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A regular analogue of the Smilansky model
(and related models)

Diana Barseghyan

Nuclear Physics Institute of the ASCR, Řež near Prague & University of Ostrava

joint work with Pavel Exner, Milos Tater, Andrii Khrabustovskyi, Olga Rossi

We consider the spectral properties of Schrödinger operators

$$-\Delta + V$$

$$(i\nabla + A)^2 + V$$

with the potentials unbounded from below.

One of the most celebrated elementary results on Schrödinger operators is that their spectrums are purely discrete if

$$\lim_{|x|\to\infty} V(x) = \infty.$$

But it is not necessary condition.

Various examples of systems which have purely discrete spectrum despite the fact the volume of the region where the potential is bounded is infinite were constructed in the last three decades. A classical one belongs to [Simon, 83] and describes a two-dimensional Schrödinger operator

$$-\Delta + |xy|^p \quad \text{on} \quad L^2(\mathbb{R}^2).$$

Similar behavior one can observe for Dirichlet Laplacians in regions with hyperbolic cusps, see [Geisinger-Weidl, 11] for recent results and a survey.

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In the model suggested by Uzy Smilansky one studies an operator describing by the following self-adjoint two dimensional operator

$$H_{\mathrm{Sm}} = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \lambda y \delta(x)\,.$$

$$H_{\mathrm{Sm}}\, u = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2 u\right)$$

$$\frac{\partial u}{\partial x}(+0, y) - \frac{\partial u}{\partial x}(-0, y) = \lambda\, y\, u(0, y) \quad \text{for every} \quad y \in \mathbb{R}.$$

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It was shown by Solomyak and Evans [Evans, Solomyak, 05] that the behavior of the spectrum of H_{Sm} depends on the coupling parameter:

if $\lambda < \sqrt{2}$ then

$$\sigma_{\mathrm{ess}}(H_{\mathrm{Sm}}) = [1/2, \infty).$$

The spectrum of H_{Sm} below the threshold $\lambda_0 = 1/2$ belongs to interval $(0, 1/2)$, always non empty and consists of finite number of eigenvalues.

It was proved by Solomyak [Solomyak, 04] that

$$N_-(1/2, H_{\mathrm{Sm}}) \sim \frac{1}{4\sqrt{2(s(\lambda)-1)}}, \quad s(\lambda) = \frac{\sqrt{2}}{\lambda}, \quad \lambda \to \sqrt{2}.$$

Theorem (Evans-Solomyak 05)

If $\lambda = \sqrt{2}$ *then*

$$\sigma(H_{\mathrm{Sm}}) = [0, \infty)$$

and if $\lambda > \sqrt{2}$ *then*

$$\sigma(H_{\mathrm{Sm}}) = \mathbb{R}.$$

We are going to investigate a model in which the potential with δ function multiplied by y is replaced by a smooth potential unbounded from below, and to show that it exhibits the analogous spectral transition as the coupling parameter exceeds a critical value.

We replace the δ by a family of shrinking potentials whose mean matches the δ coupling constant, $\int_{\mathbb{R}} U(x,y)\,\mathrm{d}x \sim y$.

This can be achieved, e.g., by choosing $U(x,y) = \lambda y^2 V(xy)$ for a fixed function V.

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This can be achieved, e.g., by choosing $U(x,y) = \lambda y^2 V(xy)$ for a fixed function V.

We investigate the model described by the partial differential operator on $L^2(\mathbb{R}^2)$ acting as

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \lambda y^2 V(xy),$$

where ω, a are positive constants and the potential V with $\mathrm{supp}\, V \subset [-c, c]$, $c > 0$, is a nonnegative function with bounded first derivative.

By Faris-Lavine theorem the above operator is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$.

Our aim in work is to demonstrate existence of a **critical coupling** separating two different situations: below it the spectrum is **bounded from below** while above it **covers the whole real line.**

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To state the result we will employ a one-dimensional comparison operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2 - \lambda V(x)$$

on $L^2(\mathbb{R})$ with the domain $H^2(\mathbb{R})$.

The important property will be the sign of its spectral threshold; since V is supposed to be nonnegative, the latter is a monotonous function of λ and there is a $\lambda_{\text{crit}} > 0$ at which the sign changes.

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Theorem (B.-Exner 16)

Under the stated assumptions if $\inf \sigma(L) = 0$

$$\sigma(H) = \sigma_{\text{ess}}(H) = [0, \infty).$$

Theorem (B.-Exner 16)

Let $\inf \sigma(L) > 0$ *then*

$$\sigma_{\mathrm{ess}}(H) = [\omega, \infty).$$

Theorem (B.-Exner 16)

Let $\inf \sigma(L) > 0$*, then the spectrum of operator* H *below* ω *is discrete, non-empty and is contained in the interval* $[0, \omega)$*.*

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Theorem (B.-Exner 16)

Let $\inf \sigma(L) = \gamma_0 > 0$ *then for any* $\sigma > \frac{1}{2}$ *the inequality*

$$\operatorname{tr}(\omega - H)_+^\sigma \le 2\lambda^{2\sigma} \|V\|_\infty^{2\sigma} c^{4\sigma} \sum_{n=1}^{\infty} \frac{1}{\alpha_1^{2\sigma} \left(\sqrt{\lambda \|V\|_\infty} c + (n-1)\pi\right)^{2\sigma}} + \left(\frac{2\alpha_1 \sqrt{\omega + \lambda \alpha_1^2 \|V\|_\infty}}{\pi} + 1\right)^2 \left(\omega + \lambda \alpha_1^2 \|V\|_\infty\right)^\sigma$$

holds, where ...

Theorem (continued)

Let l_k be the Neumann restriction of L to the interval $[-k, k]$, $k > 0$. We denote

$$\kappa := \min\left\{k : \inf \sigma(l_k) \geq \gamma_0/2\right\} .$$

$$\alpha_1 := \max\left\{\sqrt{\kappa}, \frac{2\omega}{\gamma_0}, \frac{\sqrt{\lambda \|V\|_\infty c}}{\sqrt{2\omega}}\right\}.$$

The existence of κ is guaranteed by the result of P.B. Bailey, W.N. Everitt, J. Weidmann, A. Zettl, Regular approximations of singular Sturm-Liouville problems, Results in Mathematics 22 (1993), 3–22.

Theorem (Exner-B. 14)

Under our hypotheses, $\sigma(H) = \mathbb{R}$ *holds if* $\inf \sigma(L) < 0$.

We consider a **regular analogue** of the Smilanksy model with the presence **of a constant magnetic field** given as follows

$$H(A) = (i\nabla + A)^2 + \omega^2 y^2 - \lambda y^2 V(xy),$$

where V is a nonnegative smooth enough function with $\mathrm{supp}(V) \subset [-c, c]$, $c > 0$, $\omega > 0$, and the magnetic potential A corresponds to a constant magnetic field $B > 0$.

To show the essentially self-adjointness one needs to construct a sequence of non-overlapping annular regions

$$A_m = \{z \in \mathbb{R}^2 : a_m < |z| < b_m\}$$

and a sequence of positive numbers ν_m such that

$$(b_m - a_m)^2 \nu_m > K, \ \ V(z) \geq -k\nu_m^2 (b_m - a_m)^2 \quad \text{for} \quad z \in A_m$$
$$\text{and} \quad \sum_{m=1}^{\infty} \nu_m^{-1} = \infty,$$

where K and k are positive constants independent of m.

It can be easily checked that for $a_m = m, \ b_m = m+1$ and $\mu_m = m+1, \ \ m = 0, 1, 2, \dots$ the requirement is satisfied with $K = \frac{1}{2}$ and $k = \lambda \|V\|_\infty$.

A. Iwatzuka, Essential Self-Adjointness of the Schrödinger Operators with Magnetic Fields Diverging at Infinity, Publications of the Research Institute for Mathematical Sciences, 26 (1990), 841-860.

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$$H_{\mathrm{Sm}}(A) = (i\nabla + A)^2 + \omega^2 y^2 + \lambda y \delta(x),$$

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$$L_p(\lambda)\ :\ L_p(\lambda)\psi = -\Delta\psi + \left(|xy|^p - \lambda(x^2+y^2)^{p/(p+2)}\right)\psi\,, \quad p \geq 1\,,$$

on $L^2(\mathbb{R}^2)$; where $\lambda \geq 0$.

Note that $\frac{2p}{p+2} < 2$ so the operator is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ by Faris-Lavine theorem; the symbol L_p or $L_p(\lambda)$ will always mean its closure.

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The spectral properties of $L_p(\lambda)$ depend crucially on the value of λ.

Let us start with the case of small values of λ.

To characterize the smallness quantitatively we need an auxiliary operator on line

$$H_p \; : \; H_p u = -u'' + |t|^p u$$

with the standard domain.

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$$\gamma_2 = 1, \quad \gamma_p \to \frac{\pi^2}{4} \quad \text{as} \quad p \to \infty$$

minimum value $\gamma_p \approx 0.998995$ at $p \approx 1.788$.

Theorem (B-Exner 12)

For any $\lambda < \lambda_{\text{crit}}$, where $\lambda_{\text{crit}} = \gamma_p$, the operator $L_p(\lambda)$ is bounded from below for any $p \geq 1$; and its spectrum is purely discrete.

The situation is different for large values of λ.

Theorem (B-Exner-Khrabustovskyi-Tater 16)

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Let us now pass to the subcritical case $\lambda < \gamma_p$.

Theorem (B-Exner-Khrabustovskyi-Tater 16)

Let $\lambda < \gamma_p$, then for any $\Lambda \geq 0$ and $\sigma \geq 3/2$ the following trace inequality holds,

$$\mathrm{tr}\,(\Lambda - L_p(\lambda))_+^\sigma \leq C_{p,\sigma} \frac{(\Lambda+1)^{\sigma+(p+1)/p}}{(\gamma_p-\lambda)^{\sigma+(p+1)/p}} \left(\left| \ln \left(\frac{\Lambda+1}{\gamma_p-\lambda} \right) \right| + 1 \right) + C_{p,\sigma}\, C_\lambda^2 \left(\Lambda + C_\lambda^{2p/(p+2)} \right)^{\sigma+1},$$

where the constant $C_{p,\sigma}$ depends on p and σ only and

$$C_\lambda = \max\left\{ \frac{1}{(\gamma_p-\lambda)^{(p+2)/(p(p+1))}}, \frac{1}{(\gamma_p-\lambda)^{(p+2)^2/(4p(p+1))}} \right\}.$$

Let us now pass to the case when the parameter value is critical

$$L_p(\gamma_p) = -\Delta + (|xy|^p - \gamma_p(x^2+y^2)^{p/(p+2)}),\ p \geq 1, \quad on \quad L^2(\mathbb{R}^2).$$

Theorem (B-Exner-Khrabustovskyi-Tater 16)

The essential spectrum of $L_p(\gamma_p)$ equals to the interval $[0,\infty)$.

Theorem (B-Exner-Khrabustovskyi-Tater 16)

The negative spectrum of $L_p(\gamma_p)$, $p \geq 1$, is discrete.

The question of existence of a negative discrete spectrum is addressed numerically. We show that there a range of values of p for which the critical operator $L_p(\gamma_p)$ has at least one negative eigenvalue.

We consider first the operator $L_2(\gamma_2)$, $\gamma_2 = 1$ defined on a circle of radius R circled at the origin with Dirichlet boundary condition, and find the corresponding first eigenvalue using the Finite Element Method.

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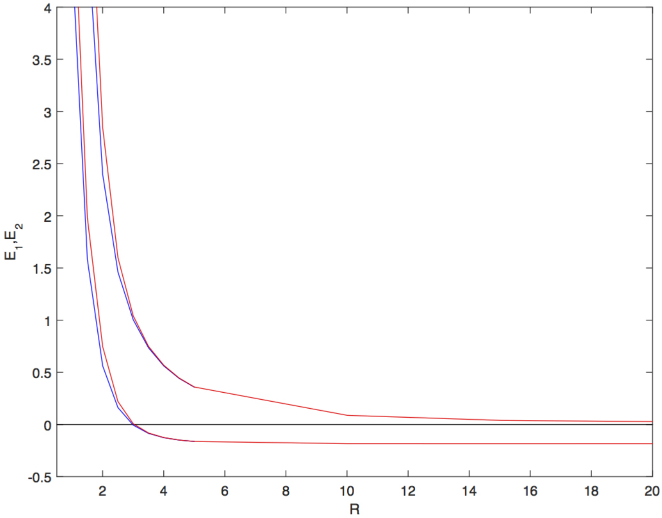
4
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The lowest Dirichlet eigenvalue is negative starting from some R_0 which by an elementary bracketing argument indicates that $L_2(\gamma_2)$ has a negative eigenvalue.

By continuity, the negative ground-state eigenvalue of $L_p(\lambda)$ exists in the vicinity of the point $p = 2$; one is naturally interested what one can say about a broader range of the parameter.

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$$L_p(\lambda) \,:\, L_p(\lambda)\psi = -\Delta\psi + \left(|xy|^p - \lambda(x^2+y^2)^{p/(p+2)}\right)\psi\,, \quad p \geq 1\,,$$

it is natural to ask about the limit $p \to \infty$.

Let $D = \{|xy| \leq 1\}$. We shall consider the operator

$$H_D(\lambda) : H_D(\lambda)\psi = -\Delta\psi - \lambda(x^2 + y^2)\psi$$

with a non-negative parameter λ initially defined on the set $\widetilde{C}_0^2(\overline{D}) = \{u \in C^2(\overline{D}) : \ u = 0 \text{ on } \partial D, \ \mathrm{supp}(u) \text{ is a compact set}\}$.

We show that for $\lambda \leq \frac{\pi^2}{4}$ the operator $H_D(\lambda)$ is non-negative and therefore one can construct its self-adjoint extension using the Friedrichs method.

Using the fact that densily defined and symmetric operator is always closable, in case if $\lambda > \frac{\pi^2}{4}$ we deal with its closure $\overline{H_D}(\lambda)$.

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The first important observation is that spectral properties of the operator $H_D(\lambda)$ depend crucially on the value of the parameter λ.

Theorem (B-Rossi 16)

For any $\lambda \in \left[0, \frac{\pi^2}{4}\right]$ the operator $H_D(\lambda)$ initially defined on $\widetilde{C}_0^2(\overline{D})$ is non-negative.

Theorem (B-Rossi 16)

If $\lambda < \frac{\pi^2}{4}$ the spectrum of $H_D(\lambda)$ is purely discrete. Moreover, for the corresponding eigenvalues, denoted by $\{\beta_j(\lambda)\}_{j=1}^{\infty}$, $\lambda < \frac{\pi^2}{4}$, the following expression holds

$$\beta_j(\lambda) = c_j(\lambda)\,\mu_j, \qquad j = 1, 2, \dots\,,$$

where $\left(1 - \frac{4\lambda}{\pi^2}\right) \leq c_j(\lambda) \leq 1$ and μ_j, $j = 1, 2, \dots$ are the eigenvalues of the Dirichlet Laplacian $-\Delta_D$ arranged in the ascending order and $\mu_j \sim \frac{\pi j}{\ln j}$.

Theorem (B-Rossi 16)

Let $\lambda < \frac{\pi^2}{4}$. Then for any $\epsilon > 0$ there exists a natural number $M(\epsilon)$ such that for the eigenvalue sum of operator $H_D(\lambda)$ the following lower bound holds true:

$$\sum_{j=1}^{N} \beta_j(\lambda) \geq (1-\epsilon)\frac{(\pi^2 - 4\lambda)}{4\pi}\frac{(N-2)^2}{\ln N}, \qquad N > M(\epsilon).$$

On the other hand, the following upper bound is valid

$$\sum_{j=1}^{N} \beta_j(\lambda) \leq (1+\epsilon)\pi\frac{N^2}{\ln N}, \qquad N > M(\epsilon).$$

Now we consider the critical case $\lambda = \lambda_{\rm crit} = \frac{\pi^2}{4}$. The following theorem holds true.

Theorem (B-Rossi 16)

The spectrum of $H_D(\lambda_{\rm crit})$ coincides with the half line $[0, \infty)$.

Let $\lambda > \frac{\pi^2}{4}$ and let $\overline{H_D}(\lambda)$ denote the closure of the operator $H_D(\lambda)$ initially defined on $\widetilde{C}_0^2(\overline{D})$. Our next result is the following.

Theorem (B-Rossi 16)

For any $\lambda > \frac{\pi^2}{4}$ the spectrum of $\overline{H_D}(\lambda)$ contains the real line $\mathbb{R}$.

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Thank you for your attention