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Convergence analysis of a Galerkin boundary element method for the Dirichlet Laplacian eigenvalue problem

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Abstract

In this paper, a rigorous convergence and error analysis of a Galerkin boundary element method for the Dirichlet Laplacian eigenvalue problem is presented. The formulation of the eigenvalue problem in terms of a boundary integral equation yields a nonlinear boundary integral operator eigenvalue problem. This nonlinear eigenvalue problem and its Galerkin approximation are analyzed in the framework of eigenvalue problems for holomorphic Fredholm operator–valued functions. The convergence of the approximation is shown and quasi–optimal error estimates are presented. Numerical experiments are given confirming the theoretical results.

1 Introduction

We consider a Galerkin boundary element method for the Dirichlet eigenvalue problem of the Laplace operator,

$$-\Delta u(x) = \lambda u(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma = \partial\Omega, \tag{1.1}$$

where we assume that $\Omega \subset \mathbb{R}^3$ is a bounded simply connected Lipschitz domain. Although the eigenvalue problem (1.1) is linear, boundary integral formulations lead to nonlinear eigenvalue problems for related boundary integral operators. This is due to the nonlinear dependence of the eigenvalue parameter in the fundamental solution. Different to finite element approaches which require a discretization of the computational domain Ω, the use of boundary integral formulations and boundary element methods to solve the eigenvalue problem needs only a discretization of the boundary Γ.

For the discretization of the boundary integral eigenvalue problem collocation schemes [4, 5, 14, 16, 17, 22, 27, 29] and Galerkin methods [6, 7, 31, 34] are considered. Both methods yield algebraic nonlinear eigenvalue problems where the matrix entries are transcendental functions with respect to the eigenparameter. In several publications different approaches are suggested to approximate first the nonlinear boundary integral operator eigenvalue problem by a polynomial one. A Taylor polynomial approximation of the fundamental solution with respect to the eigenparameter is suggested by so-called multiple reciprocity methods [4, 5, 16, 17, 27]. In [20] a polynomial interpolation of the fundamental solution is considered to find eigenvalues in a predescribed interval.

To our knowledge, a rigorous and comprehensive numerical analysis for the discretization of boundary integral operator eigenvalue problems has not be done so far. Only in few works [6, 7, 31] the issue of the numerical analysis is addressed. In [31] we analyzed a Galerkin discretization of a Newton scheme for the approximation of a boundary integral operator eigenvalue problem and derived error estimates for eigenpairs for simple eigenvalues. However, the case of multiple eigenvalues was not considered in the numerical analysis nor the questions if discrete spurious eigenvalues are excluded and how the multiplicities of the eigenvalues are effected by the discretization.

In this paper these issues will be covered for a Galerkin boundary element approximation of the Dirichlet Laplacian eigenvalue problem. We describe the boundary integral formulation of the eigenvalue problem (1.1) as eigenvalue problem for a holomorphic Fredholm operator–valued function. The concept of eigenvalue problems for holomorphic Fredholm operator–valued functions [9, 10, 21, 37] is a generalization of the theory for eigenvalue problems of bounded linear operators and provides an important tool for the numerical analysis of approximations of such eigenvalue problems. This concept is also used in boundary integral approaches for the theoretical study of eigenvalue perturbation problems for the Laplacian and the Lamé system with respect to the variation of the domain and small inclusions [1]. The analysis of the approximation of eigenvalue problems for holomorphic Fredholm operator functions has a long tradition [12, 15, 18, 19, 35, 36] and is usually done in the framework of the concepts of the discrete approximation scheme [32] and the regular convergence of operators [2, 11]. In this framework a comprehensive numerical analysis is given by Karma [18, 19]. These results are valid for the Galerkin setting in the case that the underlying operators of the eigenvalue problem are compact perturbations of an elliptic operator. This enables us to apply these results to the numerical analysis of the Galerkin boundary element approximation of the Dirichlet Laplacian eigenvalue problem.

The focus of the present paper is the rigorous analysis of the boundary integral formulation of the Dirichlet Laplacian eigenvalue problem and its Galerkin discretization. Algorithms for the solution of the discretized boundary integral eigenvalue problem will be not discussed here but have been addressed in [34, Ch. 6, 7]. However, robust and efficient algorithms for such nonlinear eigenvalue problems remain an important issue of the ongoing work.

This paper is organized as follows: in Section 2 we introduce an equivalent boundary integral formulation of the Dirichlet Laplacian eigenvalue problem and show that it is a holomorphic eigenvalue problem for a Fredholm operator–valued function. The concept of eigenvalue problems for holomorphic Fredholm operator–valued functions is introduced in Section 3 and important properties of such eigenvalue problems are summarized. In

Section 4 the results of the numerical analysis for the Galerkin approximation of a class of such eigenvalue problems are presented. The properties of the boundary integral operator eigenvalue problem are analyzed in Section 5. The general results of the numerical analysis of Section 4 are applied to the Galerkin boundary element discretization of the Dirichlet Laplacian eigenvalue problem in Section 6. Numerical experiments are presented in Section 7 which confirm the theoretical results.

2 Boundary integral operator formulations

The Dirichlet eigenvalue problem (1.1) of the Laplace operator can be written in terms of the Helmholtz equation with $\lambda = \kappa^2$,

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma. \tag{2.1}$$

Any solution $u \in H_0^1(\Omega)$ of the boundary value problem (2.1) can be described by using the representation formula [23, 30],

$$u(x) = (\widetilde{V}(\kappa)\frac{\partial}{\partial n}u)(x) := \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}\frac{\partial}{\partial n_y}u(y)ds_y \quad \text{for } x \in \Omega, \tag{2.2}$$

where $\widetilde{V}(\kappa) \in \mathcal{L}(H^{-1/2}(\Gamma), H^1(\Omega))$ is the single layer potential of the Helmholtz equation. Applying the interior trace operator γ_0^{int} to (2.2) we obtain a boundary integral equation

$$0 = \gamma_0^{\text{int}} u(x) = \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}\frac{\partial}{\partial n_y}u(y)ds_y \quad \text{for } x \in \Gamma \tag{2.3}$$

which is obviously fulfilled for any solution $(\kappa, u) \in \mathbb{C} \times H_0^1(\Omega)$ of (2.1). The associated boundary integral operator

$$(V(\kappa)t)(x) := \gamma_0^{\text{int}}(\widetilde{V}(\kappa)t)(x) = \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}t(y)ds_y \quad \text{for } x \in \Gamma$$

is the single layer boundary integral operator of the Helmholtz equation and defines a continuous mapping $V(\kappa) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ [23, 30]. The eigenvalue problem (2.1) allows an equivalent boundary integral formulation in terms of the single layer boundary integral operator $V(\kappa)$ which will be shown in the next theorem.

Theorem 2.1. *If $(\kappa^2, u) \in \mathbb{R} \times H_0^1(\Omega) \setminus \{0\}$ is an eigenpair of (2.1), then the normal derivative $\frac{\partial}{\partial n}u$ on Γ is a non trivial solution of the boundary integral equation*

$$(V(\kappa)\frac{\partial}{\partial n}u)(x) = 0 \quad \textit{for } x \in \Gamma. \tag{2.4}$$

Conversely, if $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma) \setminus \{0\}$ satisfies the boundary integral equation

$$(V(\kappa)t)(x) = 0 \quad \textit{for } x \in \Gamma,$$

then

$$u(x) = (\widetilde{V}(\kappa)t)(x) \quad \textit{for } x \in \Omega$$

defines an eigenfunction of the eigenvalue problem (2.1) *corresponding to the eigenvalue* κ^2*.*

Proof. Let $(\kappa^2, u) \in \mathbb{R} \times H_0^1(\Omega) \setminus \{0\}$ be an eigenpair of the Dirichlet Laplacian eigenvalue problem (2.1). Using the representation formula (2.2) we can write the eigenfunction u as

$$u(x) = (\widetilde{V}(\kappa)\frac{\partial}{\partial n}u)(x) \quad \text{for } x \in \Omega,$$

which implies that $\dfrac{\partial}{\partial n}u \not\equiv 0$ on Γ, since $u \not\equiv 0$ in Ω. The boundary integral equation (2.4) is obviously fulfilled, cf. (2.3).

To prove the second part, let now $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma) \setminus \{0\}$ be a solution of the boundary integral equation $(V(\kappa)t)(x) = 0$ for $x \in \Gamma$. We have to show that (κ^2, u), where

$$u(x) = (\widetilde{V}(\kappa)t)(x) \quad \text{for } x \in \Omega,$$

is an eigenpair of the Dirichlet eigenvalue problem of the Laplace operator. The function u solves the eigenvalue equation (2.1) in Ω, cf. [23, p. 202], and it is zero on the boundary Γ, since

$$\gamma_0^{\text{int}}u(x) = \gamma_0^{\text{int}}(\widetilde{V}(\kappa)t)(x) = (V(\kappa)t)(x) = 0 \quad \text{for } x \in \Gamma.$$

It remains to show that $u \not\equiv 0$ in Ω. We have

$$\frac{\partial}{\partial n}u(x) = \frac{\partial}{\partial n}(\widetilde{V}(\kappa)t)(x) = t(x) \quad \text{for } x \in \Gamma,$$

which follows from the unique solvability of the exterior boundary value problem with certain incoming and outgoing radiating conditions of the Helmholtz equation [23, 26], and the jump relation of the single layer potential operator [23, p. 218], for details see [34, Thm. 2.4.3]. Since $\frac{\partial}{\partial n}u = t \not\equiv 0$ on Γ, we have $u \not\equiv 0$ in Ω, which completes the proof. ☐

Theorem 2.1 provides an equivalent boundary integral formulation of the Dirichlet Laplacian eigenvalue problem (2.1) and it reads as follows: Find $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma) \setminus \{0\}$ such that

$$(V(\kappa)t)(x) = \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}t(y)ds_y = 0 \quad \text{for } x \in \Gamma. \tag{2.5}$$

Since the eigenvalue parameter κ appears nonlinearly in the kernel of the boundary integral, the eigenvalue problem (2.5) is a nonlinear eigenvalue problem.

So far we have analyzed the relation of the boundary integral equation (2.5) and the boundary value problem (2.1) only for real κ. Therefore let us consider now the case that $\text{Im}(\kappa) \neq 0$.

Lemma 2.2. *Let* $Im(\kappa) \neq 0$ *and suppose that* $(V(\kappa)t)(x) = 0$ *for* $x \in \Gamma$ *for some* $t \in H^{-1/2}(\Gamma) \setminus \{0\}$*. Then,* $(\widetilde{V}(\kappa)t)(x) = 0$ *for* $x \in \Omega$*.*

Proof. Let $\mathrm{Im}(\kappa) \neq 0$ and $(V(\kappa)t)(x) = 0$ for $x \in \Gamma$ for some $t \in H^{-1/2}(\Gamma) \setminus \{0\}$. Then, $u(x) = (\widetilde{V}(\kappa)t)(x)$ for $x \in \Omega$ is a solution of the boundary value problem (2.1). For $\mathrm{Im}(\kappa) \neq 0$ we have either $\kappa^2 < 0$ or $\mathrm{Im}(\kappa^2) \neq 0$. But since all eigenvalues of the Dirichlet Laplacian eigenvalue problem are positive, it follows that $u(x) = (\widetilde{V}(\kappa)t)(x) = 0$ for $x \in \Omega$. □

Remark 2.3. *The equivalence of the Dirichlet Laplacian eigenvalue problem* (2.1) *and the boundary integral formulation* (2.5) *is only valid for* $\kappa \in \mathbb{R}$. *There exist* $\kappa \in \mathbb{C} \setminus \mathbb{R}$ *and* $t \in H^{-1/2}(\Gamma) \setminus \{0\}$ *such that* $V(\kappa)t = 0$ *on* Γ *but with* $\widetilde{V}(\kappa)t = 0$ *on* Ω [33, Prop. 7.7.10]. *Such values describe the scattering frequencies of certain technical applications as the Helmholtz resonator and are called scattering poles* [1, Sect. 5.3].

The properties of the single layer boundary integral operator $V(\kappa)$ are well known [23, 28, 30]. For $\kappa = 0$ the operator $V(\kappa)$ is $H^{-1/2}(\Gamma)$-elliptic, i.e., there exists a constant $c_V > 0$ such that

$$\langle V(0)t, \overline{t}\rangle_\Gamma \geq c_V \|t\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } t \in H^{-1/2}(\Gamma), \tag{2.6}$$

where

$$\langle \cdot, \cdot \rangle_\Gamma := \langle \cdot, \cdot \rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$$

is the duality pairing with $L_2(\Gamma)$ as pivot space. The operator

$$V(\kappa) - V(0) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is compact for any $\kappa \in \mathbb{C}$, see [3, Lem. 2.1]. Hence, the operator $V(\kappa)$ is a compact perturbation of the elliptic operator $V(0)$ and therefore Fredholm of index zero.

In the next sections, the boundary integral operator eigenvalue problem (2.5) will be analyzed in the framework of eigenvalue problems of operator–valued functions. Therefore, we consider the single layer potential operator $\widetilde{V}(\kappa)$ and the single layer boundary integral operator $V(\kappa)$ also as operator–valued functions which depend on κ.

Lemma 2.4. *The operator–valued function defined by*

$$\begin{aligned} \widetilde{V} : \mathbb{C} &\to \mathcal{L}(H^{-1/2}(\Gamma), H^1(\Omega)), \\ \kappa &\mapsto \widetilde{V}(\kappa). \end{aligned} \tag{2.7}$$

is holomorphic.

Proof. According to [38, V.3 Thm.1] it is sufficient to show that $\widetilde{V}$ is weakly holomorphic, i.e., the function

$$f_{t,\ell}(\kappa) := \langle V(\kappa)t, \ell\rangle_{H^1(\Omega)\times(H^1(\Omega))^*}$$

is holomorphic on $\mathbb{C}$ for any $t \in H^{-1/2}(\Gamma)$ and for any functional $\ell \in (H^1(\Omega))^*$. For $\kappa \in \mathbb{C}$ and $\ell \in (H^1(\Omega))^*$ we can write

$$\begin{aligned} f_{t,\ell}(\kappa) &= \langle V(\kappa)t, \ell\rangle_{H^1(\Omega)\times(H^1(\Omega))^*} = \langle \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|\cdot - y|}}{|\cdot - y|} t(y) ds_y, \ell\rangle_{H^1(\Omega)\times(H^1(\Omega))^*} \\ &= \sum_{n=0}^{\infty} \kappa^n \langle \frac{1}{4\pi}\int_\Gamma \frac{i^n|\cdot - y|^{n-1}}{n!} t(y) ds_y, \ell\rangle_{H^1(\Omega)\times(H^1(\Omega))^*}. \end{aligned} \tag{2.8}$$

Here we used that the operator $A_n : H^{-1/2}(\Gamma) \to H^1(\Omega)$ defined by

$$(A_n t)(x) := \frac{1}{4\pi} \int_\Gamma \frac{i^n |x-y|^{n-1}}{n!} t(y) ds_y \quad \text{for } x \in \Omega$$

is linear and bounded for every $n \in \mathbb{N}_0$. This property of A_n can be shown for $n \geq 1$ in a similar way as it is done for the case $n = 0$ in [30, Ch. 6], since the kernel of A_n for $n \geq 1$ is smoother than the kernel of A_0. Note that A_0 is the single layer potential operator of the Laplace equation.
The representation (2.8) of the function $f_{t,\ell}$ shows that $f_{t,\ell} : \mathbb{C} \to \mathbb{C}$ is holomorphic for any $t \in H^{-1/2}(\Gamma)$ and $\ell \in (H^1(\Omega))^*$ which proves the assertion. $\square$

The holomorphy of $\widetilde{V}$ implies the holomorphy of $V : \mathbb{C} \to \mathcal{L}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$ with $\kappa \mapsto V(\kappa)$, since we have $V(\kappa) = \gamma_0^{\text{int}} \widetilde{V}(\kappa)$ and $\gamma_0^{\text{int}} \in \mathcal{L}(H^1(\Omega), H^{1/2}(\Gamma))$.

3 Eigenvalue problems for holomorphic Fredholm operator–valued functions

In this section we recall notions and properties of eigenvalue problems for holomorphic Fredholm operator–valued functions where we follow [21, Appendix]. Let X be a reflexive Hilbert space and let $\Lambda \subset \mathbb{C}$ be open and connected. We assume that $A : \Lambda \to \mathcal{L}(X, X)$ is a holomorphic operator–valued function and that $A(\lambda) : X \to X$ is Fredholm with index zero for all $\lambda \in \Lambda$. The set

$$\rho(A) := \{\lambda \in \Lambda : \exists A(\lambda)^{-1} \in \mathcal{L}(X, X)\}$$

is called the resolvent set of A. In the following we will assume that the resolvent set of A is not empty. The complement of the resolvent set $\rho(A)$ in Λ is called spectrum $\sigma(A)$. A number $\lambda_0 \in \Lambda$ is an eigenvalue of A if there exists a non trivial $x_0 \in X \setminus \{0\}$ such that

$$A(\lambda_0) x_0 = 0.$$

x_0 is called an eigenelement of A corresponding to the eigenvalue λ_0. The spectrum $\sigma(A)$ has no cluster points in Λ [9, Cor. IV.8.4] and each $\lambda \in \sigma(A)$ is an eigenvalue of A which follows from the Fredholm alternative. The dimension of the nullspace $\ker A(\lambda_0)$ of an eigenvalue λ_0 is called the geometric multiplicity. An ordered collection of elements $x_0, x_1, \ldots, x_{m-1}$ in X is called a Jordan chain of λ_0 if x_0 is an eigenelement corresponding to λ_0 and if

$$\sum_{j=0}^{n} \frac{1}{j!} A^{(j)}(\lambda_0) x_{n-j} = 0 \quad \text{for } n = 0, 1, \ldots, m-1 \tag{3.1}$$

is satisfied, where $A^{(j)}$ denotes the j-th derivative. The length of any Jordan chain of an eigenvalue is finite [21, Lem. A.8.3]. The maximal length of a Jordan chain formed by an eigenelement x_0 will be denoted by $m(A, \lambda_0, x_0)$. The maximal length of a Jordan chain of the eigenvalue λ_0 is denoted by $\varkappa(A, \lambda_0)$. Elements of any Jordan chain of an eigenvalue λ_0 are called generalized eigenelements of λ_0. The closed linear hull of all eigenelements of an eigenvalue λ_0 is called generalized eigenspace of λ_0 and is denoted by $G(A, \lambda_0)$.

A basis $x_0^1, \ldots, x_0^J$ of the eigenspace of an eigenvalue λ_0 is called canonical if

i. $m(A, \lambda_0, x_0^1) = \varkappa(A, \lambda_0)$,

ii. x_0^j is an eigenelement of the maximal possible order belonging to some direct complement M_j in $\ker A(\lambda_0)$ to the linear hull $\operatorname{span}\{x_0^1, \ldots, x_0^{j-1}\}$, i.e.,

$$m(A, \lambda_0, x_0^j) = \max_{x \in M_j \setminus \{0\}} m(A, \lambda_0, x) \quad \text{for } j = 2, \ldots, J.$$

Obviously, a canonical system of eigenelements of an eigenvalue is not unique, but the order of the eigenelements of two canonical systems coincides [21, Prop. A.4.6]. Let $x_0^1, \ldots, x_0^J$ be a canonical system of the eigenspace $\ker A(\lambda_0)$, then the numbers

$$m_i(A, \lambda_0) := m(A, \lambda_0, x_0^i) \quad \text{for } i = 1, \ldots J$$

are called partial multiplicities of λ_0. The number

$$m(\lambda_0) := \sum_{i=1}^{J} m_i(A, \lambda_0)$$

is called the algebraic multiplicity of λ_0 and coincides with the dimension of the generalized eigenspace $G(A, \lambda_0)$ [21].

Jordan chains and partial multiplicities of an eigenvalue can also be described by the concept of Jordan functions/root functions [10, 18, 25]. A function $\varphi : \Lambda \to X$ is called a Jordan function of order m for A corresponding to an eigenpair (λ_0, x_0) if φ is holomorphic in λ_0 and if

i. $\varphi(\lambda_0) = x_0$ and

ii. λ_0 is a zero of multiplicity m of the function defined by $\lambda \mapsto A(\lambda)\varphi(\lambda)$, i.e.,

$$\begin{aligned} &\frac{d^j}{d\lambda^j} \left[A(\lambda)\varphi(\lambda)\right]_{\lambda=\lambda_0} = 0 \quad \text{for } j = 0, 1, \ldots, m-1 \quad \text{and} \\ &\frac{d^m}{d\lambda^m} \left[A(\lambda)\varphi(\lambda)\right]_{\lambda=\lambda_0} \neq 0. \end{aligned} \tag{3.2}$$

Any Jordan function φ of order m defines a Jordan chain of length m by

$$\varphi(\lambda_0), \frac{1}{1!}\varphi'(\lambda_0), \frac{1}{2!}\varphi^{(2)}(\lambda_0), \ldots, \frac{1}{(m-1)!}\varphi^{(m-1)}(\lambda_0).$$

Conversely, for any Jordan chain $x_0, \ldots, x_{m-1}$ with $m = m(A, \lambda_0, x_0)$ the polynomial

$$\varphi(\lambda) = x_0 + (\lambda - \lambda_0)x_1 + \ldots + (\lambda - \lambda_0)^{m-1}x_{m-1}$$

is a Jordan function of order m of A corresponding to (λ_0, x_0).

4 Galerkin approximation of eigenvalue problems for coercive operator–valued functions

The numerical analysis of eigenvalue problems for holomorphic Fredholm operator–valued functions [12, 18, 19, 35, 36] is usually done in the framework of the concepts of the discrete approximation scheme [32] and the theory of regular convergence of operators [2, 11]. The Galerkin approximation of holomorphic eigenvalue problems where the underlying operators are compact perturbations of an elliptic operator fits into this framework which will be shown in Lemma 4.1. For the Galerkin approximation of such eigenvalue problems the same results of the numerical analysis as given in the above mentioned articles have been derived without using the concept of regular convergence of operators in [34]. Before summarizing these results we specify the required properties of the eigenvalue problem and of its approximation.

We consider the eigenvalue problem for a holomorphic coercive operator–valued function $A : \Lambda \to \mathcal{L}(X, X)$ of the from

$$A(\lambda) = D + C(\lambda), \tag{4.1}$$

where $D \in \mathcal{L}(X, X)$ is X-elliptic, i.e.,

$$(Dx, x)_X \geq c_D \|x\|_X^2 \qquad \text{for all } x \in X$$

and where $C(\lambda) \in \mathcal{L}(X, X)$ is compact for all $\lambda \in \Lambda$. Since $A(\lambda)$ is a compact perturbation of an elliptic operator, it is a Fredholm operator with index zero. For the Galerkin approximation of the eigenvalue problem

$$A(\lambda_0)x_0 = 0 \tag{4.2}$$

we consider a sequence of conforming finite dimensional subspaces $\{X_n\}_{n\in\mathbb{N}} \subset X$ with the approximation property

$$\lim_{n\to\infty} \inf_{x_n \in X_n} \|x - x_n\|_X = 0 \quad \text{for all } x \in X. \tag{4.3}$$

The Galerkin variational eigenvalue problem of (4.2) is to find $(\lambda_n, x_n) \in \Lambda \times X_n \setminus \{0\}$ such that

$$(A(\lambda_n)x_n, v_n)_X = 0 \tag{4.4}$$

is satisfied for all $v_n \in X_n$. Let $P_n : X \to X_n$ be the orthogonal projection of X into X_n, then the orthogonality relation

$$(A(\lambda_n)x_n - P_n A(\lambda_n)x_n, v_n)_X = 0 \quad \text{for all } v_n \in X_n$$

implies that (4.4) is equivalent to

$$P_n A(\lambda_n)x_n = 0. \tag{4.5}$$

In the next lemma the properties of the Galerkin approximation of eigenvalue problems for coercive operator–valued functions are specified which are required in order to apply in the following the results of the numerical analysis of [18, 19].

Lemma 4.1. *Let $A : \Lambda \to \mathcal{L}(X, X)$ be a holomorphic coercive operator function of the form* (4.1)*. Suppose that $\{X_n\}_{n\in\mathbb{N}} \subset X$ is a sequence of finite dimensional spaces which satisfies the approximation property* (4.3) *and let $P_n : X \to X_n$ be the orthogonal projection of X into X_n. Define $B_n(\lambda) : X_n \to X_n$ by*

$$B_n(\lambda)y_n := P_n A(\lambda)y_n$$

for $\lambda \in \Lambda$ and $y_n \in X_n$. Then:

i. The sequence $\{X_n\}_{n\in\mathbb{N}}$ is a discrete approximation of X in the sense of [32]*.*

ii. $B_n(\lambda)$ is Fredholm of index zero for all $\lambda \in \Lambda$.

iii. The sequence $\{B_n(\cdot)\}_{n\in\mathbb{N}}$ is equibounded on each compact $\Lambda_0 \subset \Lambda$, i.e., there exists a $C(\Lambda_0) > 0$ such that

$$\|B_n(\lambda)\|_{\mathcal{L}(X_n,X_n)} \leq C(\Lambda_0) \qquad \text{for all } \lambda \in \Lambda_0, n \in \mathbb{N}.$$

iv. The sequence $\{B_n(\lambda)\}_{n\in\mathbb{N}}$ approximates $A(\lambda)$ for all $\lambda \in \Lambda$, i.e.,

$$\lim_{n\to\infty} \|[B_n(\lambda)P_n - P_n A(\lambda)]x\|_X \to 0 \qquad \text{for all} \lambda \in \Lambda, x \in X.$$

v. The sequence $\{B_n(\lambda)\}_{n\in\mathbb{N}}$ is regular for every $\lambda \in \Lambda$, i.e., for every sequence $\{B_n(\lambda)x_n\}_{n\in\mathbb{N}}$ which has a converging subsequence in X there exists a converging subsequence of $\{x_n\}_{n\in\mathbb{N}}$ in X.

Proof. i. The approximation property (4.3) of $\{X_n\}_{n\in\mathbb{N}}$ is sufficient that $\{X_n\}_{n\in\mathbb{N}}$ is a discrete approximation of X in the sense of [32].
ii. The operator $B_n(\lambda) : X_n \to X_n$ is a linear and bounded operator which maps from a finite dimensional space into itself. This implies that $B_n(\lambda)$ is Fredholm of index zero.
iii. Using $\|P_n\|_{\mathcal{L}(X,X)} = 1$ and the holomorphy of $A : \Lambda \to \mathcal{L}(X, X)$ we get

$$\sup_{\lambda\in\Lambda_0} \|B_n(\lambda)\|_{\mathcal{L}(X_n,X_n)} \leq \sup_{\lambda\in\Lambda_0} \|A(\lambda)\|_{\mathcal{L}(X,X)} \leq C(\Lambda_0).$$

iv. The approximation property (4.3) implies that $\|(P_n - I_X)x\|_X \to 0$ as $n \to \infty$ for all $x \in X$. Hence, we conclude that

$$\|B_n(\lambda)P_n - P_n A(\lambda)]x\|_X = \|P_n A(\lambda)P_n - P_n A(\lambda)]x\|_X \leq c(\lambda)\|(P_n - I_X)x\|_X \to 0$$

for all $\lambda \in \Lambda$ and $x \in X$.
v. A proof of the regularity of the sequence $\{B_n(\lambda)\}_{n\in\mathbb{N}}$ is given in [12, Satz 32]. ☐

The last lemma enables us to apply the results of the numerical analysis in [18, 19] to the Galerkin approximation (4.4). The first result which was already proven in [13, Satz 2] and [12, Satz 32] shows that every eigenvalue of the continuous eigenvalue problem (4.2) may be approximated by a converging sequence of eigenvalues of the Galerkin eigenvalue problem (4.4) and that any limit of a converging sequence of eigenvalues of the Galerkin eigenvalue problem is an eigenvalue of the continuous eigenvalue problem [18, Thm. 1, Thm. 2].

Theorem 4.2. *For each eigenvalue $\lambda_0 \in \sigma(A)$ there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenvalues of the Galerkin eigenvalue problem $P_nA(\lambda_n)x_n = 0$ such that*

$$\lim_{n\to\infty} \lambda_n = \lambda_0.$$

On the other hand, if $\{(\lambda_n, x_n)\}_{n\in\mathbb{N}}$ is a sequence of eigenpairs of the Galerkin eigenvalue problem

$$P_nA(\lambda_n)x_n = 0,$$

where $\|x_n\|_X = 1$, then

$$\lim_{n\to\infty} \lambda_n = \lambda_0 \in \sigma(A).$$

Furthermore, there exists a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ which converges to an eigenelement corresponding to λ_0.

The next result shows that there are no discrete spurious eigenvalues [18, Thm. 1], [34, Lem. 4.2.3].

Lemma 4.3. *Let $\Lambda_c \subset \Lambda$ be a compact set in $\mathbb{C}$ such that $\rho(A) \subset \Lambda_c$. Then, $\rho(P_nA) \subset \Lambda_c$ for sufficiently large $n \in \mathbb{N}$.*

The convergence order both of the eigenvalues and the eigenelements of the Galerkin eigenvalue problem (4.4) to an eigenpair (λ_0, x_0) of the continuous eigenvalue problem (4.2) depends on the approximation property of the sequence of trial spaces $\{X_n\}_{n\in\mathbb{N}}$ with respect to the generalized eigenspace $G(A, \lambda_0)$ and with respect to the generalized eigenspace $G(A^*, \overline{\lambda}_0)$ of the adjoint eigenvalue problem

$$A^*(\overline{\lambda}_0)y_0 := [A(\lambda_0)]^*\, y_0 = 0.$$

Let us therefore define

$$\delta_n := \max_{\substack{x_0\in G(A,\lambda_0)\\ \|x_0\|_X\le 1}} \inf_{x_n\in X_n} \|x_0 - x_n\|_X, \quad \delta_n^* := \max_{\substack{y_0\in G(A^*,\overline{\lambda}_0)\\ \|y_0\|_X\le 1}} \inf_{x_n\in X_n} \|y_0 - x_n\|_X.$$

The following asymptotic error estimate for the eigenvalues can be found in [19, Thm. 3], [34, Thm. 4.3.6]. For the estimate of the eigenelements we refer to [34, Thm. 4.3.7]. Similar results for the estimates of the eigenelements are presented in [35, p. 75] and [36, Thm. 4].

Theorem 4.4. *Let $\Lambda_c \subset \Lambda$ be a compact set such that the boundary $\partial\Lambda_c \subset \rho(A)$ and $\Lambda_c \cap \sigma(A) = \{\lambda_0\}$. Then there exist a constant $C > 0$ and a $N \in \mathbb{N}$ such that*

$$|\lambda_n - \lambda_0| \le C\,(\delta_n\delta_n^*)^{1/\varkappa(A,\lambda_0)} \tag{4.6}$$

holds for all $\lambda_n \in \sigma(P_nA) \cap \Lambda_c$ and all $n \ge N$. Furthermore, for the corresponding eigenelements x_n of λ_n with $\|x_n\|_X = 1$ there exists a constant $c > 0$ such that

$$\inf_{x_0\in\ker A(\lambda_0)} \|x_n - x_0\|_X \le c\left(|\lambda_n - \lambda_0| + \max_{\substack{z_0\in\ker A(\lambda_0)\\ \|z_0\|_X\le 1}} \inf_{z_n\in X_n} \|z_0 - z_n\|_X\right) \tag{4.7}$$

holds for all $n \ge N$.

A crucial point of the approximation of eigenvalue problems is the behavior of the multiplicities of the discrete eigenvalues. Galerkin approximations of eigenvalue problems for holomorphic coercive operator–valued functions are stable with respect to the algebraic multiplicities of the eigenvalues [8, Thm. 1], [18, Thm. 5], [34, Thm. 4.4.2].

Theorem 4.5. *Let $\Lambda_c \subset \Lambda$ be compact and connected with a simple rectifiable boundary. Suppose that $\partial\Lambda_c \subset \rho(A)$ and $\Lambda_c \cap \sigma(A) = \{\lambda_0\}$. Then there exists a $N(\Lambda_c) \in \mathbb{N}$ such that for all $n \geq N(\Lambda_c)$ we have*

$$m(A, \lambda_0) = \sum_{\lambda_n \in \sigma(P_n A) \cap \Lambda_c} m(P_n A, \lambda_n). \tag{4.8}$$

Since the algebraic multiplicity of an eigenvalue λ_0 coincides with the dimension of the generalized eigenspace $G(A, \lambda_0)$, the result (4.8) implies that

$$\dim G(A, \lambda_0) = \sum_{\lambda_n \in \sigma(P_n A) \cap \Lambda_c} \dim G(P_n A, \lambda_n).$$

5 Properties of the boundary integral operator eigenvalue problem

The variational formulation of the boundary integral operator eigenvalue problem (2.5) is to find $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma) \setminus \{0\}$ such that

$$\langle V(\kappa)t, \overline{v}\rangle_\Gamma = 0 \tag{5.1}$$

is satisfied for all $v \in H^{-1/2}(\Gamma)$. In the following we will also use a representation of the variational formulation (5.1) in terms of the inner product in $H^{-1/2}(\Gamma)$. To this end, consider the Riesz map $J : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, then

$$(u, w)_{H^{1/2}(\Gamma)} = \langle u, Jw\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \langle u, Jw\rangle_\Gamma$$

holds for all $u, w \in H^{1/2}(\Gamma)$. The operator $\mathcal{I} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ defined by the complex conjugate

$$\mathcal{I}w := \overline{Jw} \quad \text{for } w \in H^{1/2}(\Gamma)$$

is an isomorphism. Hence, we can write

$$(u, w)_{H^{1/2}(\Gamma)} = \langle u, Jw\rangle_\Gamma = \langle u, \overline{\mathcal{I}w}\rangle_\Gamma$$

for all $u, w \in H^{1/2}(\Gamma)$. Using that $\mathcal{I}^{-1} = \mathcal{I}^*$, we can represent the sesquilinear form $\langle \cdot, \bar{\cdot}\rangle_\Gamma$ by the inner products of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$

$$\langle u, \overline{v}\rangle_\Gamma = \langle u, \overline{\mathcal{I}\mathcal{I}^* v}\rangle_\Gamma = (u, \mathcal{I}^* v)_{H^{1/2}(\Gamma)} = (\mathcal{I}u, v)_{H^{-1/2}(\Gamma)} \tag{5.2}$$

for all $u \in H^{1/2}(\Gamma)$ and $v \in H^{-1/2}(\Gamma)$. The variational formulation (5.1) of the eigenvalue problem (2.5) can therefore be written as

$$\langle V(\kappa)t, \overline{v}\rangle_\Gamma = (\mathcal{I}V(\kappa)t, v)_{H^{-1/2}(\Gamma)} = 0 \tag{5.3}$$

for all $v \in H^{-1/2}(\Gamma)$.

Theorem 5.1. *Consider the operator–valued function*

$$\begin{aligned} \mathcal{I}V : \mathbb{C} &\to \mathcal{L}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma)), \\ \kappa &\mapsto \mathcal{I}V(\kappa). \end{aligned}$$

Then:

i. The function $\mathcal{I}V : \mathbb{C} \to \mathcal{L}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma))$ is holomorphic and the operator $\mathcal{I}V(\kappa)$ is a compact perturbation of the $H^{-1/2}(\Gamma)$-elliptic operator $\mathcal{I}V(0)$ for all $\kappa \in \mathbb{C}$.

ii. The spectra of V and $\mathcal{I}V$ coincide and

$$\ker V(\kappa) = \ker \mathcal{I}V(\kappa)$$

for any $\kappa \in \mathbb{C}$. Further, for any eigenvalue $\kappa_0 \in \sigma(V)$ the maximal length of a Jordan chain and the algebraic multiplicity are equal for V and $\mathcal{I}V$,

$$\varkappa(V, \kappa_0) = \varkappa(\mathcal{I}V, \kappa_0), \qquad m(V, \kappa_0) = m(\mathcal{I}V, \kappa_0).$$

Proof. The assertions follow from the fact that $\mathcal{I} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is an isomorphism, see [21, Appendix]. ☐

For the error estimates of the Galerkin approximation of the boundary integral operator eigenvalue problem (5.3) we have to consider the Hilbert space adjoint operator of $\mathcal{I}V(\kappa)$.

Lemma 5.2. *Let $\kappa \in \mathbb{R}$, then*

$$[\mathcal{I}V(\kappa)]^* = \mathcal{I}V(-\kappa).$$

Proof. Let $\kappa \in \mathbb{R}$ and $t, w \in H^{-1/2}(\Gamma)$, then

$$\begin{aligned} (\mathcal{I}V(\kappa)t, v)_{H^{-1/2}(\Gamma)} = \langle V(\kappa)t, \overline{v}\rangle_\Gamma &= \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|} t(y)ds_y\overline{v(x)}ds_x \\ &= \frac{1}{4\pi}\int_\Gamma\int_\Gamma \overline{\frac{e^{-i\kappa|x-y|}}{|x-y|} v(x)ds_x t(y)} ds_y \\ &= \frac{1}{4\pi}\int_\Gamma\int_\Gamma \overline{\frac{e^{-i\kappa|x-y|}}{|x-y|} v(x)ds_x \overline{t(y)}} ds_y = \overline{\langle V(-\kappa)v, \overline{t}\rangle}_\Gamma \\ &= \overline{(\mathcal{I}V(-\kappa)v, t)}_{H^{-1/2}(\Gamma)} = (t, \mathcal{I}V(-\kappa)v)_{H^{-1/2}(\Gamma)}. \end{aligned}$$

Hence, $[\mathcal{I}V(\kappa)]^* = \mathcal{I}V(-\kappa)$. ☐

The maximal length of a Jordan chain of an eigenvalue affects the rate of convergence of its approximation. An optimal convergence rate can be achieved only if the maximal length of a Jordan chain of an eigenvalue is one, cf. Theorem 4.4.

Lemma 5.3. *Let $\kappa_0 \in \mathbb{R}$ be an eigenvalue of the eigenvalue problem* (2.5)*. Then, for the maximal length of a Jordan chain of κ_0 we have*

$$\varkappa(V, \kappa_0) = 1.$$

Proof. Let $(\kappa_0, t_0) \in \mathbb{R} \times H^{-1/2}(\Gamma)$ be an eigenpair of the boundary integral operator eigenvalue problem (2.5). Define $u_0(x) = (\widetilde{V}(\kappa_0)t_0)(x)$ for $x \in \Omega$, then (κ_0^2, u_0) is by Theorem 2.1 an eigenpair of the Dirichlet Laplacian eigenvalue problem (2.1). Let $\varphi(\kappa)$ be a Jordan function of V corresponding to (κ_0, t_0), i.e., in a neighborhood $U_{\varepsilon_0}(\kappa_0)$ of κ_0 the function $\varphi : U_{\varepsilon_0}(\kappa_0) \to H^{-1/2}(\Gamma)$ is holomorphic, $\varphi(\kappa_0) = t_0$, and $V(\kappa_0)\varphi(\kappa_0) = 0$. The maximal length of a Jordan chain of κ_0 beginning with t_0 is the order of κ_0 as zero of the function $V(\kappa)\varphi(\kappa)$. Therefore we have to show that the multiplicity of the zero κ_0 of $V(\kappa)\varphi(\kappa)$ is one. Since $V(\kappa)\varphi(\kappa)$ is holomorphic and $V(\kappa_0)\varphi(\kappa_0) = 0$, there exists a neighborhood $U_{\delta_0}(\kappa_0) \subset U_{\varepsilon_0}(\kappa_0)$ of κ_0 such that

$$V(\kappa)\varphi(\kappa) = (\kappa^2 - \kappa_0^2)\psi(\kappa) \tag{5.4}$$

with a holomorphic function $\psi : U_{\delta_0}(\kappa_0) \to H^{1/2}(\Gamma)$. Note that $\kappa_0 \neq 0$ and that therefore the representation (5.4) is possible. Define

$$u(\kappa)(x) = (\widetilde{V}(\kappa)\varphi(\kappa))(x) \quad \text{for } x \in \Omega,$$

then $u(\kappa)$ is a solution of the boundary value problem

$$-\Delta u(\kappa)(x) - \kappa^2 u(\kappa)(x) = 0 \;\text{ for } x \in \Omega, \quad \gamma_0(u(\kappa))(x) = (\kappa^2 - \kappa_0^2)\psi(\kappa)(x) \;\text{ for } x \in \Gamma.$$

Using Green's second identity we obtain

$$\int_\Omega u(\kappa)(x)\overline{u_0}(x)dx = \frac{1}{\kappa^2 - \kappa_0^2}\int_\Gamma \gamma_0 u(\kappa)(x)\gamma_1\overline{u_0}(x)ds_x = \int_\Gamma \psi(\kappa)(x)\gamma_1\overline{u_0}(x)ds_x.$$

By Theorem 2.4, the function $\widetilde{V} : \mathbb{C} \to \mathcal{L}(H^{-1/2}(\Gamma), H^1(\Omega))$ is holomorphic. Together with the holomorphy of the function $\varphi : U_{\varepsilon_0}(\kappa_0) \to H^{-1/2}(\Gamma)$ it follows that $u(\kappa) = \widetilde{V}(\kappa)\varphi(\kappa)$ is holomorphic in $U_{\varepsilon_0}(\kappa_0)$. This implies that $u(\kappa) \to u(\kappa_0) = u_0$ as $\kappa \to \kappa_0$. Therefore we have

$$0 \neq \int_\Omega u_0(x)\overline{u_0}(x)dx = \int_\Gamma \psi(\kappa_0)(x)\gamma_1 u_0(x)ds_x,$$

since $\psi(\kappa)$ is also holomorphic. Hence, $\psi(\kappa_0) \not\equiv 0$. Using the definition (5.4) of $\psi(\kappa)$, we conclude that κ_0 is a simple zero of $V(\kappa)\varphi(\kappa)$. Thus, the maximal length of a Jordan chain of κ_0 beginning with t_0 is one. ∎

6 Galerkin approximation of the boundary integral operator eigenvalue problem

In the following we will apply the results of the numerical analysis of Section 4 to the Galerkin discretization of the eigenvalue problem (5.3). Let $\{X_n\}_{n\in\mathbb{N}} \subset H^{-1/2}(\Gamma)$ be some sequence of conforming boundary element spaces with

$$\lim_{n\to\infty} \inf_{z_n\in X_n} \|z - z_n\|_{H^{-1/2}(\Gamma)} = 0 \quad \text{for all } z \in H^{-1/2}(\Gamma).$$

The Galerkin approximation of the eigenvalue problem (5.3) is to find $(\kappa_n, t_n) \in \mathbb{C}\times X_n\backslash\{0\}$ such that

$$(\mathcal{I}V(\kappa_n)t_n, v_n)_{H^{-1/2}(\Gamma)} = \langle V(\kappa_n)t_n, \overline{v_n}\rangle_\Gamma = 0 \tag{6.1}$$

is satisfied for all $v_n \in X_n$. Let $P_n : X_n \to H^{-1/2}(\Gamma)$ be the orthogonal projection of X into X_n, then the Galerkin variational formulation (6.1) is equivalent to

$$P_n\mathcal{I}V(\kappa_n)t_n = 0. \tag{6.2}$$

Since $\mathcal{I}V : \mathbb{C} \to \mathcal{L}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma))$ is a holomorphic operator–valued function where the underlying operator is a compact perturbation of an elliptic operator, the numerical analysis of Section 4 can be applied to (6.2).

Lemma 6.1.

i. For each eigenvalue κ_0 of $\mathcal{I}V$ there exists a sequence of eigenpairs $\{(\kappa_n, t_n)\}_{n\in\mathbb{N}}$ of the Galerkin variational problem (6.2) *with $\|t_n\|_{H^{-1/2}(\Gamma)} = 1$ such that*

$$\lim_{n\to\infty} |\kappa_0 - \kappa_n| = 0$$

and

$$\lim_{n\to\infty} \inf_{t_0\in\ker \mathcal{I}V(\kappa_0)} \|t_0 - t_n\|_{H^{-1/2}(\Gamma)} = 0.$$

ii. Let $\{\kappa_n\}_{n\in\mathbb{N}}$ be a sequence of eigenvalues of the Galerkin variational problem (6.2) *with*

$$\lim_{n\to\infty} \kappa_n = \mu_0.$$

Then μ_0 is an eigenvalue of $\mathcal{I}V$.

The error estimate for an eigenpair (κ_n, t_n) of (6.2) depends on the approximation property ε_n of the trial space X_n with respect to the eigenspace of the corresponding continuous eigenvalue κ_0,

$$\varepsilon_n = \max_{\substack{t_0\in\ker \mathcal{I}V(\kappa_0)\\ \|t_0\|_{H^{-1/2}(\Gamma)}\le 1}} \inf_{z_n\in X_n} \|t_0 - z_n\|_{H^{-1/2}(\Gamma)}.$$

Theorem 6.2. *Let $\kappa_0 \in \sigma(\mathcal{I}V) \cap \mathbb{R}$. Suppose that $\Lambda_c \subset \mathbb{C}$ is a compact set such that $\partial\Lambda_c \subset \rho(\mathcal{I}V)$ and $\Lambda_c \cap \sigma(\mathcal{I}V) = \{\kappa_0\}$. Then there exist a constant $C > 0$ and a $N \in \mathbb{N}$ such that for all $n \geq N$ the estimate*

$$|\kappa_0 - \kappa_n| \leq C\varepsilon_n^2 \tag{6.3}$$

holds for all $\kappa_n \in \sigma(P_n\mathcal{I}V) \cap \Lambda_c$. Further, for any $t_n \in \ker P_n\mathcal{I}V(\kappa_0)$ with $\|t_n\|_{H^{-1/2}(\Gamma)} = 1$ there exists a constant $c > 0$ such that for all $n \geq N$ we have

$$\inf_{t_0 \in \ker \mathcal{I}V(\kappa_0)} \|t_0 - t_n\|_{H^{-1/2}(\Gamma)} \leq c\,(|\kappa_0 - \kappa_n| + \varepsilon_n)\,. \tag{6.4}$$

Proof. Applying Theorem 4.4 to (6.2) we obtain for sufficiently large $n \in \mathbb{N}$ the error estimate

$$|\kappa_n - \kappa_0| \leq C(\delta_n\delta_n^*)^{1/\varkappa(\mathcal{I}V,\kappa_0)} \quad \text{for all } \kappa_n \in \sigma(P_n\mathcal{I}V) \cap \Lambda_c,$$

where δ_n and δ_n^* represent the approximation property of the trial space X_n with respect to the generalized eigenspace $G(\mathcal{I}V, \kappa_0)$ and $G((\mathcal{I}V)^*, \overline{\kappa}_0)$, respectively. Lemma 5.3 shows that $\varkappa(\mathcal{I}V, \kappa_0) = 1$ which implies that the generalized eigenspace $G(\mathcal{I}V\kappa_n)$ coincides with the eigenspace $\ker \mathcal{I}V(\kappa_n)$. Hence, $\delta_n = \varepsilon_n$. It remains to consider the adjoint of $\mathcal{I}V(\kappa_0)$. Since κ_0 is assumed to be real, we have for the adjoint $[\mathcal{I}V(\kappa_0)]^* = \mathcal{I}V(-\kappa_0)$ by Lemma 5.2. Using that $\ker \mathcal{I}V(\kappa_0) = \ker \mathcal{I}V(-\kappa_0)$, the error estimates (6.3) and (6.4) follow. ☐

Note that the error estimate for the eigenfunctions (6.4) is quasi–optimal. As already used in the last proof, the generalized eigenspace $G(\mathcal{I}V, \kappa_0)$ coincides with the eigenspace $\ker \mathcal{I}V(\kappa_0)$. This implies that the geometric multiplicity of an eigenvalue is equal to its algebraic multiplicity. Using Theorem 4.5 we get the following stability result for the multiplicities of the eigenvalues of the Galerkin approximation (6.2).

Theorem 6.3. *Let $\kappa_0 \in \sigma(\mathcal{I}V) \cap \mathbb{R}$. Suppose that Λ_c is compact and connected with a simple rectifiable boundary such that $\partial\Lambda_c \subset \rho(\mathcal{I}V)$ and $\sigma(\mathcal{I}V) \cap \Lambda_c = \{\kappa_0\}$. Then, there exists a $N \in \mathbb{N}$ such that for all $n \geq N$*

$$\dim\ker \mathcal{I}V(\kappa_0) = \sum_{\kappa_n \in \sigma(\mathcal{I}P_nV) \cap \Lambda_c} \dim\ker P_n\mathcal{I}V(\kappa_n).$$

Next, we want to consider the approximation property of the space of piecewise constant functions $S_h^0(\Gamma) \subset H^{-1/2}(\Gamma)$ for the discretization of the boundary integral operator eigenvalue problem (2.5). Let $S_h^0(\Gamma)$ be defined with respect to some globally quasi–uniform boundary element mesh with mesh size h. Using the approximation property of $S_h^0(\Gamma)$, see [28, Ch. 4], [30, Ch. 10], we get for a discrete eigenpair (κ_h, t_h) with $\|t_h\|_{H^{-1/2}(\Gamma)} = 1$ the following asymptotic error estimates,

$$|\kappa_0 - \kappa_h| \leq ch^{2s+1} \max_{w_0 \in \ker \mathcal{I}V(\kappa_0),\, \|w_0\|_{H^{-1/2}(\Gamma)}=1} |w_0|_{H^s_{\mathrm{pw}}(\Gamma)} \tag{6.5}$$

and

$$\inf_{\substack{t_0 \in \ker \mathcal{I}V(\kappa_0), \\ \|t_0\|_{H^{-1/2}(\Gamma)}=1}} \|t_0 - t_h\|_{H^{-1/2}(\Gamma)} \leq Ch^{s+1/2} \max_{w_0 \in \ker \mathcal{I}V(\kappa_0), \|w_0\|_{H^{-1/2}(\Gamma)}=1} |w_0|_{H^s_{\mathrm{pw}}(\Gamma)}$$

when assuming that $\ker \mathcal{I}V(\kappa_0) \subset H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in (-1/2, 1]$. The Sobolev spaces $H^s_{\mathrm{pw}}(\Gamma)$ are defined with respect to a piecewise smooth boundary Γ, see, e.g., [30, Ch. 2].

Remark 6.4. *The spectrum $\sigma(\mathcal{I}V)$ contains two types of eigenvalues. The real eigenvalues of $\sigma(\mathcal{I}V)$ correspond to the eigenvalues of the Laplace operator with Dirichlet boundary conditions and whereas the nonreal ones correspond to the scattering poles, see Remark* 2.3. *According to Lemma* 4.3*, the Galerkin approximation* (6.2) *of the eigenvalue problem for the function $\mathcal{I}V$ exhibits no discrete spurious eigenvalues. In practical computations, the decision weather an approximation κ_n of an eigenvalue of $\sigma(\mathcal{I}V)$ which lies close to the real axis is an eigenvalue of the Laplacian or a scattering pole can be done by computing $(\widetilde{V}(\kappa_n)t_n)(x)$ for $x \in \Omega$. If κ_n is an approximation of an eigenvalue of the Laplacian, then $(\widetilde{V}(\kappa_n)t_n)(x) \not\equiv 0$ for $x \in \Omega$. In the case that κ_n is an approximation of a scattering pole, Lemma* 2.2 *implies that $\widetilde{V}(\kappa_n)t_n \approx 0$ on Ω.*

7 Numerical results

In this section we present some numerical results of the Galerkin approximation of the boundary integral formulation (2.5) of the Dirichlet Laplacian eigenvalue problem. As domain Ω we choose the cube $\Omega = (0, \frac{1}{2})^3$. The eigenvalues are given by

$$\lambda_k \; = \; 4\pi^2 \left[k_1^2 + k_2^2 + k_3^2\right]$$

and the associated eigenfunctions are

$$u_k(x) \; = \; (\sin 2\pi k_1 x_1)(\sin 2\pi k_2 x_2)(\sin 2\pi k_3 x_3).$$

It turns out that the first eigenvalue ($k_1 = k_2 = k_3 = 1$)

$$\lambda_1 \; = \; 12\pi^2, \quad \kappa_1 \; = \sqrt{\lambda_1} = \; 2\sqrt{3}\pi$$

is simple, while the second eigenvalue ($k_1 = 2, k_2 = k_3 = 1$)

$$\lambda_2 \; = \; 24\pi^2, \quad \kappa_2 \; = \sqrt{\lambda_2} = \; 2\sqrt{6}\pi$$

is multiple.

We use the space of piecewise constant functions $S^0_h(\Gamma)$ as trial space where the boundary Γ is decomposed into n_h uniform triangles τ_ℓ. Let $\left\{\psi^h_\ell\right\}_{\ell=1}^{n_h}$ be a basis of $S^0_h(\Gamma)$ such that ψ^h_ℓ is constant one on the boundary element τ_ℓ and elsewhere zero. The Galerkin variational eigenvalue problem reads then as follows: Find $(\kappa_h, t_h) \in \mathbb{C} \times S^0_h(\Gamma) \setminus \{0\}$ such that

$$(\mathcal{I}V(\kappa_h)t_h, v_h)_{H^{-1/2}(\Gamma)} = \langle V(\kappa_h)t_h, \overline{v_h}\rangle_\Gamma = 0 \tag{7.1}$$

is satisfied for all $v_h \in S_h^0(\Gamma)$. Set

$$t_h = \sum_{\ell=1}^{n_h} t_\ell \psi_\ell^h,$$

then the variational problem (7.1) is equivalent to the algebraic nonlinear eigenvalue problem: Find $(\kappa_h, \underline{t}) \in \mathbb{C} \times \mathbb{C}^{n_h} \setminus \{\underline{0}\}$ such that

$$V_h(\kappa_h)\underline{t} = 0, \tag{7.2}$$

where

$$V_h(\kappa_h)[k,\ell] := \frac{1}{4\pi} \int_{\tau_\ell} \int_{\tau_k} \frac{e^{i\kappa_h|x-y|}}{|x-y|} ds_y ds_x \qquad \text{for } k, \ell = 1, \ldots, n_h.$$

We use the inverse iteration for nonlinear eigenvalue problems [24] to solve the algebraic nonlinear eigenvalue problem (7.2). The numerical results of the boundary element approximations for the eigenvalue κ_1 and κ_2 are presented in Table 1 and Table 2. A cubic convergence order with respect to mesh size h of the boundary discretization can be observed which confirms the theoretical error estimate (6.5).

L	h	n_h	$\kappa_{1,h}^{\text{BEM}}$	$\lvert\kappa_1 - \kappa_{1,h}^{\text{BEM}}\rvert$	eoc
2	2^{-3}	384	10.8768 - 1.1e-05i	5.986e-03	-
3	2^{-4}	1536	10.8821 - 2.4e-07i	6.962e-04	3.1
4	2^{-5}	6144	10.8827 - 6.0e-09i	8.619e-05	3.0

Table 1: BEM approximation of $\kappa_1 = 2\sqrt{3}\pi \approx 10.8828$, simple eigenvalue.

L	h	n_h	$\kappa_{2_1,h}^{\text{BEM}}$	$\lvert\kappa_2 - \kappa_{2_1,h}^{\text{BEM}}\rvert$	eoc
2	2^{-3}	384	15.373851 - 5.1e-05i	1.7e-02	-
3	2^{-4}	1536	15.3887048 - 9.4e-07i	1.9e-03	3.1
4	2^{-5}	6144	15.39037160 - 2.1e-08i	2.3e-04	3.1
L	h	n_h	$\kappa_{2_2,h}^{\text{BEM}}$	$\lvert\kappa_2 - \kappa_{2_2,h}^{\text{BEM}}\rvert$	eoc
2	2^{-3}	384	15.37364 - 5.1e-05i	1.7e-02	-
3	2^{-4}	1536	15.3887060 - 9.4e-07i	1.9e-03	3.1
4	2^{-5}	6144	15.39037171 - 2.1e-08i	2.3e-04	3.1
L	h	n_h	$\kappa_{2_3,h}^{\text{BEM}}$	$\lvert\kappa_2 - \kappa_{2_3,h}^{\text{BEM}}\rvert$	eoc
2	2^{-3}	384	15.373876 - 5.1e-05i	1.7e-02	-
3	2^{-4}	1536	15.3887071 - 9.4e-07i	1.9e-03	3.1
4	2^{-5}	6144	15.39037180 - 2.1e-08i	2.3e-04	3.1

Table 2: BEM approximation of $\kappa_2 = 2\sqrt{6}\pi \approx 15.3906$, multiple eigenvalue.

For a discussion and comparison of different algorithms for the solution of the algebraic nonlinear eigenvalue (7.2) problem we refer to [34, Ch. 6, 7]. A comparsion with the accuracy of a FEM approximation is presented in [31] which shows that with a comparatively coarse mesh of the boundary element discretization the same accuracy as for the FEM discretization can be achieved. However, the inverse iteration which is used for the solution of the discretized boundary integral eigenvalue problem is only locally convergent and in one run only one eigenvalue can be approximated. Therefore, robust and efficient algorithms are important issuses for the ongoing work.

8 Conclusions

The formulation of the Dirichlet Laplacian eigenvalue problem in terms of boundary integral operators yields a nonlinear boundary integral eigenvalue problem. In this paper we have analyzed a Galerkin discretization of this eigenvalue problem by using the concept of eigenvalue problems for holomorphic Fredholm operator–valued functions. The convergence of the Galerkin discretization of the boundary integral eigenvalue problem has been shown and quasi–optimal error estimates have been derived. Furthermore, a stability result for the multiplicities of the eigenvalues is given. We want to mention that the Neumann eigenvalue problem may be treated in the same way and that similar results of the numerical analysis can be obtained [34].

The solution of the discretization of the boundary integral operator eigenvalue problems requires numerical algorithms for algebraic nonlinear eigenvalue problems which has been not discussed in this paper. Such algorithms have been adressed in [34] and are a topic of the ongoing work.

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