
Problem 1. Let $1 \leq p, q \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} \geq 1$ and $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Set $\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1$ with $1 \leq r \leq \infty$. Show that $f * g \in L^r(\mathbb{R}^d)$ and $\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}$.

Hint: Show the assertion first for the case $p = 1$, and then for arbitrary $1 < p \leq \infty$ by considering a decomposition of the form

$$|f(x-y)g(y)| = |f(x-y)|^{\alpha}|g(y)|^{\beta}\Big(|f(x-y)|^{1-\alpha}|g(y)|^{1-\beta}\Big),$$

with $\alpha = p/q'$ and $\beta = q/p'$ and the conjugated exponents[†] $p', q' \geq 1$ of $p, q \geq 1$. The generalization of the Hölder inequality to more than two functions may also be helpful.

Problem 2. Let $\mathcal{O} \in \mathbb{R}^d$ be an open set with finite Lebesgue measure. Show the following:

(i) There exist at most countably many pairwise disjoint compact cuboids $Q_n = [a_1^{(n)}, b_1^{(n)}] \times \cdots \times [a_d^{(n)}, b_d^{(n)}]$ with $a_j^{(n)}, b_j^{(n)} \in \mathbb{R}$ for all $j = 1, \ldots, d$, such that

$$\mathcal{O} = \bigcup_{n=1}^{\infty} Q_n.$$

(ii) Let $N \in \mathbb{N}$ and define $s_N := \sum_{n=1}^{N} \mathbb{1}_{Q_n}$. Prove that s_N converges to $\mathbb{1}_{\mathcal{O}}$ in $L^2(\mathbb{R}^d)$.

Problem 3. Let

$$\rho(x) := \begin{cases} Ce^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $C \in \mathbb{R}$ is chosen so that $\int_{\mathbb{R}^d} \rho(x)\mathrm{d}x = 1$. Prove that $\rho \in \mathcal{D}(\mathbb{R}^d)$ and $\mathrm{supp}(\rho) = \overline{B(0,1)}$.

Hint: Write ρ as the composition of a function defined on $\mathbb{R}$ and a function that maps $\mathbb{R}^d$ to $\mathbb{R}$.

Problem 4. Show that the function space

$$C_0(\mathbb{R}^d) := \left\{ f \in C(\mathbb{R}^d) \;\middle|\; \lim_{|x| \to \infty} f(x) = 0 \right\},$$

equipped with the supremum norm $\|\cdot\|_{\infty}$, is a Banach space.

[†] $p' \geq 1$ is called the conjugate exponent of $p \geq 1$ if $\frac{1}{p} + \frac{1}{p'} = 1$ applies.

Problem 5. Let $\Omega \subseteq \mathbb{R}^d$ be nonempty and open. Show the following formulae of *partial integration*:

(i) Assume additionally that Ω is bounded with a C^1-smooth boundary. Then, for any $u, v \in C^1(\overline{\Omega})$

$$\int_\Omega (\partial_j u) v \mathrm{d}x = \int_{\partial\Omega} u v \nu_j \mathrm{d}\sigma - \int_\Omega u \partial_j v \mathrm{d}x, \quad j = 1, \dots, d,$$

holds, where $\nu := (\nu_1, \dots, \nu_d)^T$ denotes the exterior unit normal vector field of Ω.

(ii) For $u \in C^1(\Omega)$ and $v \in \mathcal{D}(\Omega)$ one has

$$\int_\Omega (\partial_j u) v \mathrm{d}x = -\int_\Omega u \partial_j v \mathrm{d}x, \quad j = 1, \dots, d.$$

Hint: Of course, it is allowed to use the classical Gauß divergence theorem in the following (or another) form:
Let $G \subseteq \mathbb{R}^d$ be a bounded domain with C^1-smooth boundary and $F \in C^1(\overline{G}, \mathbb{R}^d)$. Then

$$\int_G \mathrm{div} F \mathrm{d}x = \int_{\partial G} F \cdot \nu \mathrm{d}\sigma.$$

Problem 6. Let $d = 1$ and $\rho \in \mathcal{D}(\mathbb{R})$ be the test function from Exercise 3. As in the lecture, we define $\rho_n(x) := n\rho(nx)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Show the following:

(i) $\int_\mathbb{R} \rho_n(x)\, dx = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \rho_n(x) = \begin{cases} \infty, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

(ii) The regular distributions T_{ρ_n} converge "*pointwise*" to the δ-distribution δ_0, i.e.

$$\lim_{n\to\infty} \int_\mathbb{R} \rho_n(x)\psi(x)\, dx = \psi(0), \qquad \psi \in \mathcal{D}(\mathbb{R}).$$

(iii) The δ-distribution δ_0 is not a regular distribution.

Problem 7. For $\varphi \in \mathcal{D}(0, \infty)$ define

$$T\varphi := \sum_{m=1}^{\infty} \varphi^{(m)}\left(\frac{1}{m}\right).$$

Show that T belongs to $\mathcal{D}'(0, \infty)$ and determine its derivative.

Problem 8. Let $\alpha > 0$ be real and $B(0, 1) \subseteq \mathbb{R}^d$ be the d-dimensional unit sphere. Furthermore, let the function $u : B(0, 1) \to [0, \infty]$ be defined by $u(x) := |x|^{-\alpha}$. For which α is $u \in H^1(B(0, 1))$?

Problem 9. Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in L^2(\Omega)$ such that the weak derivative $D^\alpha u$ exists in $L^2(\Omega)$ for some $\alpha \in \mathbb{N}_0$. Let $\widetilde{u} \in L^2(\mathbb{R}^d)$ be the extension by zero of u. For $n \in \mathbb{N}$ let the mollifier ρ_n be defined as in the lecture and $U \Subset \Omega$. Show that for each sufficiently large n the following identity holds almost everywhere on U:

$$D^\alpha(\rho_n * \widetilde{u}) = \rho_n * \widetilde{D^\alpha u}.$$

Hint: Use $\frac{\partial}{\partial x_j}\rho_n(x-y) = -\frac{\partial}{\partial y_j}\rho_n(x-y)$.

Problem 10. Let $\Omega \subseteq \mathbb{R}^d$ be open. Show the following:

(i) Let $u \in H^1(\Omega)$. Then, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ and $D^{e_j}u_n \to D^{e_j}u$ in $L^2(U)$ for $j \in \{1,\dots,d\}$ and any $U \Subset \Omega$, where e_j is the j-th unit vector in $\mathbb{R}^d$.

(ii) Let $u, f_1, \dots, f_d \in L^2(\Omega)$. Assume that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^d)$ such that $u_n \to u$ and $\frac{\partial u_n}{\partial x_j} \to f_j$ in $L^2(U)$ for any $U \Subset \Omega$ and all $j \in \{1,\dots,d\}$. Then, $u \in H^1(\Omega)$ and $D^{e_j}u = f_j$, $j \in \{1,\dots,d\}$.

Hint for (i): One can use the same sequence $(u_n)_{n\in\mathbb{N}}$ as in the proof of the statement that $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$.

Problem 11. Let $\Omega \subseteq \mathbb{R}^d$ be open and $u, v \in H^1(\Omega)$. Verify that uv is weakly differentiable and that $D^{e_j}(uv) = (D^{e_j}u)v + u(D^{e_j}v)$, where e_j is the j-th unit vector in $\mathbb{R}^d$.

Problem 12. Show the following:

(i) A generalization of the chain rule: let $\Omega \subseteq \mathbb{R}^d$ be a open set and let $f \in C^1(\mathbb{R})$ such that $f(0) = 0$ and assume that there exists a constant $M > 0$ such that $|f'(r)| \le M$ holds for all $r \in \mathbb{R}$. Then $(f \circ u) \in H^1(\Omega)$ and $D^{e_j}(f \circ u) = (f' \circ u) \cdot D^{e_j}u$ for any $u \in H^1(\Omega)$ and $j = 1, \ldots, d$, where e_j denotes the j-th unit normal vector in $\mathbb{R}^d$.

(ii) The condition $f(0) = 0$ can be dropped for bounded Ω.

Hint: A result from measure theory sometimes known as the "reverse theorem of Lebesgue" can be useful.†

Problem 13. Assume that $\Omega_1, \Omega_2 \subseteq \mathbb{R}^d$ are open sets and $F : \Omega_2 \to \Omega_1$ is bijective with $F \in C^1(\Omega_2, \mathbb{R}^d)$ and $F^{-1} \in C^1(\Omega_1, \mathbb{R}^d)$. Moreover, assume that the Jacobians DF and DF^{-1} of F and F^{-1}, respectively, satisfy $\sup\{\|(DF)(x)\| \mid x \in \Omega_2\} < \infty$ and $\sup\{\|(DF^{-1})(y)\| \mid y \in \Omega_1\} < \infty$. Finally, let $u \in H^1(\Omega_1)$. Show that $u \circ F \in H^1(\Omega_2)$ and

$$D^{e_j}(u \circ F) = \sum_{k=1}^{d} \big((D^{e_k}u) \circ F\big) \cdot (D^{e_j}F_k), \quad j \in \{1, \ldots, d\}.$$

Problem 14. Consider $d = 1$, $\Omega = (0,1)$ and $u(x) := x \in H^1(\Omega)$. Show that the zero extension $\widetilde{u}$ is not in $H^1(\mathbb{R})$.

Problem 15. Show the following:

(i) Let $\Omega \subseteq \mathbb{R}^d$ be open and set

$$H^1_c(\Omega) := \big\{u \in H^1(\Omega) \,\big|\, \operatorname{supp}(u) \text{ is compact in } \Omega\big\}.$$

Prove that $H^1_c(\Omega) \subseteq H^1_0(\Omega)$.

(ii) Let $\Omega = B(0,1) \setminus \{0\} \subseteq \mathbb{R}^3$ and let $\varphi \in \mathcal{D}(B(0,1))$. Verify that $\varphi \in H^1_0(\Omega)$.

Hint for (i): Let $(\rho_n)_{n\in\mathbb{N}}$ be the sequence of mollifiers. Verify that then for a fixed $u \in H^1_c(\Omega)$ and all sufficiently large n the relation $\rho_n * \widetilde{u} \in \mathcal{D}(\Omega)$ holds.

†Let $f_n, f \in L^2(\Omega)$ such that $f_n \to f$ with respect to the L^2-norm. Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and $g \in L^2(\Omega)$ such that $f_{n_k}(x) \to f(x)$ and $|f_{n_k}(x)| \le |g(x)|$ are true for almost all $x \in \Omega$.

Problem 16. Let $k > \frac{d}{2}$, $m \in \mathbb{N}_0$ and $u \in H^{k+m}(\mathbb{R}^d)$. Sobolev embedding yields that $u \in C^m(\mathbb{R}^d)$. Prove that there exists a constant $C > 0$ depending only on d and k such that

$$\|D^\alpha u\|_{L^\infty(\mathbb{R}^d)} \le C\|u\|_{H^{k+m}(\mathbb{R}^d)}$$

is satisfied for any multi index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$.

Hint: Make use of the definition of the Sobolev spaces by means of the Fourier transform.

Problem 17. Let $\Omega \subseteq \mathbb{R}^d$ be a open set, $u \in H^1_{\mathrm{loc}}(\Omega)$ and $f \in H^k_{\mathrm{loc}}(\Omega)$ for some $k \in \mathbb{N}_0$, with $-\Delta u = f$ in the sense of distributional derivatives. As usual denote by $\tilde{}$ the extension by zero on $\mathbb{R}^d$. Let $\eta \in \mathcal{D}(\Omega)$. Show that

$$\Delta(\widetilde{\eta u}) = \big(u\Delta\eta + 2\nabla\eta\cdot\nabla u - \eta f\big)^{\sim}$$

holds on $\mathbb{R}^d$ in the sense of distributional derivatives.

Problem 18. Let $\Omega \subseteq \mathbb{R}^d$, $d \ge 1$, be open, bounded and nonempty, and consider for $f \in L^2(\Omega)$ the boundary value problem

$$(0.1)\qquad -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\alpha^{jk}\frac{\partial}{\partial x_k}u + \alpha u = f, \quad u \restriction \partial\Omega = 0.$$

Assume that $\alpha \in C(\overline{\Omega})$ satisfies $\alpha \ge 0$ and that $\alpha^{jk} \in C^1(\overline{\Omega})$ are real-valued (for all $k, j = 1, \dots, d$), symmetric (i.e. $\alpha^{jk}(x) = \alpha^{kj}(x)$, $\forall x \in \Omega$ and $\forall k, j \in \{1, \dots, d\}$) and fulfill

$$\big((\alpha^{jk}(x))_{j,k=1}^d\xi, \xi\big)_{\mathbb{C}^d} \ge E\|\xi\|^2_{\mathbb{C}^d}, \quad \forall x \in \Omega, \forall \xi \in \mathbb{C}^d,$$

for some $E > 0$. We say that $u \in H^1_0(\Omega)$ is a *weak solution* of (0.1), if u satisfies

$$\sum_{j,k=1}^{d}\int_\Omega \alpha^{jk}(x)\frac{\partial u(x)}{\partial x_j}\overline{\frac{\partial v(x)}{\partial x_k}}\mathrm{d}x + \int_\Omega \alpha(x)u(x)\overline{v(x)}\mathrm{d}x = \int_\Omega f(x)\overline{v(x)}\mathrm{d}x \quad \forall v \in H^1_0(\Omega).$$

Verify that for any given $f \in L^2(\Omega)$ there exists a uniquely determined weak solution $u \in H^1_0(\Omega)$.

Problem 19. Let $\Omega \subseteq \mathbb{R}^d$, $d \ge 1$, be open, bounded and nonempty. According to Exercise 18 there exists for any $f \in L^2(\Omega)$ a uniquely determined weak solution $u_f \in H^1_0(\Omega)$ of the boundary value problem (0.1). Hence, the solution operator

$$R : L^2(\Omega) \to L^2(\Omega), \quad Rf = u_f,$$

mapping a given right hand side to the associated weak solution is well defined. Show that R is a compact linear operator.

Hint: Write R as the product of a bounded and a compact operator. You are allowed to use (without proof, see Exercise 18) that the sesquilinear form associated to (0.1) is coercive.

Problem 20. Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded and let $f : \Omega \to \mathbb{R}$ be Lipschitz continuous. Verify, that $f \in H^1(\Omega)$. For this, proceed as follows:

(i) Show that f is bounded. Hence, we have $f \in L^2(\Omega)$.

(ii) Prove that the weak partial derivatives of f exist and belong to $L^2(\Omega)$.

Hints: Make use of the difference quotient method and related results from the lecture to show that f is weakly differentiable. In particular, set $\Omega_h := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > h\}$ and $f_{j,h} := \chi_{\Omega_h} D_j^h f$. Show that $f_{j,h}$ is uniformly bounded in $L^2(\Omega)$ and verify that these functions converge weakly to $\partial_j f$.

Problem 21. Prove or disprove (a sketch of the arguments is sufficient) that the following sets are Lipschitz domains.

(i) $W := (0,1)^2$;
(ii) $Q := (0,1)^2 \setminus \left(\{\frac{1}{2}\} \times [0, \frac{1}{2}]\right)$;
(iii) $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

Problem 22. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded open Lipschitz domain. Show that there does **not** exist a continuous operator $T : L^2(\Omega) \to L^2(\partial\Omega)$ such that $Tu = u \restriction \partial\Omega$ holds for all $u \in L^2(\Omega) \cap C(\overline{\Omega})$.

Problem 23. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded open Lipschitz domain and let $T : H^1(\Omega) \to L^2(\partial\Omega)$ be the trace operator defined as in the lecture.† Prove that $Tu = u \restriction \partial\Omega$ for any $u \in C(\overline{\Omega}) \cap H^1(\Omega)$. For this, proceed as follows:

(i) Construct (locally) for a fixed $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ a continuous extension $\bar{u}$ of u as in the proof that a Lipschitz domain has the extension property, which is defined in an open neighborhood of Ω.
(ii) Set $u_n := (\rho_n * \widehat{u}) \restriction \Omega$, where (ρ_n) is the sequence of mollifiers and $\widehat{u}$ is an extension of $\bar{u}$ onto $\mathbb{R}^d$ by zero. Show that u_n converges to u in the $\|\cdot\|_\infty$ - norm.
(iii) Deduce that $Tu = u \restriction \partial\Omega$ for any $u \in C(\overline{\Omega}) \cap H^1(\Omega)$.

Hint: Use that functions that are continuous on compact sets are uniformly continuous.

Problem 24. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain.

(i) Show that

$$\int_\Omega (\Delta u) v dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\sigma, \tag{0.2}$$

where ν is the outer unit normal vector at $\partial\Omega$ and $\frac{\partial u}{\partial \nu} := \nu \cdot \nabla u$ is the normal derivative, is true for all $u \in C^2(\overline{\Omega})$ and all $v \in C^1(\overline{\Omega})$.
(ii) Show that the Neumann trace $T_N u = \frac{\partial u}{\partial \nu}$, $u \in C^2(\overline{\Omega})$, can be extended to a bounded operator $T_N : H^2(\Omega) \to L^2(\partial\Omega)$.
(iii) Prove that

$$\int_\Omega (\Delta u) v dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} T_N u T_D v d\sigma, \tag{0.3}$$

holds for any $u \in H^2(\Omega)$ and all $v \in H^1(\Omega)$, where T_D is the Dirichlet trace operator defined as in the lecture.

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†Recall that T was defined to be the extension by continuity of $\widetilde{T} : C^1(\overline{\Omega}) \to L^2(\partial\Omega), \widetilde{T}u = u \restriction \partial\Omega$.

Hint: You are allowed to use, without proof, that $C^2(\overline{\Omega})$ is dense in $H^2(\Omega)$ and that the classical Gauß divergence theorem is also valid for bounded Lipschitz domains.

Problem 25. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, $\lambda \in \mathbb{C}$, $f \in L^2(\Omega)$ and $\vartheta : \partial\Omega \to [0,\infty)$ a bounded and measurable function. Denote by $T : H^1(\Omega) \to L^2(\partial\Omega)$ the trace operator, as in the lecture. Show that $u \in H^1(\Omega)$ is a (distributional) solution of the Robin boundary value problem

$$\begin{aligned} (-\Delta - \lambda)u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \vartheta T u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

if and only if

$$\int_\Omega \nabla u \cdot \nabla v \; dx - \int_\Omega \lambda u v \; dx + \int_{\partial\Omega} \vartheta (Tu)(Tv) \; d\sigma = \int_\Omega f v \; dx$$

holds for all $v \in H^1(\Omega)$.

Problem 26. Let S be a densely defined and linear operator in Hilbert space $\mathcal{H}$. Show the following properties of the adjoint operator.

$$\begin{aligned} \operatorname{dom}(S^*) &= \left\{ g \in \mathcal{H} \,\middle|\, \exists g' \in \mathcal{H} \text{ such that } (Sf, g)_{\mathcal{H}} = (f, g')_{\mathcal{H}} \quad \forall f \in \operatorname{dom}(S) \right\} \\ S^* g &= g'. \end{aligned}$$

(i) S^* is well-defined, i.e. the element $g' \in \mathcal{H}$ in the definition of S^* is unique.
(ii) S^* is a linear operator.
(iii) S^* is closed.
(iv) $\operatorname{ran}(S)^\perp = \ker(S^*)$.

Problem 27. Let S be a symmetric operator in a Hilbert space $\mathcal{H}$ and assume that there exists a $\lambda \in \mathbb{R}$ such that $\operatorname{ran}(S - \lambda) = \mathcal{H}$.[†] Show the following statements:

(i) λ is not an eigenvalue of S.
(ii) S is self-adjoint.

Problem 28. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a smooth bounded domain. We consider in $L^2(\Omega)$ the operator

$$\begin{aligned} \operatorname{dom}(S) &= H_0^2(\Omega) = \left\{ f \in H^2(\Omega) \,\middle|\, u \restriction \partial\Omega = \nu \cdot (\nabla u \restriction \partial\Omega) = 0 \right\}, \\ Sf &= -\Delta f, \end{aligned}$$

where ν is the normal vector in Ω and traces are build in the sense of trace operators. Show the following statements:

[†]Recall that the range of a linear operator A is $\operatorname{ran}(A) = \{Ax \mid x \in \operatorname{dom}(A)\}$, where $\operatorname{dom}(A)$ denotes the domain of definition of A.

(i) S is symmetric.
(ii) The adjoint of S is given by

$$\begin{aligned} \operatorname{dom}(S^*) &= \left\{ f \in L^2(\Omega) \,\middle|\, \Delta f \in L^2(\Omega) \text{ in the distributional sense} \right\}, \\ S^* f &= -\Delta f. \end{aligned}$$

Recall: $\Delta f \in L^2(\Omega)$ in the distributional sense means that there is an $g \in L^2(\Omega)$ such that

$$\int_\Omega f \Delta \varphi \mathrm{d}x = \int_\Omega g \varphi \mathrm{d}x$$

holds for all $\varphi \in \mathcal{D}(\Omega)$. In this case, we define $\Delta f = g$.

Problem 29. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a smooth, bounded domain and S be the linear operator of Exercise 28. Show that S is a closed operator. For this, proceed as follows:

(i) Show that the mapping $\|\cdot\|_\Delta : H_0^2(\Omega) \to \mathbb{R}$, defined by

$$\|u\|_\Delta^2 := \|u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2, \quad \forall u \in H_0^2(\Omega),$$

is a norm in $H_0^2(\Omega)$ which is equivalent to the $H^2(\Omega)$-norm.
(ii) Conclude from item (i) that S is a closed operator.

Problem 30. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain. The Dirichlet Laplacian in Ω is defined by

$$\begin{aligned} \operatorname{dom}(A_0) &= H^2(\Omega) \cap H_0^1(\Omega) \\ A_0 f &= -\Delta f. \end{aligned}$$

Show $\sigma(A_0) = \sigma_p(A_0) = \{\lambda_n \mid n \in \mathbb{N}\}$, where $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ is a sequence of real numbers with $\lambda_n \to \infty$ for $n \to \infty$. Furthermore, show that the corresponding eigenvectors $(u_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom}(A_0)$ form an orthonormal basis of $L^2(\Omega)$.

Hint: Show that there exists an $\lambda \in \rho(A_0)$ such that the resolvent $(A_0 - \lambda)^{-1}$ is a compact operator in $L^2(\Omega)$. Then deduce the form of $\sigma(A_0)$ from the knowledge of the spectrum of $(A_0 - \lambda)^{-1}$.