Space–Time Methods

15. Consider the variational formulation to find $u \in X_T := \{u \in Y : \partial_t u \in Y^*, u(x,T) = 0, x \in \Omega\}$ such that

$$b(u,v) := -\langle \partial_t u, v\rangle_Q + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} \, = \, \langle f, v\rangle_Q$$

is satisfied for all $v \in Y := L^2(0,T;H^1_0(\Omega))$. Prove the inf-sup stability condition

$$c_S \, \|u\|_X \leq \sup_{0 \neq v \in Y} \frac{b(u,v)}{\|v\|_Y} \quad \text{for all } u \in X_T,$$

and the surjectivity of the bilinear form $b(\cdot,\cdot)$, i.e., for all $v \in Y \backslash \{0\}$ there exists a $u_v \in X_T$ such that $b(u_v, v) \neq 0$ is satisfied.

16. Define

$$X_{T,h} := W^1_{h_t} \otimes V_{h_x} \quad \text{with} \quad W^1_{h_t} = \text{span}\{\psi_i\}_{i=0}^{N_t-1} \subset H^1_{,0}(0,T), \quad V_{h_x} = \text{span}\{\varphi_k\}_{k=1}^{M_x} \subset H^1_0(\Omega),$$

and

$$Y_h := W^0_{h_t} \otimes V_{h_x} \quad \text{with} \quad W^0_{h_t} = \text{span}\{\psi_i\}_{i=1}^{N_t} \subset L^2(0,T).$$

Prove the discrete inf-sup stability condition

$$c_S \sqrt{\|\partial_t u_h\|^2_{Y^*} + \|Q^0_{h_t} u_h\|^2_Y} \, \leq \sup_{0 \neq v_h \in Y_h} \frac{b(u_h, v_h)}{\|v_h\|_Y} \quad \text{for all } u_h \in X_{T,h}.$$

17. Prove the discrete inf-sup stability constant

$$c \, \min\{1, h_t^{-1} h_x^2\} \, \|u_h\|_X \, \leq \sup_{0 \neq v_h \in Y_h} \frac{b(u_h, v_h)}{\|v_h\|_Y} \quad \text{for all } u_h \in X_{T,h}.$$