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Robust boundary element domain decomposition solvers in acoustics*

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Abstract

A stable boundary element tearing and interconnecting domain decomposition method is considered for the parallel solution of the Helmholtz equation. In particular, we discuss the preconditioned iterative solution of the resulting linear system and present some numerical results.

1 Introduction

Tearing and interconnecting domain decomposition methods [2, 3] are well established for an efficient and parallel solution of various elliptic partial differential equations by using finite and boundary element methods. But in the case of the Helmholtz equation, additional difficulties may appear. Although the global boundary value problem admits a unique solution, local subdomain solvers as used in the tearing and interconnecting approch may fail due to spurious modes. In a recent paper [7] we have introduced a boundary element tearing and interconnecting domain decomposition approach which is robust for all local wave numbers. The aim of the present paper is the discussion of some efficient preconditioners which are needed in the iterative solution of the resulting linear system. In particular we will use preconditioners of the opposite order [6] for the solution of the local boundary value problems, while the construction of the global preconditioner is based on the use of planar waves following the FETI–H method as introduced in [1]. Numerical results confirm the efficiency and the robustness of the proposed solution strategies.

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2 Formulation of the domain decomposition approach

As a model problem we consider the Neumann boundary value problem of the Helmholtz equation

$$\Delta u(x) + [\kappa(x)]^2 u(x) = 0 \quad \text{for } x \in \Omega, \quad \frac{\partial}{\partial n_x} u(x) = g(x) \quad \text{for } x \in \Gamma, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. We assume that the boundary value problem (2.1) admits a unique solution. Since the wave number $\kappa(x)$ is assumed to be piecewise constant, i.e. $\kappa(x) = \kappa_i$ for $x \in \Omega_i$, $i = 1, \ldots, p$, instead of (2.1) we consider the local boundary value problems

$$\Delta u_i(x) + \kappa_i^2 u_i(x) = 0 \quad \text{for } x \in \Omega_i, \quad \frac{\partial}{\partial n_i} u_i(x) = g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma, \tag{2.2}$$

together with the transmission or interface boundary conditions, see Fig. 1,

$$u_i(x) = u_j(x) \quad \text{for } x \in \Gamma_{ij}, \tag{2.3}$$

$$\frac{\partial}{\partial n_i} u_i(x) + \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \tag{2.4}$$

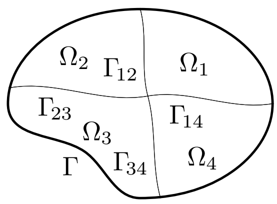

Figure 1: Decomposition

To avoid non–unique solutions of either local Dirichlet or Neumann boundary value problems, instead of the Neumann transmission boundary condition in (2.4) we consider a Robin type interface condition given as

$$\frac{\partial}{\partial n_i} u_i(x) + \frac{\partial}{\partial n_j} u_j(x) + i\eta_{ij} R_{ij}[u_i(x) - u_j(x)] \;=\; 0 \quad \text{for } x \in \Gamma_{ij}, i < j, \tag{2.5}$$

together with the Dirichlet transmission conditions (2.3). Note that the operators $R_{ij} : H^{1/2}(\Gamma_{ij}) \to \widetilde{H}^{-1/2}(\Gamma_{ij})$ are assumed to be self–adjoint and $H^{1/2}(\Gamma_{ij})$–elliptic, and $\eta_{ij} \in \mathbb{R}\backslash\{0\}$.
The local subdomain boundary $\Gamma_i = \partial\Omega_i$ of a subdomain Ω_i is considered as the union

$$\Gamma_i = (\Gamma_i \cap \Gamma) \cup \bigcup_{\Gamma_{ij}} \Gamma_{ij},$$

where $\Gamma_i \cap \Gamma$ corresponds to the original boundary where Neumann boundary conditions are given, while Γ_{ij} denotes the coupling boundary with an adjacent subdomain. We define

$$(R_i u_{|\Gamma_i})(x) \;:=\; (R_{ij} u_{|\Gamma_{ij}})(x) \quad \text{for } x \in \Gamma_{ij} \tag{2.6}$$

and

$$\eta_i(x) \;:=\; \begin{cases} \eta_{ij} & \text{for } x \in \Gamma_{ij},\ i < j, \\ -\eta_{ij} & \text{for } x \in \Gamma_{ij},\ i > j, \\ 0 & \text{for } x \in \Gamma_i \cap \Gamma\,. \end{cases} \tag{2.7}$$

We assume, that $\eta_i(x)$ for $x \in \Gamma_i$ does not change its sign. This can be guaranteed either when considering a checker board domain decomposition [1], or when enforcing Robin type boundary conditions only on a part of the local boundary Γ_i, i.e. setting $\eta_{ij} = 0$ on some coupling boundaries Γ_{ij}.
The solutions of the local boundary value problems (2.2) are given by using the representation formulae [4, 5]

$$u_i(x) = \int_{\Gamma_i} U^*_{\kappa_i}(x,y) t_i(y) ds_y - \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*_{\kappa_i}(x,y) u_i(y) ds_y \quad \text{for } x \in \Omega_i, \tag{2.8}$$

where

$$U^*_{\kappa_i}(x,y) = \frac{1}{4\pi} \frac{e^{i\kappa_i|x-y|}}{|x-y|}, \quad t_i(y) := \frac{\partial}{\partial n_y} u_i(y), \ y \in \Gamma$$

are the fundamental solution of the Helmholtz equation and the associated normal derivative of the solution u_i, respectively. By taking the Dirichlet and Neumann traces of the representation formulae (2.8) we obtain systems of local boundary integral equations which can be written by means of the Calderon projector on Γ_i as

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_{\kappa_i} & V_{\kappa_i} \\ D_{\kappa_i} & \frac{1}{2}I + K'_{\kappa_i} \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix}. \tag{2.9}$$

In (2.9),

$$(V_{\kappa_i} t_i)(x) = \int_{\Gamma_i} U^*_{\kappa_i}(x,y) t_i(y) ds_y, \ (D_{\kappa_i} u)(x) = -\frac{\partial}{\partial n_x} \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*_{\kappa_i}(x,y) u_i(y) ds_y$$

are the single layer potential and the hypersingular integral operator, and

$$(K_{\kappa_i} u_i)(x) = \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*_{\kappa_i}(x,y) u_i(y) ds_y, \ (K'_{\kappa_i} t_i)(x) = \int_{\Gamma_i} \frac{\partial}{\partial n_x} U^*_{\kappa_i}(x,y) t_i(y) ds_y$$

are the double layer and the adjoint double layer potential, respectively. The mapping properties of all boundary integral operators as introduced above are well known [4, 5].
By considering a Galerkin boundary element discretisation of the boundary integral equations (2.9) by using piecewise constant basis functions for the local Neumann data t_i and piecewise linear basis functions for the local Dirichlet data u_i, by inserting the transmission boundary conditions (2.3) and (2.5), and when applying a tearing and interconnecting approach, we finally obtain the linear system [7]

$$\begin{pmatrix} V_{\kappa_1,h} & -\widetilde{K}_{\kappa_1,h} & & & & \\ \widetilde{K}'_{\kappa_1,h} & D_{\kappa_1,h} + i\eta R_{1,h} & & & & -B_1^\top \\ & & \ddots & & & \vdots \\ & & & V_{\kappa_p,h} & -\widetilde{K}_{\kappa_p,h} & \\ & & & \widetilde{K}'_{\kappa_p,h} & D_{\kappa_p,h} + i\eta R_{p,h} & -B_p^\top \\ & B_1 & \dots & & B_p & \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \underline{u}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{g}_1 \\ \vdots \\ \underline{0} \\ \underline{g}_p \\ \underline{0} \end{pmatrix}. \tag{2.10}$$

Note that $\widetilde{K}_{\kappa_i,h} := (\frac{1}{2}M_{i,h} + K_{\kappa_i,h})$, and B_i are boolean matrices which ensure the continuity of the Dirichlet data. Since the related global bilinear form is coercive satisfying a Gårding inequality, unique solvability of the linear system (2.10) follows if the mesh size h is sufficiently small [7]. After eliminating the primal degrees of freedom we end up with the Schur complement system

$$\begin{aligned} F\underline{\lambda} &= \sum_{i=1}^{p} \begin{pmatrix} 0 & B_i \end{pmatrix} \begin{pmatrix} V_{\kappa_i,h} & -\widetilde{K}_{\kappa_i,h} \\ \widetilde{K}'_{\kappa_i,h} & D_{\kappa_i,h} + i\eta_i R_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{0} \\ B_i^\top \underline{\lambda} \end{pmatrix} \\ &= -\sum_{i=1}^{p} \begin{pmatrix} 0 & B_i \end{pmatrix} \begin{pmatrix} V_{\kappa_i,h} & -\widetilde{K}_{\kappa_i,h} \\ \widetilde{K}'_{\kappa_i,h} & D_{\kappa_i,h} + i\eta_i R_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{0} \\ \underline{g}_i \end{pmatrix} = \underline{d}\,. \end{aligned} \tag{2.11}$$

3 Construction of preconditioners

For an iterative solution of the linear system (2.11) by using a GMRES method in parallel, we need to have efficient preconditioners. This involves the construction of a global preconditioner C_F for the assembled stiffness matrix F, and the derivation of local preconditioners C_{A_i} for the local matrices

$$A_i = \begin{pmatrix} -V_{\kappa_i,h} & \widetilde{K}_{\kappa_i,h} \\ \widetilde{K}'_{\kappa_i,h} & D_{\kappa_i,h} + i\eta_i R_{i,h} \end{pmatrix}, \; i = 1, \ldots, p. \tag{3.1}$$

3.1 Local preconditioners

We first describe local preconditioners C_{A_i} for the local systems (3.1). For this we use a block diagonal preconditioner which is based on the idea of operators of opposite order [6],

$$C_{A_i}^{-1} := \begin{pmatrix} -M_{0,i,h}^{-1}\overline{D}_{i,h}M_{0,i,h}^{-1} & \\ & M_{1,i,h}^{-1}\overline{V}_{i,h}M_{1,i,h}^{-1} \end{pmatrix}, \tag{3.2}$$

where $M_{0,i,h}$ and $M_{1,i,h}$ are the mass matrices using constant and linear basis functions, respectively. The matrix $\overline{V}_{i,h}$ is the Galerkin discretisation of the single layer potential by using piecewise linear and continuous basis functions,

$$\overline{V}_{i,h}[\ell,k] = \frac{1}{4\pi}\int_{\Gamma_i} \phi_{i,\ell}(x) \int_{\Gamma_i} \frac{1}{|x-y|}\phi_{i,k}(y)ds_y ds_s.$$

Accordingly, $\overline{D}_{i,h}$ is the Galerkin discretisation of the stabilised hypersingular boundary integral operator. When using integration by parts, the matrix entries are given as

$$\overline{D}_{i,h}[\ell,k] = \frac{1}{4\pi}\int_{\Gamma_i}\int_{\Gamma_i} \frac{\mathbf{curl}_\Gamma\,\psi_{i,k}(y)\cdot \mathbf{curl}_\Gamma\,\psi_{i,\ell}(x)}{|x-y|} ds_y ds_s + \langle 1, \psi_{i,k}\rangle_{\Gamma_i}\langle 1, \psi_{i,\ell}\rangle_{\Gamma_i}.$$

Since the local single layer potential $V_{\kappa_i,h}$ is discretised by using piecewise constant basis functions, also the preconditioning matrix $\overline{D}_{i,h}$ has to be discretised by using the same piecewise constant basis functions. The application of $\mathbf{curl}_\Gamma$ on a constant function can be interpreted as a distribution on the edges which leads to a formulation based on line integrals

$$\overline{D}_{i,h}[\ell,k] = \frac{1}{4\pi}\int_{\partial\tau_{i,k}}\int_{\partial\tau_{i,\ell}} \frac{r_{i,k}\cdot r_{i,\ell}}{|x-y|} ds_y ds_x + \langle 1,\psi_{i,k}\rangle_{\Gamma_i}\langle 1,\psi_{i,\ell}\rangle_{\Gamma_i},$$

where $r_{i,k}$ and $r_{i,\ell}$ are the direction vectors of the edges of the triangles $\tau_{i,k}$ and $\tau_{i,\ell}$. Note that the described Galerkin discretisation of the hypersingular boundary integral operator by using piecewise constant basis functions is non–conform, and requires special techniques when evaluating singular line integrals involved.

3.2 Global preconditioners

For the construction of a global preconditioner we follow an idea of [1]. Let $\underline{r}$ be the residual of the global problem (2.11), i.e.

$$\underline{r} := \underline{d} - F\underline{\lambda}.$$

The solution algorithm is modified in such a way that the residual $\underline{r}$ is orthogonal to a given m–dimensional subspace which is represented by the columns of an orthogonal matrix Q, i.e.

$$Q^\top \underline{r} = Q^\top(\underline{d} - F\underline{\lambda}) = 0. \tag{3.3}$$

This restriction implies a solution constraint, since the residual represents the jump of the Dirichlet data on the interface,

$$\underline{r} = \underline{d} - F\underline{\lambda} = \sum_{i=1}^{p} B_i \underline{u}_i.$$

To enforce the orthogonality relation (3.3), we first introduce a new iterate

$$\underline{\widetilde{\lambda}} := \underline{\lambda} + Q\underline{\gamma} \tag{3.4}$$

and obtain

$$Q^\top F Q \underline{\gamma} = Q^\top(\underline{d} - F\underline{\lambda}).$$

By solving this equation we get from (3.4) the alternative representation

$$\underline{\widetilde{\lambda}} = P\underline{\lambda} + \underline{\lambda}^0$$

with the projector

$$P := I - Q(Q^\top F Q)^{-1} Q^\top F$$

and

$$\underline{\lambda}^0 = Q(Q^\top FQ)^{-1}Q^\top \underline{d}\,.$$

From $F\underline{\lambda} = \underline{d}$ we then obtain the linear system

$$FP\underline{\lambda} + F\underline{\lambda}^0 = \underline{d}\,,$$

and after multiplication with the transposed projector $P^\top$ we have to solve the linear system

$$P^\top FP\underline{\lambda} = P^\top \underline{d}\,.$$

Note that $P^\top F\underline{\lambda}^0 = 0$. Moreover, due to

$$P^\top FP = (I - FQ(Q^\top FQ)^{-1}Q^\top)F(I - Q(Q^\top FQ)^{-1}Q^\top F) = FP$$

we can save one application of P, and therefore one application of F in each iteration step. It remains to discuss the choice of the subspace which is spanned by the orthonormal matrix Q. As in [1] we consider planar waves, which are evaluated locally. In particular, for each subdomain Ω_i we consider a set of m_i directions θ_j, and evaluate the planar wave with the wave number κ_i locally at nodes x_{ℓ_i} to obtain

$$Q^i[\ell_i, j] = e^{i\kappa_i(\theta_j, x_{\ell_i})}\,.$$

The global matrix is finally constructed by

$$Q = [Q^1 \dots Q^i \dots Q^p].$$

By using this local approach one speeds up the construction of $Q^\top FQ$, since only a few local subproblems have to be solved for every direction θ_j. However, we still have to ensure that $Q^\top FQ$ is invertible. For this one has to eliminate certain columns of Q which can be realized when considering a LU factorization of $Q^\top FQ$.

4 Numerical examples

4.1 Local preconditioners

We first test the local preconditioner as defined in (3.2) to solve a linear system with the stiffness matrix (3.1) where $R_{i,h} = D_{1,h}$ is the Galerkin discretisation of the Yukawa hypersingular integral operator, $\eta_i = 1$, and $\Omega_i = (0,1)^3$ is the unit cube. As solver we use a standard GMRES algorithm with a relative accuracy of $\varepsilon = 10^{-8}$. The right hand side of the linear system to be solved is given by an evaluation of the sinus function, i.e. no geometric information is used. In Table 1, the iteration numbers are given for several wave numbers κ when solving the preconditioned local system (3.1). Note that N is the number of triangles, and M is the number of nodes.

Table 1: Number of iterations for preconditioned local system.

N	M	$\kappa = 1.0$	$\kappa = 2.0$	$\kappa = 4.0$	$\kappa = 6.0$	$\kappa = 8.0$
12	8	13	13	15	16	17
48	26	20	22	30	37	46
192	98	24	25	39	52	69
768	386	26	27	42	58	80
3072	1538	28	29	43	57	81
12288	6146	29	29	42	56	79

The number of iterations indicate that the proposed preconditioner is optimal with respect to the boundary element mesh size h, as predicted in theory [6]. However, the spectral condition number of the preconditioned system depends on the properties of the double layer potential, which involves the dependency on the domain Ω, and on the wave number κ. In particular, for an increasing wave number κ also the number of iterations increases mildly.

4.2 Global preconditioners

As numerical example we consider the Neumann boundary value problem (2.1) for the unit cube $\Omega = (0,1)^3$ which is decomposed into $p = n^3$ subdomains Ω_i. The subdomain boundaries $\Gamma_i = \partial\Omega_i$ are discretised uniformly by using 24 plane triangular elements and 14 nodes on the coarsest level ($L = 0$), and are refined uniformly on the next levels. As iterative solver we use a projected GMRES algorithm with a relative accuracy of $\varepsilon = 10^{-8}$. The boundary datum g is chosen such that the exact solution is the fundamental solution $U_\kappa^*(x, \overline{x})$ of the Helmholtz equation with the source $\overline{x} = (-0.2, 0, 0)^\top$.
In Tables 2–4 we present the number of global GMRES iterations for different wave numbers κ, and different numbers p of subdomains. By m_i we denote the number of local planar waves as used in the construction of the global preconditioner. We see, that the number of iterations decreases as the number of planar waves increases, while the number of iterations increases with an increasing wave number, as expected.

Table 2: Number of iterations in the case $\kappa = 2$.

	p=8					p=27					p=64					p=125				
$L \setminus m_i$	0	2	4	6	8	0	2	4	6	8	0	2	4	6	8	0	2	4	6	8
0	23	15	11	7	8	54	25	17	12	13	100	32	19	13	14	165	37	21	14	14
1	29	21	19	17	16	58	32	26	24	22	105	39	28	25	24	156	43	31	26	26
2	31	24	23	21	21	59	34	28	27	25	104	41	32	23	25	137	44	35	30	30
3	35	29	27	26	24	62	39	35	33	30	105	45	37	34	35	137	47	40	31	31

Table 3: Number of iterations in the case $\kappa = 4$.

	p=8					p=27					p=64					p=125				
$L \setminus m_i$	0	2	4	6	8	0	2	4	6	8	0	2	4	6	8	0	2	4	6	8
0	27	19	13	6	12	69	36	23	17	22	130	47	28	18	19	215	59	31	18	17
1	32	23	21	18	16	70	38	33	28	24	128	52	39	33	29	205	60	40	35	33
2	35	27	24	22	21	68	39	33	29	27	121	50	41	36	29	191	56	44	38	34
3	40	31	27	25	24	71	44	39	36	31	119	52	45	40	36	181	57	49	43	38

Table 4: Number of iterations in the case $\kappa = 8$.

	p=8					p=27					p=64					p=125				
$L \setminus m_i$	0	2	4	6	8	0	2	4	6	8	0	2	4	6	8	0	2	4	6	8
0	43	31	16	2	1	88	56	39	25	2	157	93	69	49	24	254	138	88	57	35
1	49	39	28	21	17	80	53	42	36	31	162	89	70	59	50	267	128	95	74	64
2	54	41	31	25	22	81	55	44	37	32	145	77	61	53	47	252	124	81	64	57

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