Galerkin boundary element methods, see, e.g., Hsiao and Wendland [1990]. Appropriate preconditioned iterative strategies were later considered in Carstensen et al. [1998], while quite general preconditioners based on operators of the opposite order were introduced in Steinbach and Wendland [1998]. Boundary element tearing and interconnecting (BETI) methods were described in Langer and Steinbach [2003] as counterpart of FETI methods while in Langer et al. [2006] these methods were combined with a fast multipole approximation of the local boundary integral operators involved. For an alternative approach to boundary integral domain decomposition methods see also Khoromskij and Wittum [2004].

Here we will give a quite general setting of tearing and interconnecting, or more general, hybrid domain decomposition methods. The local partial differential equation is rewritten as a local Dirichlet to Neumann map which can be realized either by domain variational formulations or by using boundary integral formulations. Since the related function spaces are fractional Sobolev spaces, one may ask for the right definition of the associated norms. It turns out that the used norms which are induced by the local single layer potential or its inverse allows for almost explicite spectral equivalence inequalities, and an appropriate stabilisation of the singular Steklov–Poincaré operators. The modified Dirichlet to Neumann map is then used to obtain a mixed variational formulation allowing a weak coupling of the local Dirichlet data. However, staying with a globally conform method and using biorthogonal basis functions we end up with the standard tearing and interconnecting approach as in FETI and in BETI.

The aim of this paper is to sketch some ideas to obtain advanced formulations in boundary integral domain decomposition methods, to propose to use special norms in the numerical analysis, and to state some challenging tasks in the implementation of fast boundary element domain decomposition algorithms to solve challenging problems from engineering and industry.

## 2 Boundary Integral Equation DD Methods

As a model problem we consider the Dirichlet boundary value problem of the potential equation,

$$-\mathrm{div}[\alpha(x)\nabla u(x)] \;=\; f(x) \quad \text{for } x\in\Omega, \quad u(x)=g(x) \quad \text{for } x\in\Gamma \tag{1}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with Lipschitz boundary $\Gamma=\partial\Omega$. We assume that there is given a non–overlapping domain decomposition

$$\overline{\Omega}=\bigcup_{i=1}^{p}\overline{\Omega}_i, \quad \Omega_i\cap\Omega_j=\emptyset \quad \text{for } i\neq j, \quad \Gamma_i=\partial\Omega_i. \tag{2}$$

The domain decomposition as considered in (2) may arise from a piecewise constant coefficient function $\alpha(x)$ due to the physical model, in particular we may assume $\alpha(x)=\alpha_i$ for $x\in\Omega_i$. However, to construct efficient solution strategies in parallel, one may also introduce a domain decomposition (2) when considering the Laplace or Poisson equation in a complicated three–dimensional structure. A challenging task is to find a domain decomposition (2) which is based on boundary informations only, i.e., without any additional volume meshes. Using ideas as used in fast boundary element methods, i.e. by a bisection algorithm it is possible to decompose a given boundary mesh into two separate surface meshes. While this step seems to be simple, the delicate task is the definition of the new interface mesh which should take care of the given geometric situation, i.e. one should avoid the intersection of the new interface with the original boundary. We have already applied this algorithm to find a suitable domain decomposition of the Lake St. Wolfgang domain as shown in Figure 1.

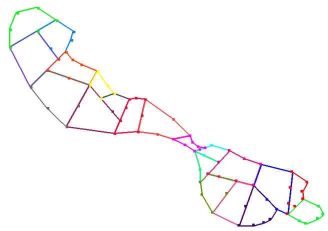

Figure 1: Domain Decomposition of the Lake St. Wolfgang Domain.

It seems to be an open problem to find and to implement a robust algorithm for an automatic domain decomposition of complicated three–dimensional strutures which is based on surface informations only. Such a tool is essentially needed when considering boundary element domain decomposition methods. Preliminary results on this topic will be published elsewhere (Of and Steinbach [2007]). A similar approach was already used in Ivanov et al. [2006] to design an automatic domain decomposition approach for unstructured grids in three dimensions. There, the remeshing of the new interface is done within the splitting hyperplane without considering the robustness of the algorithm for complicated geometries.
Instead of the global boundary value problem (1) we now consider the local boundary value problems

$$-\alpha_i\Delta u_i(x)=f_i(x) \quad \text{for } x\in\Omega_i, \quad u_i(x)=g(x) \quad \text{for } x\in\Gamma_i\cap\Gamma \tag{3}$$

together with the transmission boundary conditions

$$u_i(x) = u_j(x), \quad \alpha_i t_i(x) + \alpha_j t_j(x) = 0 \quad \text{for } x \in \Gamma_{ij} = \Gamma_i \cap \Gamma_j, \tag{4}$$

where $t_i = n_i \cdot \nabla u_i$ is the exterior normal derivative of $u_i$ on $\Gamma_i$. Since the solution $u_i$ of the local boundary value problem (3) is given via the representation formula

$$u_i(x) = \frac{1}{4\pi}\int\limits_{\Gamma_i} \frac{t_i(y)}{|x-y|} ds_y - \frac{1}{4\pi}\int\limits_{\Gamma_i} u_i(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|} ds_y + \frac{1}{\alpha_i}\frac{1}{4\pi}\int\limits_{\Omega_i} \frac{f_i(y)}{|x-y|} dy$$

for $x \in \Omega_i$, it is sufficient to find the complete Cauchy data $[u_i, t_i]_{|\Gamma_i}$ which are related to the solutions $u_i$ of the local boundary value problems (3). The appropriate boundary integral equations to derive a boundary integral representation of the involved Dirichlet to Neumann map are given by means of the Calderon projector

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_i & V_i \\ D_i & \frac{1}{2}I + K_i' \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix} + \frac{1}{\alpha_i}\begin{pmatrix} N_0 f_i \\ N_1 f_i \end{pmatrix},$$

where $V_i$ is the single layer potential, $K_i$ is the double layer potential, $D_i$ is the hypersingular boundary integral operator, and $N_j f_i$ are some Newton potentials, respectively. Hence, we find the Dirichlet to Neumann map as

$$\alpha_i t_i(x) = \alpha_i (S_i u_i)(x) - (N_i f_i)(x) \quad \text{for } x \in \Gamma_i \tag{5}$$

with the Steklov–Poincaré operator

$$\begin{aligned} (S_i u_i)(x) &= V_i^{-1}(\frac{1}{2}I + K_i)u_i(x) & (6)\\ &= \left[D_i + (\frac{1}{2}I + K_i')V_i^{-1}(\frac{1}{2}I + K_i)\right] u_i(x) \quad \text{for } x \in \Gamma_i. & (7) \end{aligned}$$

Note that $N_i f_i = V_i^{-1} N_0 f_i$. Replacing the partial differential equation in (3) by the related Dirichlet to Neumann map (5) this results in a coupled formulation to find the local Cauchy data $[u_i, t_i]_{|\Gamma_i}$ such that

$$\begin{array}{rcll} \alpha_i t_i(x) &=& \alpha_i (S_i u_i)(x) - (N_i f_i)(x) & \text{for } x \in \Gamma_i, \\ u_i(x) &=& g(x) & \text{for } x \in \Gamma_i \cap \Gamma, \\ u_i(x) &=& u_j(x) & \text{for } x \in \Gamma_{ij}, \\ \alpha_i t_i(x) + \alpha_j t_j(x) &=& 0 & \text{for } x \in \Gamma_{ij}. \end{array} \tag{8}$$

In what follows we first have to analyze the local Steklov–Poincare operators $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$. Since we are dealing with fractional Sobolev spaces $H^{\pm 1/2}(\Gamma_i)$ one may ask for appropriate norms to be used. It turns out that norms which are induced by the local single layer potentials $V_i$ may be advantageous. In particular,

$$\|\cdot\|_{V_i} = \sqrt{\langle V_i \cdot, \cdot\rangle_{\Gamma_i}}, \qquad \|\cdot\|_{V_i^{-1}} = \sqrt{\langle V_i^{-1} \cdot, \cdot\rangle_{\Gamma_i}}$$

are equivalent norms in $H^{-1/2}(\Gamma_i)$ and in $H^{1/2}(\Gamma_i)$, respectively. With the contraction property of the double layer potential (Steinbach and Wendland [2001]),

$$\|(\frac{1}{2}I + K_i)v_i\|_{V_i^{-1}} \leq c_{K,i}\|v_i\|_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i) \tag{9}$$

where the constant

$$c_{K,i} = \frac{1}{2} + \sqrt{\frac{1}{4} - c_1^{D_i} c_1^{V_i}} < 1$$

is only shape sensitive, we have

$$\|S_i v_i\|_{V_i} \,=\, \|(\frac{1}{2}I + K_i)v_i\|_{V^{-1}} \,\leq\, c_{K,i}\,\|v_i\|_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i)$$

as well as

$$\langle S_i v_i, v_i\rangle_{\Gamma_i} \,\geq\, (1 - c_{K,i})\,\|v_i\|^2_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i), v_i \perp 1.$$

In particular, the constants form the non–trivial kernel of the local Steklov–Poincare operators $S_i$, i.e., $S_i 1 = 0$ in the sense of $H^{-1/2}(\Gamma_i)$. To realize the related orthogonal splitting of $H^{1/2}(\Gamma_i)$ we introduce the natural density $w_{eq,i} \in H^{-1/2}(\Gamma_i)$ as the unique solution of the local boundary integral equation $V_i w_{eq,i} = 1$. Then we may define the stabilized hypersingular boundary integral operator $\widetilde{S}_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ via the Riesz representation theorem by the bilinear form

$$\langle \widetilde{S}_i u_i, v_i\rangle_{\Gamma_i} \,=\, \langle S_i u_i, v_i\rangle_{\Gamma_i} + \beta_i \langle u_i, w_{eq,i}\rangle_{\Gamma_i} \langle v_i, w_{eq,i}\rangle_{\Gamma_i}, \quad \beta_i \in \mathbb{R}_+. \tag{10}$$

**Theorem 2.1** *Let $\widetilde{S}_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ be the stabilized Steklov–Poincaré operator as defined in (10). Then there hold the spectral equivalence inequalities*

$$c_1^{\widetilde{S}_i}\,\langle V_i^{-1} v_i, v_i\rangle_{\Gamma_i} \,\leq\, \langle \widetilde{S}_i v_i, v_i\rangle_{\Gamma_i} \,\leq\, c_2^{\widetilde{S}_i}\,\langle V_i^{-1} v_i, v_i\rangle_{\Gamma_i}$$

*for all $v_i \in H^{1/2}(\Gamma_i)$ with positive constants*

$$c_1^{\widetilde{S}_i} = \min\{1 - c_{K,i}, \beta_i\langle 1, w_{eq,i}\rangle_{\Gamma_i}\}, \quad c_2^{\widetilde{S}_i} = \max\{c_{K,i}, \beta_i\langle 1, w_{eq,i}\rangle_{\Gamma_i}\}.$$

*Therefore, an optimal scaling is given for*

$$\beta_i = \frac{1}{2\langle 1, w_{eq,i}\rangle_{\Gamma_i}}, \quad c_1^{\widetilde{S}_i} = 1 - c_{K,i}, \quad c_2^{\widetilde{S}_i} = c_{K,i}.$$

Hence, the Dirichlet to Neumann map (5) can be written in a modified variational formulation as

$$\alpha_i \langle t_i, v_i\rangle_{\Gamma_i} \,=\, \langle \widetilde{S}_i \widetilde{u}_i, v_i\rangle_{\Gamma_i} - \langle N_i f_i, v_i\rangle_{\Gamma_i} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i) \tag{11}$$

when assuming the local solvability conditions

$$\alpha_i \langle t_i, 1\rangle_{\Gamma_i} + \langle N_i f_i, 1\rangle_{\Gamma_i} \,=\, 0\,. \tag{12}$$

In particular, inserting $v_i = 1$ into the modified Dirichlet to Neumann map (11), we obtain from the solvability condition (12)

$$0 \,=\, \alpha_i \langle t_i, 1\rangle_{\Gamma_i} + \langle N_i f_i, 1\rangle_{\Gamma_i} \,=\, \langle S_i \widetilde{u}_i, 1\rangle_{\Gamma_i} + \beta_i \langle \widetilde{u}_i, w_{eq,i}\rangle_{\Gamma_i} \langle 1, w_{eq,i}\rangle_{\Gamma_i}$$

and therefore the scaling condition

$$\langle \widetilde{u}_i, w_{eq,i}\rangle_{\Gamma_i} \,=\, 0 \tag{13}$$

due to

$$\langle S_i \widetilde{u}_i, 1\rangle_{\Gamma_i} \,=\, \langle \widetilde{u}_i, S_i 1\rangle_{\Gamma_i} \,=\, 0, \quad \langle 1, w_{eq,i}\rangle_{\Gamma_i} \,=\, \langle 1, V_i^{-1} 1\rangle_{\Gamma_i} \,>\, 0.$$

In fact, the scaling condition (13) is the natural characterization of functions $\widetilde{u}_i \in H^{1/2}(\Gamma_i)$ which are orthogonal to the constants in the sense of $H^{-1/2}(\Gamma_i)$. Hence, the local Dirichlet datum is given via

$$u_i \,=\, \widetilde{u}_i + \gamma_i, \quad \gamma_i \in \mathbb{R}.$$

Now, the coupled formulation (8) can be rewritten as

$$\begin{array}{rcll}
\alpha_i t_i(x) & = & \alpha_i(\widetilde{S}_i \widetilde{u}_i)(x) - (N_i f_i)(x) & \text{for } x \in \Gamma_i, \\
\widetilde{u}_i(x) + \gamma_i & = & g(x) & \text{for } x \in \Gamma_i \cap \Gamma, \\
\widetilde{u}_i(x) + \gamma_i & = & \widetilde{u}_j(x) + \gamma_j & \text{for } x \in \Gamma_{ij}, \\
\alpha_i t_i(x) + \alpha_j t_j(x) & = & 0 & \text{for } x \in \Gamma_{ij}, \\
\alpha_i \langle t_i, 1\rangle_{\Gamma_i} + \langle N_i f_i, 1\rangle_{\Gamma_i} & = & 0 &
\end{array} \tag{14}$$

where we have to find $\widetilde{u}_i \in H^{1/2}(\Gamma_i)$, $t_i \in H^{-1/2}(\Gamma_i)$, and $\gamma_i \in \mathbb{R}$, $i = 1, \ldots, p$. Hereby, the variational formulation of the modified Dirichlet to Neumann map reads: Find $\widetilde{u}_i \in H^{1/2}(\Gamma_i)$ such that

$$\alpha_i \langle \widetilde{S}_i \widetilde{u}_i, v_i \rangle_{\Gamma_i} - \alpha_i \langle t_i, v_i \rangle_{\Gamma_i} \;=\; \langle N_i f_i, v_i \rangle_{\Gamma_i} \tag{15}$$

is satisfied for all $v_i \in H^{1/2}(\Gamma_i)$, $i = 1, \ldots, p$. The Neumann transmission conditions in weak form are

$$\langle \alpha_i t_i + \alpha_j t_j, v_{ij} \rangle_{\Gamma_{ij}} \;=\; \int\limits_{\Gamma_{ij}} [\alpha_i t_i(x) + \alpha_j t_j(x)] v_{ij}(x) ds_x \;=\; 0 \tag{16}$$

for all $v_{ij} \in H^{1/2}(\Gamma_{ij})$. Taking the sum over all interfaces $\Gamma_{ij}$, this is equivalent to

$$\sum_{i=1}^{p} \alpha_i \langle t_i, v_{|\Gamma_i} \rangle_{\Gamma_i \setminus \Gamma} \;=\; 0 \quad \text{for all } v \in H^{1/2}(\Gamma_S), \tag{17}$$

where $\Gamma_S = \cup_{i=1}^{p} \Gamma_i$ is the skeleton of the given domain decomposition. The Dirichlet transmission conditions in (14) can be written as

$$\langle [\widetilde{u}_i + \gamma_i] - [\widetilde{u}_j + \gamma_j], \tau_{ij} \rangle_{\Gamma_{ij}} \;=\; 0 \quad \text{for all } \tau_{ij} \in \widetilde{H}^{-1/2}(\Gamma_{ij}) = [H^{1/2}(\Gamma_{ij})]', \tag{18}$$

while the Dirichlet boundary conditions in weak form read

$$\langle \widetilde{u}_i + \gamma_i, \tau_{i0} \rangle_{\Gamma_i \cap \Gamma} \;=\; \langle g, \tau_{i0} \rangle_{\Gamma_i \cap \Gamma} \quad \text{for all } \tau_{i0} \in \widetilde{H}^{-1/2}(\Gamma_i \cap \Gamma). \tag{19}$$

In addition we need to have the local solvability conditions

$$\alpha_i \langle t_i, 1 \rangle_{\Gamma_i} + \langle N_i f_i, 1 \rangle_{\Gamma_i} \;=\; 0. \tag{20}$$

The coupled variational formulation (15)–(20) is in fact a mixed (saddle point) domain decomposition formulation of the original boundary value problem (1). Hence we have to ensure a certain stability (BBL) condition to be satisfied, i.e., a stable duality pairing between the primal variables $\widetilde{u}_i$ and the dual Lagrange variable $t_i$ along the interfaces $\Gamma_{ij}$. Note that we also have to incorporate the additional constraints (20) and their associated Lagrange multipliers $\gamma_i$. While the unique solvability of the continuous variational formulation (15)–(20) follows in a quite standard way, as, e.g. in Steinbach [2003], the stability of an associated discrete scheme is not so obvious. Clearly, the Galerkin discretization of the coupled problem (15)–(20) depends on the local trial spaces to approximate the local Cauchy data $[\widetilde{u}, t_i]$. In particular, the variational formulation (15)–(20) may serve as a starting point for Mortar domain decomposition or three–field formulations as well (see Steinbach [2003] and the references given therein). However, here we will consider only an approach which is globally conform.

Let $S_h^1(\Gamma_S)$ be a suitable trial space on the skeleton $\Gamma_S$, e.g., of piecewise linear basis functions $\varphi_k$, $k = 1, \ldots, M$, and let $S_h^1(\Gamma_i)$ denote its restriction onto $\Gamma_i$, where the local basis functions are $\varphi_k^i$, $k = 1, \ldots, M_i$. In particular, $A_i \in \mathbb{R}^{M_i \times M}$ are connectivity matrices linking the global degrees of freedom $\underline{u} \in \mathbb{R}^M \leftrightarrow u_h \in S_h^1(\Gamma_S)$ to the local ones, $\underline{u}_i = A_i \underline{u} \in \mathbb{R}^{M_i} \leftrightarrow u_{h|\Gamma_i} \in S_h^1(\Gamma_i)$. Moreover, let $S_h^0(\Gamma_{ij})$ be some trial space to approximate the local Neumann data $t_i$ and $t_j$ along the interface $\Gamma_{ij}$, for example we may use piecewise constant basis functions $\psi_s^{ij}$. In the same way we introduce basis functions $\psi_s^0 \in S_h^0(\Gamma)$ to approximate the Neumann data along the Dirichlet boundary $\Gamma$. The trial spaces $S_h^0(\Gamma_{ij})$ and $S_h^0(\Gamma)$ define a global trial space $S_h^0(\Gamma_S)$ of piecewise constant basis functions $\psi_\iota$ implying $\lambda_h \in S_h^0(\Gamma_S) \leftrightarrow \underline{\lambda} \in \mathbb{R}^N$, i.e., we have $\lambda_{h|\Gamma_{ij}} \in S_h^0(\Gamma_{ij}) \leftrightarrow \underline{\lambda}_{ij} \in \mathbb{R}^{N_{ij}}$ and $\lambda_{h|\Gamma} \in S_h^0(\Gamma) \leftrightarrow \underline{\lambda}_0 \in \mathbb{R}^{N_0}$. For the global trial space

$$S_h^0(\Gamma_S) \;=\; \bigcup_{i<j} S_h^0(\Gamma_{ij}) \cup S_h^0(\Gamma) \;=\; \text{span}\{\psi_\iota\}_{\iota=0}^{N},$$

we define the restrictions $\psi_s^{ij} = r_\iota^{ij} \psi_\iota$ with $r_\iota^{ij} = 1$, $r_\iota^{ji} = -1$ for $i < j$, and $\psi_s^0 = r_\iota^0 \psi_\iota$, $r_\iota^0 = 1$ for $x \in \Gamma$. Hence we can also introduce a local mapping

$$\underline{t}_i = \frac{1}{\alpha_i} R_i \underline{\lambda} \in \mathbb{R}^{N_i} \quad \text{for } \underline{\lambda} \in \mathbb{R}^N$$

satisfying

$$R_i[s_i, \iota] = r_\iota^{ij} = 1, \quad R_j[s_j, \iota] = r_\iota^{ji} = -1$$

for $\iota = 1, \ldots, N, s_i = 1, \ldots, N_i, i < j$, and $R_i[s_i, \iota] = r_\iota^0 = 1$ for $x \in \Gamma$. For the associated approximations $t_{i,h} \in S_h^0(\Gamma_i) \leftrightarrow \underline{t}_i \in \mathbb{R}^{N_i}$, we then find

$$\alpha_i t_{i,h}(x) + \alpha_j t_{j,h}(x) \,=\, 0 \quad \text{for } x \in \Gamma_{ij},$$

i.e., the Neumann transmission conditions (16) are satisfied in a strong sense.
The Galerkin approximation of the Dirichlet transmission condition (18) can now be written as

$$\int\limits_{\Gamma_{ij}} \left\{ \left[ \sum_{k=1}^{M_i} \widetilde{u}_{i,k} \varphi_k^i(x) + \gamma_i \right] - \left[ \sum_{k=1}^{M_j} \widetilde{u}_{j,k} \varphi_k^j(x) + \gamma_j \right] \right\} \psi_\sigma^{ij}(x) ds_x \,=\, 0$$

for $\sigma = 1, \ldots, N_{ij}$, and $i < j$, or for $\iota = 1, \ldots, N$

$$\int\limits_{\Gamma_{ij}} \left\{ \left[ \sum_{k=1}^{M_i} \widetilde{u}_{i,k} \varphi_k^i(x) + \gamma_i \right] r_\iota^{ij} \psi_\iota(x) + \left[ \sum_{k=1}^{M_j} \widetilde{u}_{j,k} \varphi_k^j(x) + \gamma_j \right] r_\iota^{ji} \psi_\iota(x) ds_x \right\} \,=\, 0.$$

Correspondingly, the Galerkin discretization of the Dirichlet boundary condition (19) reads

$$\int\limits_{\Gamma_i \cap \Gamma} \left[ \sum_{k=1}^{M_i} \widetilde{u}_{i,k} \varphi_k^i(x) + \gamma_i \right] r_\iota^0 \psi_\iota(x) ds_x \,=\, \int\limits_{\Gamma_i \cap \Gamma} g(x) r_\iota^0 \psi_\iota(x) ds_x.$$

Combining both the Galerkin discretization of the Dirichlet transmission and of the Dirichlet boundary conditions, we can write

$$\sum_{i=1}^{p} B_i \underline{\widetilde{u}}_i + G \underline{\gamma} \,=\, \underline{g} \tag{21}$$

where $B_i \in \mathbb{R}^{M \times M_i}$ are defined by

$$B_i[\iota, k] = \int\limits_{\Gamma_{ij}} \varphi_k^i(x) r_\iota^{ij} \psi_\iota(x) ds_x, \quad B_i[\iota, k] = \int\limits_{\Gamma_i \cap \Gamma} \varphi_k^i(x) r_\iota^0 \psi_\iota(x) ds_x.$$

In addition, the matrix $G = (G_1, \ldots, G_p) \in \mathbb{R}^{M \times p}$ and the vector $\underline{g} \in \mathbb{R}^M$ of the right hand side are defined correspondingly, i.e.

$$G_i[\iota, i] = \int\limits_{\Gamma_{ij}} r_\iota^{ij} \psi_\iota(x) ds_x, \quad G_i[\iota, i] = \int\limits_{\Gamma_i \cap \Gamma_i} r_\iota^0 \psi_\iota(x) ds_x.$$

In particular, we have $G_i = B_i \underline{1}_i$ where $\underline{1}_i \in \mathbb{R}^{M_i}$ is the coefficient vector which is related to the constant function $1 \in H^{1/2}(\Gamma_i)$. Moreover, from the solvability conditions (20) we obtain

$$G_i^\top \underline{\lambda} = q_i = -\langle N_i f_i, 1 \rangle_{\Gamma_i} \quad \text{for } i = 1, \ldots, p.$$

The Galerkin discretization of the local Dirichlet to Neumann map (15) finally gives

$$\alpha_i \widetilde{S}_{i,h} \underline{\widetilde{u}}_i - B_i^\top \underline{\lambda} \,=\, \underline{f}_i,$$

where we have to approximate the exact stiffness matrix $\widetilde{S}_{i,h}$ including the local Steklov–Poincaré operator $S_i$, e.g., in the symmetric representation (7), by some boundary element discretization,

$$\widetilde{S}_{i,h} = D_{i,h} + (\frac{1}{2} M_{i,h}^\top + K_{i,h}^\top) V_{i,h}^{-1} (\frac{1}{2} M_{i,h} + K_{i,h}) + \beta_i \underline{a}_i \underline{a}_i^\top,$$

where the local boundary element matrices are given as

$$\begin{aligned} D_{i,h}[\ell,k] &= \langle D_i\varphi_k^i,\varphi_\ell^i\rangle_{\Gamma_i}, & K_{i,h}[\nu,k] &= \langle K_i\varphi_k^i,\vartheta_\nu^i\rangle_{\Gamma_i}, \\ V_{i,h}[\nu,\mu] &= \langle V_i\vartheta_\mu^i,\vartheta_\nu^i\rangle_{\Gamma_i}, & M_{i,h}[\nu,k] &= \langle \varphi_k^i,\vartheta_\nu^i\rangle_{\Gamma_i}, \\ a_{i,k} &= \langle \varphi_k^i, w_{eq,i}\rangle_{\Gamma_i} \end{aligned}$$

for $k,\ell = 1,\ldots,M_i,\ \mu,\nu = 1,\ldots,\bar{N}_i$ where $\operatorname{span}\{\vartheta_\mu^i\}_{\mu=1}^{\bar{N}_i} \subset H^{-1/2}(\Gamma_i)$ is some local boundary element trial space to approximate the local Neumann data which are needed in the definition of the approximate Steklov–Poincaré operator. Note that the basis functions $\vartheta_\mu^i$ can be defined in an almost arbitrary way, we only have to assume some approximation property to ensure convergence of the discrete scheme. The most simplest choice would be to identify the basis functions $\vartheta_\mu^i$ with $\psi_s^{ij}$ which are defined along the skeleton. In an analogue manner, one may even define an approximate Steklov–Poincaré operator by using local finite elements, see, e.g., Langer and Steinbach [2004]. Summarizing the above, we end up with a global system of linear equations,

$$\begin{pmatrix} \alpha_1\widetilde{S}_{1,h} & & & -B_1^\top & \\ & \ddots & & \vdots & \\ & & \alpha_p\widetilde{S}_{p,h} & -B_p^\top & \\ B_1 & \cdots & B_p & & G \\ & & & G^\top & \end{pmatrix} \begin{pmatrix} \underline{\widetilde{u}}_1 \\ \vdots \\ \underline{\widetilde{u}}_p \\ \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_p \\ \underline{g} \\ \underline{q} \end{pmatrix}. \tag{22}$$

The unique solvability of the linear system (22) and therefore of the coupled variational problem (15)–(20) follows from some stability (LBB) condition linking the local trial spaces $S_h^1(\Gamma_i)$ and $S_h^0(\Gamma_{ij})$ along the coupling interface $\Gamma_{ij}$. Here, we only consider the special case where the basis functions $\varphi_k^i$ and $\psi_s^{ij}$ are biorthogonal, i.e.

$$\int\limits_{\Gamma_{ij}} \varphi_k^i(x)\psi_s^{ij}(x)ds_x = \begin{cases} 1 & \text{for } s = k, \\ 0 & \text{for } s \neq k. \end{cases}$$

Then, the entries of the matrices $B_i$ consist just of zeros and $\pm 1$ describing a nodal coupling of the associated primal degrees of freedom. In particular, the use of biorthogonal basis functions to discretize the coupled variational problem (15)–(20) is equivalent to a redundant finite or boundary element tearing and interconnecting approach for a standard domain decomposition formulation, see, e.g., Langer and Steinbach [2004].

While for matching grids the described formulation is a conform discretization scheme, it may be generalized to different local grids and different local trial spaces as well. This leads immediately to hybrid or mortar domain decomposition methods where the choice of local trial spaces is essential to ensure the local stability conditions, see, e.g., Wohlmuth [2001] and the references given therein. Since the approximation of the local Dirichlet to Neumann maps can be done by any available discreization scheme, the presented formulation allows the coupling of different discretization schemes such as finite and boundary element methods, and the coupling of locally different meshes and trial spaces. However, when considering a boundary element approximation of the Steklov–Poincaré operator

$$S_iu_i = [D_i + (\frac{1}{2}I + K_i')V_i^{-1}(\frac{1}{2}I + K_i)]u_i = D_iu_i + (\frac{1}{2}I + K_i')w_i$$

the local Neumann data $w_i = V_i^{-1}(\frac{1}{2}I + K_i)$ coincide with the Neumann data $t_i$ as used in the coupled formulation (14). It seems to be an open problem how this relation can be used to find further advanced boundary element domain decomposition formulations, in particular to find more efficient preconditioned iterative strategies to solve the resulting linear systems of equations in parallel.

## 3 Conclusions

For the numerical analysis of standard boundary element methods see, for example, Steinbach [2007]. Since the discretization of non–local boundary integral operators with singular kernel functions leads to dense stiffness matrices, the use of fast boundary element methods is an issue. For an overview of those methods, and for the implementation and for the application of the Adaptive Cross Approximation approach, see Rjasanow and Steinbach [2007]. Other possible fast boundary element methods are the Fast Multipole Method, see, e.g., Of et al. [2006] and the references therein, or Hierarchical Matrices, see, e.g., Hackbusch et al. [2005]. The iterative solution of the linear system (22) of the boundary element tearing and interconnecting approach can be done by a projected preconditioned conjugate gradient method in a special inner product since the system matrix has a two fold saddle point structure, see also Langer et al. [2006], where we have also describe appropriate preconditioning strategies. Since the potential equation (1) is just a model problem, the methodology given in this paper can be extended to more advanced problems in a straight forward way, e.g., for problems in linear elastostatics, for almost incombressible materials and for the Stokes problem. More challenging are the handling of the Helmholtz equation or of the Maxwell system where more advanced formulations are needed to obtain boundary integral equations which are unique solvable for all wave numbers.

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