Partial Differential Equations and Boundary Value Problems

Lecture Notes

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Contents

Chapter 1

Distributions and Sobolev spaces

In this chapter we provide preliminaries on distributions, weak derivatives, Sobolev spaces, and the Fourier transform. We start with a standard procedure for the approximation of integrable functions by smooth ones.

1.1 Regularization

Let $\Omega \subset \mathbb{R}^d$ be open, nonempty, $d \geq 1$. Recall that the *support* of a function $\varphi : \Omega \to \mathbb{C}$ is defined as

$$\operatorname{supp} \varphi := \overline{\{x \in \Omega : \varphi(x) \neq 0\}} \subset \mathbb{R}^d$$

(closure with respect to standard norm in $\mathbb{R}^d$). If not explicitly stated different, in this lecture all functions are **complex-valued**. Recall further that the linear space $L^2(\Omega)$ equipped with the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)\overline{v(x)}dx, \quad u, v \in L^2(\Omega),$$

and the corresponding norm $\| \cdot \|_{L^2(\Omega)}$ is a Hilbert space. We use the usual multi-index notation: For $\alpha = (\alpha_1, \dots, \alpha_d)^\top \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ we write

$$|\alpha| := \sum_{j=1}^d \alpha_j, \quad x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}, \quad \text{and} \quad D^\alpha := \prod_{j=1}^d \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}.$$

Example (for $d = 5$): For $\alpha = (2, 1, 0, 3, 0)^\top$ we have $|\alpha| = 6$, $x^\alpha = x_1^2 x_2 x_4^3$ and $D^\alpha \varphi = \frac{\partial^6 \varphi}{\partial x_1^2 \partial x_2 \partial x_4^3}$.

We write

$$\mathscr{D}(\Omega) := \big\{\varphi \in C^\infty(\Omega) : \operatorname{supp}\varphi \subset \Omega, \operatorname{supp}\varphi \text{ compact}\big\}.$$

Then $\mathscr{D}(\Omega)$ is a linear space. Sometimes we call the functions in $\mathscr{D}(\Omega)$ *test functions*.

Define

$$\rho(x) := \begin{cases} Ce^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & \text{else,} \end{cases}$$

where $C > 0$ is chosen such that $\int_{\mathbb{R}^d} \rho(x)dx = 1$. Then $\rho \in \mathscr{D}(\mathbb{R}^d)$ (easy to check; exercises) and $\operatorname{supp}\rho = \overline{B(0,1)}$. For $n \in \mathbb{N}$ define

$$\rho_n(x) := n^d \rho(nx), \quad x \in \mathbb{R}^d. \tag{1.1}$$

Then $\rho_n \in \mathscr{D}(\Omega)$, $\int_{\mathbb{R}^d} \rho_n(x)dx = 1$, and $\operatorname{supp}\rho_n = \overline{B(0,\frac{1}{n})}$. The functions ρ_n are called *mollifiers*.

Definition 1.1. Let $u, v \in L^2(\mathbb{R}^d)$. Then the function

$$(u * v)(x) := \int_{\mathbb{R}^d} u(x-y)v(y)dy, \quad x \in \mathbb{R}^d,$$

is called *convolution of* u *and* v.

Note that $u * v$ (in contrast to u and v) can be evaluated at each point $x \in \mathbb{R}^d$, and a substitution yields $u * v = v * u$ for all $u, v \in L^2(\mathbb{R}^d)$. We will see a little later that $u * v$ is in fact continuous. Note further that by the Cauchy–Schwarz inequality we have

$$|(u * v)(x)| \leq \|u\|_{L^2(\mathbb{R}^d)} \|v\|_{L^2(\mathbb{R}^d)}, \quad x \in \mathbb{R}^d. \tag{1.2}$$

In the following lemma we write as usual

$$K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$$

for two sets $K_1, K_2 \subset \mathbb{R}^d$.

Lemma 1.2. *Let $u, v \in L^2(\mathbb{R}^d)$ and let K_1, K_2 be compact sets such that $u(x) = 0$ for almost all $x \in \mathbb{R}^d \setminus K_1$ and $v(x) = 0$ for almost all $x \in \mathbb{R}^d \setminus K_2$. Then*

$$(u * v)(x) = 0 \quad \textit{for all } x \in \mathbb{R}^d \setminus (K_1 + K_2).$$

Proof. Consider *functions* u, v, i.e. representatives of the equivalence classes u, v, such that $\operatorname{supp} u \subset K_1$ and $\operatorname{supp} v \subset K_2$. Let $x \in \mathbb{R}^d$ be arbitrary. If there exists $y \in \mathbb{R}^d$ such that $u(x-y)v(y) \neq 0$ then $x - y \in K_1$ and $y \in K_2$ and thus $x = x - y + y \in K_1 + K_2$. Hence for any $x \in \mathbb{R}^d \setminus (K_1 + K_2)$ we have

$$(u * v)(x) = \int_{\mathbb{R}^d} u(x-y)v(y)dy = 0.$$

□

In the following the space

$$C_0(\mathbb{R}^d) := \left\{ u \in C(\mathbb{R}^d) : \lim_{|x|\to\infty} u(x) = 0 \right\}$$

will be used. It is a closed subspace of the set of all bounded, measurable functions on $\mathbb{R}^d$ equipped with the supremum norm

$$\|u\|_\infty := \sup_{x\in\mathbb{R}^d} |u(x)|$$

(see the exercises). In particular, $C_0(\mathbb{R}^d)$ with the supremum norm is a Banach space.

In the proof of the next proposition we will use that the step functions are dense in $L^2(\mathbb{R}^d)$. A step function s is the finite linear combination of characteristic functions for cuboids, i.e.

$$s = \sum_{n=1}^{N} a_n \mathbb{1}_{Q_n},$$

where $a_n \in \mathbb{C}, n = 1, \dots N$, and Q_n is a cuboid of the form

$$Q_n = \left[a_1^{(n)}, b_1^{(n)}\right] \times \cdots \times \left[a_d^{(n)}, b_d^{(n)}\right] \tag{1.3}$$

with some $a_i^{(n)} < b_i^{(n)}$, $i = 1, \dots, d$. The fact that step functions are dense in $L^2(\Omega)$ will be verified in the following lemma.

Lemma 1.3. *Let $\Omega \subset \mathbb{R}^d$ be open. The step functions are dense in $L^2(\Omega)$.*

Proof. Let $f \in L^2(\mathbb{R}^d)$ and fix $\varepsilon > 0$. It will be shown that there is a step function s such that

$$\|f - s\|_{L^2(\Omega)} < \varepsilon.$$

First, since simple functions are dense in $L^2(\Omega)$, there exists

$$\widetilde{s} := \sum_{i=1}^{M} a_i \mathbb{1}_{X_i},$$

where $a_i \in \mathbb{C}$ and $X_i \subset \Omega$ is measurable, $i = 1, \dots, M$, such that

$$\|\widetilde{s} - f\|_{L^2(\Omega)} < \frac{\varepsilon}{3}.$$

Without loss of generality, we assume that the sets X_i are pairwise disjoint and $a_i \neq 0$. Hence, the measure of X_i is finite as otherwise $\widetilde{s} \notin L^2(\Omega)$.

Next, since the Lebesgue measure is regular, there exists for any $i \in \{1, \dots, M\}$ an open set O_i with $X_i \subset O_i \subset \Omega$ such that

$$\lambda(O_i \setminus X_i) < \frac{\varepsilon^2}{9|a_i|^2 M^2},$$

where λ is the Lebesgue measure on Ω. Define $\widehat{s} := \sum_{i=1}^{M} a_i \mathbb{1}_{O_i}$. Then, using the triangle inequality and $X_i \subset O_i$ we find

$$\|\widetilde{s} - \widehat{s}\|_{L^2(\Omega)} \leq \sum_{i=1}^{M} |a_i| \cdot \|\mathbb{1}_{O_i} - \mathbb{1}_{X_i}\|_{L^2(\Omega)} = \sum_{i=1}^{M} |a_i| \left(\int_{O_i \setminus X_i} \mathrm{d}x \right)^{1/2} < \frac{\varepsilon}{3}.$$

Finally, any open set O_i can be written as the at most countable union of pairwise disjoint cuboids

$$O_i = \bigcup_{n=1}^{\infty} Q_{i,n},$$

where each $Q_{i,n}$ is like in (1.3). Hence, the step functions

$$s_N := \sum_{i=1}^{M} a_i \sum_{n=1}^{N} \mathbb{1}_{Q_{i,n}}$$

converge pointwise to $\widehat{s}$. By dominated convergence we see that s_N tends also to $\widehat{s}$ with respect to the L^2-norm. Therefore, there exists an $N_0 \in \mathbb{N}$ such that $\|\widehat{s} - s_{N_0}\|_{L^2(\Omega)} < \frac{\varepsilon}{3}$. Eventually, we set $s := s_{N_0}$. Then, by applying the triangle inequality we find

$$\|f - s\|_{L^2(\Omega)} \leq \|f - \widetilde{s}\|_{L^2(\Omega)} + \|\widetilde{s} - \widehat{s}\|_{L^2(\Omega)} + \|\widehat{s} - s\|_{L^2(\Omega)} < \varepsilon,$$

which is the claimed result. □

Proposition 1.4. *Let $u, v \in L^2(\mathbb{R}^d)$. Then $u * v \in C_0(\mathbb{R}^d)$.*

Proof. Let us first consider the case that $u = \mathbb{1}_{Q_1}$ and $v = \mathbb{1}_{Q_2}$ for compact cuboids $Q_1, Q_2 \subset \mathbb{R}^d$. Let $x \in \mathbb{R}^d$ be arbitrary and $(x_n)_n \subset \mathbb{R}^d$ with $x_n \to x$ as $n \to \infty$. Then for each $y \in \mathbb{R}^d$ such that $x - y \notin \partial Q_1$ we have

$$\mathbb{1}_{Q_1}(x_n - y) \longrightarrow \mathbb{1}_{Q_1}(x - y).$$

Since the y for which $x - y \in \partial Q_1$ form a set of Lebesgue measure zero it follows

$$\begin{aligned}(\mathbb{1}_{Q_1} * \mathbb{1}_{Q_2})(x_n) = \int_{\mathbb{R}^d} \mathbb{1}_{Q_1}(x_n - y)\mathbb{1}_{Q_2}(y)dy &\longrightarrow \int_{\mathbb{R}^d} \mathbb{1}_{Q_1}(x - y)\mathbb{1}_{Q_2}(y)dy \\ &= (\mathbb{1}_{Q_1} * \mathbb{1}_{Q_2})(x)\end{aligned}$$

as $n \to \infty$ by the dominated convergence theorem. Moreover, by Lemma 1.2 $\mathbb{1}_{Q_1} * \mathbb{1}_{Q_2}$ vanishes identically outside the compact set $Q_1 + Q_2$. Thus $u * v \in C_0(\mathbb{R}^d)$ if u, v are characteristic functions of cuboids. The same argument is true if u and v are finite linear combinations of characteristic functions of cuboids, i.e. step functions.

Let now $u, v \in L^2(\mathbb{R}^d)$ be arbitrary and let $(u_n)_n$ and $(v_n)_n$ be sequences of step functions with $\|u_n - u\|_{L^2(\mathbb{R}^d)} \to 0$ and $\|v_n - v\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$. Then for any $x \in \mathbb{R}^d$ we have

$$\begin{aligned}&|(u_n * v_n)(x) - (u * v)(x)| \\ &\qquad \le |(u_n * v_n)(x) - (u_n * v)(x)| + |(u_n * v)(x) - (u * v)(x)| \\ &\qquad = |(u_n * (v_n - v))(x)| + |((u_n - u) * v)(x)| \\ &\qquad \overset{(1.2)}{\le} \|u_n\|_{L^2(\mathbb{R}^d)}\|v_n - v\|_{L^2(\mathbb{R}^d)} + \|u_n - u\|_{L^2(\mathbb{R}^d)}\|v\|_{L^2(\mathbb{R}^d)} \\ &\qquad \longrightarrow 0\end{aligned}$$

as $n \to \infty$. Thus $\|u_n * v_n - u * v\|_\infty \to 0$ as $n \to \infty$ and hence $u * v \in C_0(\mathbb{R}^d)$. $\square$

Proposition 1.5. *Let $u \in L^2(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$. Then $\varphi * u \in C^\infty(\mathbb{R}^d)$ and*

$$D^\alpha(\varphi * u) = (D^\alpha \varphi) * u \quad \textit{for all } \alpha \in \mathbb{N}_0^d.$$

Proof. It suffices to prove that $\frac{\partial}{\partial x_j}(\varphi * u)$ exists and equals $\frac{\partial \varphi}{\partial x_j} * u$ for $j = 1, \dots, d$; the assertion of the proposition follows from a repeated application of this fact and Proposition 1.4. Note first that

$$\frac{1}{h}\big((\varphi * u)(x + he_j) - (\varphi * u)(x)\big) = \int_{\mathbb{R}^d} \frac{1}{h}\big(\varphi(x + he_j - y) - \varphi(x - y)\big)u(y)dy \tag{1.4}$$

holds for any $h \neq 0$, $x \in \mathbb{R}^d$ and $j = 1, \dots, d$, where e_j is the j-th unit vector in $\mathbb{R}^d$. Therefore the claim follows if we can show that the dominated convergence theorem is applicable to the integrand as $h \to 0$. Indeed, for $w, z \in \mathbb{R}^d$ we have

$$\varphi(w+z) - \varphi(w) = \int_0^1 \frac{d}{dt}\varphi(w+tz)dt = \int_0^1 (\nabla\varphi)(w+tz)\cdot z dt$$

and hence

$$|\varphi(w+z) - \varphi(w)| \leq \int_0^1 |\nabla\varphi(w+tz)\cdot z|dt \leq \int_0^1 |\nabla\varphi(w+tz)||z|dt \leq \|\nabla\varphi\|_\infty |z|,$$

which implies

$$\Big|\frac{1}{h}\big(\varphi(x+he_j-y) - \varphi(x-y)\big)\Big| \leq \frac{1}{|h|}\|\nabla\varphi\|_\infty |h| = \|\nabla\varphi\|_\infty < \infty \tag{1.5}$$

for any $x \in \mathbb{R}^d$, $h \neq 0$ and $j = 1, \dots, d$. As the integrand in (1.4) has compact support and the integral of u over this compact support is finite, the claim follows from (1.4) and (1.5). □

Theorem 1.6. *Let $u \in L^2(\mathbb{R}^d)$ and ρ_n as in* (1.1)*. Then $\rho_n * u \in C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $\|\rho_n * u - u\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$.*

Proof. By Proposition 1.5 we have $\rho_n * u \in C^\infty(\mathbb{R}^d)$. Let us first verify the identity

$$\int_{\mathbb{R}^d} |(\rho_n * u)(x)|^2 dx \leq \int_{\mathbb{R}^d} |u(x)|^2 dx, \tag{1.6}$$

which implies, in particular, $\rho_n * u \in L^2(\mathbb{R}^d)$. Indeed, for any $x \in \mathbb{R}^d$ we have

$$\begin{aligned}|(\rho_n * u)(x)| &\leq \int_{\mathbb{R}^d} \rho_n(x-y)^{1/2}\rho_n(x-y)^{1/2}|u(y)|dy \\ &\leq \Big(\underbrace{\int_{\mathbb{R}^d} \rho_n(x-y)dy}_{=1}\Big)^{1/2}\Big(\int_{\mathbb{R}^d} \rho_n(x-y)|u(y)|^2 dy\Big)^{1/2}\end{aligned}$$

by the Cauchy–Schwarz inequality. It follows

$$\int_{\mathbb{R}^d} |(\rho_n * u)(x)|^2 dx \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_n(x-y)|u(y)|^2 dy dx$$

$$= \int_{\mathbb{R}^d} \Big(\underbrace{\int_{\mathbb{R}^d} \rho_n(x-y)dx}_{=1} \Big) |u(y)|^2 dy,$$

which leads to (1.6). Let us now show the desired convergence property. Again we consider first the case that $u = \mathbb{1}_Q$ for some compact cuboid Q. Pointwise, for $x \in \mathbb{R}^d \setminus Q$ and each sufficiently large $n \in \mathbb{N}$ we have $x \notin Q + \overline{B(0, \frac{1}{n})}$ and thus $(\rho_n * \mathbb{1}_Q)(x) = 0$ by Lemma 1.2. On the other hand, for each inner point x of Q and each sufficiently large $n \in \mathbb{N}$ we have

$$(\rho_n * \mathbb{1}_Q)(x) = \int_{|y|<\frac{1}{n}} \mathbb{1}_Q(x-y)\rho_n(y)dy = \int_{|y|<\frac{1}{n}} \rho_n(y)dy = 1.$$

Thus $(\rho_n * \mathbb{1}_Q)(x) \to \mathbb{1}_Q(x)$ as $n \to \infty$ for each $x \in \mathbb{R}^d \setminus \partial Q$ and the dominated convergence theorem yields

$$\|\rho_n * u - u\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |(\rho_n * \mathbb{1}_Q)(x) - \mathbb{1}_Q(x)|^2 dx \longrightarrow 0$$

as $n \to \infty$ whenever u is the characteristic function of a cuboid. From this the assertion follows immediately for each step function u.

Let now $u \in L^2(\mathbb{R}^d)$ be arbitrary, let $\varepsilon > 0$, and let v be a step function with $\|v - u\|_{L^2(\mathbb{R}^d)} < \varepsilon/4$. By the above reasoning there exists $N(\varepsilon) \in \mathbb{N}$ such that $\|\rho_n * v - v\|_{L^2(\mathbb{R}^d)} < \varepsilon/2$ for each $n \geq N(\varepsilon)$. Hence for each $n \geq N(\varepsilon)$ we have

$$\begin{aligned} \|\rho_n * u - u\|_{L^2(\mathbb{R}^d)} &\leq \|\rho_n * (u-v)\|_{L^2(\mathbb{R}^d)} + \|\rho_n * v - v\|_{L^2(\mathbb{R}^d)} + \|v-u\|_{L^2(\mathbb{R}^d)} \\ &\overset{(1.6)}{\leq} \|u-v\|_{L^2(\mathbb{R}^d)} + \|\rho_n * v - v\|_{L^2(\mathbb{R}^d)} + \|v-u\|_{L^2(\mathbb{R}^d)} \\ &< \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon, \end{aligned}$$

which completes the proof. □

We have shown that any $u \in L^2(\mathbb{R}^d)$ can be approximated in $L^2(\mathbb{R}^d)$ by smooth functions. Our final goal of this section is to approximate any $u \in L^2(\Omega)$ on any open set Ω by functions in $\mathscr{D}(\Omega)$.

Lemma 1.7. *Let $\Omega \subset \mathbb{R}^d$ be open and nonempty and let $K \subset \Omega$ be compact. Then there exists a function $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$,* $\operatorname{supp} \varphi \subset \Omega$, *and $\varphi(x) = 1$ for all $x \in K$.*

Proof. As $\operatorname{dist}(K, \partial\Omega)$ is positive there exists $n \in \mathbb{N}$ such that $\frac{2}{n} < \operatorname{dist}(K, \partial\Omega)$. Define the compact set

$$K_n := K + \overline{B(0, 1/n)} = \{y \in \mathbb{R}^d : \exists x \in K \text{ with } |x - y| \leq 1/n\}$$

and set $\varphi := \mathbb{1}_{K_n} * \rho_n$. Then $\varphi \in C^\infty(\mathbb{R}^d)$ by Proposition 1.5. Moreover, by Lemma 1.2 we have

$$\operatorname{supp} \varphi \subset K_n + \overline{B(0, 1/n)} \subset \Omega$$

by the choice of n, and for $x \in K$ we have

$$\varphi(x) = \int_{|y|<\frac{1}{n}} \mathbb{1}_{K_n}(x - y)\rho_n(y)dy = \int_{|y|<\frac{1}{n}} \rho_n(y)dy = 1,$$

and in the same way $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^d$. □

We define a sequence of cut-off functions. For $n \in \mathbb{N}$ let

$$\Omega_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/n\} \cap B(0, n). \tag{1.7}$$

Then Ω_n is open and bounded with $\overline{\Omega_n} \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. By Lemma 1.7 for each $n \in \mathbb{N}$ there exists $\eta_n \in \mathscr{D}(\mathbb{R}^d)$ such that

$$0 \leq \eta_n \leq 1, \quad \operatorname{supp} \eta_n \subset \Omega_{n+1}, \quad \text{and} \quad \eta_n(x) = 1 \text{ for all } x \in \overline{\Omega}_n.$$

In the following for any $u \in L^2(\Omega)$ we denote by $\widetilde{u}$ the corresponding function in $L^2(\mathbb{R}^d)$ defined as

$$\widetilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & \text{else.} \end{cases}$$

Theorem 1.8. *Let $u \in L^2(\Omega)$. Then $\eta_n(\rho_n * \widetilde{u}) \in \mathscr{D}(\mathbb{R}^d)$ with $\operatorname{supp}(\eta_n(\rho_n * \widetilde{u})) \subset \Omega$ and $\|\eta_n(\rho_n * \widetilde{u}) - \widetilde{u}\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$.*

Proof. We have $\rho_n * \widetilde{u} \in C^\infty(\mathbb{R}^d)$ by Proposition 1.5. Thus $\eta_n(\rho_n * \widetilde{u}) \in \mathscr{D}(\mathbb{R}^d)$ and $\operatorname{supp}(\eta_n(\rho_n * \widetilde{u})) \subset \Omega$ follows from the properties of η_n. Moreover,

$$\begin{aligned} \|\eta_n(\rho_n * \widetilde{u}) - \widetilde{u}\|_{L^2(\mathbb{R}^d)} &\leq \|\eta_n(\rho_n * \widetilde{u} - \widetilde{u})\|_{L^2(\mathbb{R}^d)} + \|\eta_n\widetilde{u} - \widetilde{u}\|_{L^2(\mathbb{R}^d)} \\ &\leq \|\rho_n * \widetilde{u} - \widetilde{u}\|_{L^2(\mathbb{R}^d)} + \Big(\int_\Omega |\eta_n(x)u(x) - u(x)|^2 dx\Big)^{1/2}. \end{aligned}$$

The first term on the right-hand side tends to zero as $n \to \infty$ by Theorem 1.6. Moreover, as $\eta_n(x) \to 1$ as $n \to \infty$ for any $x \in \Omega$, dominated convergence implies that also the integral tends to 0. □

Corollary 1.9. *$\mathscr{D}(\Omega)$ is dense in $L^2(\Omega)$.*

For completeness we note in this context the following more general result, which may be proved with a similar strategy; cf. [6, Lemma V.1.10].

Proposition 1.10. *$\mathscr{D}(\Omega)$ is dense in $L^p(\Omega)$ for all $p \in [1,\infty)$.*

In the following we write $U \Subset \Omega$ if U is bounded and open with $\overline{U} \subset \Omega$. For $p \in [1,\infty)$ we define the spaces of locally (that is, on each compact subset) p-integrable functions

$$L^p_{\text{loc}}(\Omega) := \left\{ u : \Omega \to \mathbb{C} \text{ measurable} : \int_U |u(x)|^p dx < \infty \ \forall U \Subset \Omega \right\},$$

and the space of locally bounded functions $L^\infty_{\text{loc}}(\Omega)$ is defined accordingly. Mostly we will be using $L^2_{\text{loc}}(\Omega)$ only. Note however that $L^q_{\text{loc}}(\Omega) \subset L^p_{\text{loc}}(\Omega)$ for $q \geq p$ and that, in particular, $L^q_{\text{loc}}(\Omega) \subset L^1_{\text{loc}}(\Omega)$ for all $q \in [1,\infty]$.

Example 1.11. $x \mapsto 1/\sqrt{x}$ belongs to $L^2_{\text{loc}}(0,\infty)$ but not to $L^2(0,\infty)$.

Corollary 1.12. *Let $u \in L^1_{\text{loc}}(\Omega)$ such that*

$$\int_\Omega u(x)\varphi(x)dx = 0 \quad \textit{for all } \varphi \in \mathscr{D}(\Omega).$$

Then $u(x) = 0$ for almost all $x \in \Omega$.

Proof. We shall verify the result in the special case $u \in L^2_{\text{loc}}(\Omega)$ only. For a proof of the $L^1_{\text{loc}}(\Omega)$ case, see e.g. [1, Lemma 3.31]. Let $u \in L^2_{\text{loc}}(\Omega)$ and $U \Subset \Omega$. Then $u|_U \in L^2(U)$ and for each $\varphi \in \mathscr{D}(U)$ we have

$$\int_U u(x)\varphi(x)dx = \int_\Omega u(x)\widetilde{\varphi}(x)dx = 0$$

since $\widetilde{\varphi}|_\Omega \in \mathscr{D}(\Omega)$. Hence Corollary 1.9 yields $u|_U = 0$ in $L^2(U)$ and thus $u(x) = 0$ for almost all $x \in U$. Applying this to $U = \Omega_n$, $n \in \mathbb{N}$, with Ω_n defined in (1.7) we get the claim. □

1.2 Distributions and weak derivatives

In order to establish a satisfactory theory for partial differential equations the classical notion of differentiable functions is not sufficient. An appropriate generalization can be done by using *distributions*. This is motivated by the δ-calculus in theoretical physics (Dirac, Heaviside), where a (differentiable) "function" δ with the properties

$$\int_{\mathbb{R}} \delta(x)dx = 1 \quad \text{and} \quad \delta = 0 \text{ on } \mathbb{R} \setminus \{0\}$$

is required. The idea is to introduce a class of "generalized functions" (distributions) such that

- (at least) each continuous function is a distribution;
- each distribution can be differentiated (arbitrarily often) and for differentiable functions this is compatible with the classical definition of differentiability;
- usual rules for derivatives hold.

This is done in the following way. We equip the space $\mathscr{D}(\Omega)$ with a notion of convergence:

Definition 1.13. Let $\varphi_n, \varphi \in \mathscr{D}(\Omega)$. We say that $(\varphi_n)_n$ *converges to* φ *in* $\mathscr{D}(\Omega)$ if

(i) there exists a compact set $K \subset \Omega$ such that $\operatorname{supp}\varphi \subset K$ and $\operatorname{supp}\varphi_n \subset K$ for all $n \in \mathbb{N}$, and

(ii) $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on Ω for all $\alpha \in \mathbb{N}_0^d$.

We write shortly $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$.

Recall that uniform convergence is equivalent to convergence with respect to the supremum norm.

Definition 1.14. A *distribution on* Ω is a linear mapping $T : \mathscr{D}(\Omega) \to \mathbb{C}$ which is continuous with respect to convergence in $\mathscr{D}(\Omega)$, i.e.,

$$\varphi_n \to \varphi \text{ in } \mathscr{D}(\Omega) \quad \Longrightarrow \quad T\varphi_n \to T\varphi \text{ in } \mathbb{C}.$$

Sums and multiples of distributions are defined via

$$(T+\hat{T})\varphi := T\varphi + \hat{T}\varphi \quad \text{and} \quad (\lambda T)\varphi := \lambda(T\varphi), \qquad \varphi \in \mathscr{D}(\Omega),$$

for $\lambda \in \mathbb{C}$ and distributions $T, \hat{T}$. The set of all distributions on Ω equipped with these operations is a linear space, which we denote by $\mathscr{D}'(\Omega)$. It is the dual space of $\mathscr{D}(\Omega)$ with respect to the topology induced by the notion of convergence in Definition 1.13.

In the first example the so-called δ-distribution is discussed.

Example 1.15. For $x \in \Omega$ define

$$T_x\varphi := \varphi(x), \quad \varphi \in \mathscr{D}(\Omega).$$

This is the mathematical formalization of the "δ-function" (see exercises) and is called δ-distribution. Usually one writes δ_x instead of T_x. Is it a distribution? Linearity:

$$T_x(\lambda\varphi + \mu\psi) = (\lambda\varphi + \mu\psi)(x) = \lambda\varphi(x) + \mu\psi(x) = \lambda(T_x\varphi) + \mu(T_x\psi)$$

for $\lambda, \mu \in \mathbb{C}$ and $\varphi, \psi \in \mathscr{D}(\Omega)$. Continuity: Let $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$. Then, in particular, $\varphi_n(x) \to \varphi(x)$ in $\mathbb{C}$ and, hence,

$$T_x\varphi_n = \varphi_n(x) \to \varphi(x) = T_x\varphi.$$

It follows that $T_x \in \mathscr{D}'(\Omega)$.

Next it will be shown that any $u \in L^1_{\mathrm{loc}}(\Omega)$ gives rise to a distribution. These distributions are typically refered to as regular distributions; cf. Definition 1.17.

Example 1.16. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and define

$$T_u\varphi := \int_\Omega u(x)\varphi(x)dx, \quad \varphi \in \mathscr{D}(\Omega). \tag{1.8}$$

Then T_u is a distribution: linearity is an immediate consequence of the linearity of the integral in (1.8). For the continuity let $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$ and let $K \subset \Omega$ be compact such that $\operatorname{supp}\varphi_n, \operatorname{supp}\varphi \subset K$. Then

$$\begin{aligned} |T_u\varphi_n - T_u\varphi| &= \left| \int_\Omega u(x)\big(\varphi_n(x) - \varphi(x)\big)dx \right| \le \int_K |u(x)||\varphi_n(x) - \varphi(x)|dx \\ &\le \|\varphi_n - \varphi\|_\infty \underbrace{\int_K |u(x)|dx}_{<\infty} \longrightarrow 0 \end{aligned}$$

as $n \to \infty$. It follows that $T_u \in \mathscr{D}'(\Omega)$.

Definition 1.17. A distribution on Ω is called *regular* if there exists $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $T = T_u$ as defined in (1.8).

Note that not every distribution is regular! For instance, the δ-distribution cannot be represented in the form (1.8), see the exercises.

Lemma 1.18. *Let T be a regular distribution. Then there exists a* **unique** $u \in L^1_{\mathrm{loc}}(\Omega)$ *such that $T = T_u$.*

Proof. Assume there exist $u, v \in L^1_{\mathrm{loc}}(\Omega)$ with $T_u = T = T_v$. Then

$$\int_\Omega \big(u(x) - v(x)\big)\varphi(x)dx = 0 \quad \text{for all } \varphi \in \mathscr{D}(\Omega),$$

and hence Corollary 1.12 implies $u(x) = v(x)$ for almost all $x \in \Omega$. □

Definition 1.19. Let $T \in \mathscr{D}'(\Omega)$. The *distributional derivative of T with respect to the multi-index $\alpha \in \mathbb{N}_0^d$* is defined as

$$(D^\alpha T)\varphi := (-1)^{|\alpha|}T(D^\alpha \varphi), \quad \varphi \in \mathscr{D}(\Omega).$$

Remark 1.20. (i) $D^\alpha T \in \mathscr{D}'(\Omega)$: Linearity follows from the definition and the linearity of T and D^α. For the continuity let $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$. Then also $D^\alpha \varphi_n \to D^\alpha \varphi$ in $\mathscr{D}(\Omega)$ and, thus,

$$(D^\alpha T)\varphi_n = (-1)^{|\alpha|}T(D^\alpha \varphi_n) \longrightarrow (-1)^{|\alpha|}T(D^\alpha \varphi) = (D^\alpha T)\varphi \quad \text{as} \quad n \to \infty$$

since $T \in \mathscr{D}'(\Omega)$.

(ii) Each distribution can be differentiated arbitrarily often. In particular, each $u \in L^1_{\mathrm{loc}}(\Omega)$ can be differentiated arbitrarily often (when identified with the corresponding regular distribution). However, its derivatives may be non-regular distributions, see Example 1.21 (iii) below.

(iii) For $u \in C^1(\overline{\Omega})$ the distributional derivative is in line with the classical derivative: let, e.g., $\Omega = (a, b) \subset \mathbb{R}$ and $u \in C^1([a, b])$. Then integration by parts yields

$$\begin{aligned}(T_u)'\varphi = -T_u(\varphi') &= -\int_a^b u(x)\varphi'(x)dx \\ &= \int_a^b u'(x)\varphi(x)dx = T_{u'}\varphi, \qquad \varphi \in \mathscr{D}(a, b),\end{aligned}$$

(boundary evaluations vanish as $\varphi(a) = \varphi(b) = 0$). Accordingly it follows

$$D^\alpha T_u = T_{D^\alpha u} \quad \text{for all } \alpha \in \mathbb{N}_0^d$$

whenever $D^\alpha u$ exists in the classical sense.

Example 1.21. $\Omega = \mathbb{R}$, $T = \delta_0$:

$$\delta_0' \varphi = -\delta_0(\varphi') = -\varphi'(0), \quad \varphi \in \mathscr{D}(\mathbb{R}),$$

and, analogously,

$$\delta_0^{(k)} \varphi = (-1)^k \delta_0(\varphi^{(k)}) = (-1)^k \varphi^{(k)}(0), \quad \varphi \in \mathscr{D}(\mathbb{R}).$$

Example 1.22. $\Omega = (-1,1)$, $T = T_f$ with $f(x) = |x|$, $x \in (-1,1)$:

$$\begin{aligned} T_f' \varphi = -T_f(\varphi') &= -\int_{-1}^{1} |x| \varphi'(x) dx = \int_{-1}^{0} x\varphi'(x) dx - \int_{0}^{1} x\varphi'(x) dx \\ &= x\varphi(x)\Big|_{-1}^{0} - \int_{-1}^{0} \varphi(x) dx - x\varphi(x)\Big|_{0}^{1} + \int_{0}^{1} \varphi(x) dx \\ &= -\int_{-1}^{0} \varphi(x) dx + \int_{0}^{1} \varphi(x) dx \end{aligned}$$

where we have used integration by parts and the facts that φ has compact support and that $f(0) = 0$. Hence $T_f' = T_s$, where

$$s(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0, \end{cases}$$

is the sign function.

Example 1.23. $\Omega = (-1,1)$, $T = T_s$:

$$T_s' \varphi = -T_s(\varphi') = \int_{-1}^{0} \varphi'(x) dx - \int_{0}^{1} \varphi'(x) dx = 2\varphi(0) = 2\delta_0 \varphi, \quad \varphi \in \mathscr{D}(-1,1),$$

that is, the derivative of the sign function is twice the δ-distribution.

Definition 1.24. Let $u \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. If there exists $v \in L^1_{\text{loc}}(\Omega)$ such that $D^\alpha T_u = T_v$ then v is called *weak derivative of u with respect to α*. In this case we simply write $D^\alpha u = v$. If $\alpha = e_k$ we shall also use the notation $\frac{\partial}{\partial x_k}$ or simply ∂_k instead of D^α.

In words: If the regular distribution associated with v is the distributional derivative of the regular distribution associated with u (with respect to α) then we say that v is the weak derivative (with respect to α) of u.

Example 1.25. We have seen in Example 1.22 that the sign function s is the weak derivative of the absolute value function.

Note: An equivalent formulation of the definition of the weak derivative is the following: $v = D^\alpha u$ if and only if

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\varphi(x) dx \quad \forall \varphi \in \mathscr{D}(\Omega) \tag{1.9}$$

holds. Moreover, the weak derivative is determined uniquely by (1.9) as Corollary 1.12 shows.

Lemma 1.26. *Let Ω be connected and let $u \in L^2(\Omega)$ such that the weak derivative $\frac{\partial u}{\partial x_j}$ exists and is zero almost everywhere on Ω for each $j \in \{1, \dots, d\}$. Then there exists $c \in \mathbb{C}$ such that $u(x) = c$ for almost all $x \in \Omega$.*

Proof. Let $u_n := \eta_n(\rho_n * \widetilde{u})$, cf. Theorem 1.8. Then $u_n \in \mathscr{D}(\mathbb{R}^d)$ with $\operatorname{supp} u_n \subset \Omega$ and $u_n \to \widetilde{u}$ in $L^2(\mathbb{R}^d)$. Moreover, let $U \Subset \Omega$. Then for each sufficiently large $n \in \mathbb{N}$

$$\frac{\partial}{\partial x_j}(\rho_n * \widetilde{u}) = \rho_n * \widetilde{D^{e_j} u} = 0, \quad j = 1, \dots, d,$$

(exercise; cf. also Proposition 1.5) almost everywhere inside U. Hence for each sufficiently large $n \in \mathbb{N}$ there exists $c_n \in \mathbb{C}$ such that $u_n(x) = c_n$ for all $x \in U$ (since $\eta_n = 1$ identically on U for sufficiently large n). As

$$|U||c_n - c_m|^2 = \int_U |u_n(x) - u_m(x)|^2 dx \longrightarrow 0 \quad \text{as } m, n \to \infty,$$

$(c_n)_n$ is a Cauchy sequence in $\mathbb{C}$ and thus has a limit c. Hence

$$\int_U |u_n(x) - c|^2 dx = |U||c_n - c|^2 \to 0 \quad \text{as } n \to \infty,$$

which implies $u(x) = c$ for almost all $x \in U$. As in the proof of Corollary 1.12 it follows $u(x) = c$ for almost all $x \in \Omega$ by applying the above reasoning to $U = \Omega_n$, $n \in \mathbb{N}$, with Ω_n defined in (1.7). □

Sometimes it is useful to multiply a distribution by a smooth function.

Definition 1.27. Let $w \in C^\infty(\Omega)$ and $T \in \mathscr{D}'(\Omega)$. Then the product wT is defined as

$$(wT)\varphi := T(w\varphi), \quad \varphi \in \mathscr{D}(\Omega).$$

Note that wT is well-defined since $w\varphi \in \mathscr{D}(\Omega)$ for each $\varphi \in \mathscr{D}(\Omega)$. Note further that $wT \in \mathscr{D}'(\Omega)$ (easy exercise).

1.3 Sobolev spaces

Sobolev spaces are linear spaces consisting of (equivalence classes of) weakly differentiable functions.

Definition 1.28. Let $k \in \mathbb{N}_0$. The *Sobolev space of order k* is given by

$$H^k(\Omega) := \left\{ u \in L^2(\Omega) \colon D^\alpha u \text{ exists in } L^2(\Omega)\ \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le k \right\}.$$

Example 1.29. On $\Omega = (-1, 1)$ we have already seen in Example 1.21 (ii) that the function $f = |\cdot| \in L^2(-1, 1)$ is weakly differentiable with

$$f'(x) = s(x) = \begin{cases} -1, & x \le 0, \\ 1, & x > 0. \end{cases}$$

In particular, $f' \in L^2(-1, 1)$ and therefore $f \in H^1(-1, 1)$.

The standard inner product and norm on $H^k(\Omega)$ is provided in the next proposition. If not something different is specified explicitly, we always equip $H^k(\Omega)$ with this norm and inner product.

Proposition 1.30. *For each $k \in \mathbb{N}_0$ the mapping $(\cdot, \cdot)_{H^k(\Omega)} : H^k(\Omega) \times H^k(\Omega) \to \mathbb{C}$ given by*

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \le k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega (D^\alpha u)(x) \overline{(D^\alpha v)(x)} dx, \quad u, v \in H^k(\Omega),$$

is an inner product on $H^k(\Omega)$. Moreover, $H^k(\Omega)$ equipped with the corresponding norm

$$\|u\|_{H^k(\Omega)} = \sqrt{(u, u)_{H^k(\Omega)}} = \left(\sum_{|\alpha| \le k} \int_\Omega |(D^\alpha u)(x)|^2 dx \right)^{1/2}, \qquad u \in H^k(\Omega),$$

is a Hilbert space.

Note that for $k = 1$ the inner product can be written as

$$\begin{aligned} (u, v)_{H^1(\Omega)} &= \int_\Omega u(x) \overline{v(x)} dx + \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)} dx \\ &= (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega; \mathbb{C}^d)} \end{aligned}$$

for all $u, v \in H^1(\Omega)$.

Proof of Proposition 1.30. We show only completeness. Let $(u_n)_n$ be a Cauchy sequence in $H^k(\Omega)$, i.e., for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{|\alpha|\leq k} \|D^\alpha u_n - D^\alpha u_m\|^2_{L^2(\Omega)} < \varepsilon \quad \forall m, n \geq N(\varepsilon). \tag{1.10}$$

Then $(u_n)_n$ is a Cauchy sequence in $L^2(\Omega)$ and there exists $u \in L^2(\Omega)$ such that $\|u_n - u\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. In the same way it follows from (1.10) that for each $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$ there exists $u_\alpha \in L^2(\Omega)$ such that $\|D^\alpha u_n - u_\alpha\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. Moreover, for any α with $|\alpha| \leq k$ and any $\varphi \in \mathscr{D}(\Omega)$ we have

$$\begin{aligned}\int_\Omega u(x) D^\alpha \varphi(x) dx &= \lim_{n\to\infty} \int_\Omega u_n(x) D^\alpha \varphi(x) dx \\ &= \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega (D^\alpha u_n(x)) \varphi(x) dx \\ &= (-1)^{|\alpha|} \int_\Omega u_\alpha(x) \varphi(x) dx,\end{aligned}$$

where the continuity of the inner product in $L^2(\Omega)$ was used in order to exchange limits and integrals. It follows that $D^\alpha u$ exists and equals u_α. In particular, $u \in H^k(\Omega)$. Finally,

$$\|u_n - u\|^2_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha u_n - D^\alpha u\|^2_{L^2(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha u_n - u_\alpha\|^2_{L^2(\Omega)} \longrightarrow 0$$

as $n \to \infty$. □

The proofs of the following two lemma are left for the exercises.

Lemma 1.31. *Let $u \in H^2(\Omega)$. Then $\frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{\partial^2 u}{\partial x_k \partial x_j}$ for $j, k = 1, \dots, d$.*

Lemma 1.32. *Let $u \in H^k(\mathbb{R}^d)$. Then the identity*

$$D^\alpha(\rho_n * u) = \rho_n * D^\alpha u$$

holds for all $n \in \mathbb{N}$ and all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$.

As shown in Proposition 1.30, the space $H^k(\Omega)$ is complete. Moreover, $H^k(\Omega)$ is dense in $L^2(\Omega)$ (with respect to the norm $\|\cdot\|_{L^2(\Omega)}$) since $\mathscr{D}(\Omega) \subset H^k(\Omega)$ and

$\mathscr{D}(\Omega)$ is dense in $L^2(\Omega)$ by Corollary 1.9. On the other hand, in general $\mathscr{D}(\Omega)$ is not dense in $H^k(\Omega)$, as we will see later. We define

$$H_0^k(\Omega) := \overline{\mathscr{D}(\Omega)}^{\|\cdot\|_{H^k(\Omega)}},$$

the closure of $\mathscr{D}(\Omega)$ in $H^k(\Omega)$. Note that $H_0^k(\Omega)$ (with the norm $\|\cdot\|_{H^k(\Omega)}$) is a Hilbert space as it is a closed subspace of $H^k(\Omega)$. In general $H_0^k(\Omega)$ is a proper subspace of $H^k(\Omega)$. For $k = 1$ it consists, roughly speaking, of all functions in $H^1(\Omega)$ which vanish on the boundary of Ω in a certain sense. Lemma 1.34 and Lemma 1.35 confirm this intuition.

Proposition 1.33. *Let $u \in H_0^k(\Omega)$. Then $\widetilde{u} \in H^k(\mathbb{R}^d)$ and $D^\alpha \widetilde{u} = \widetilde{D^\alpha u}$ holds for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$.*

Proof. By definition of $H_0^k(\Omega)$ there exists a sequence $(u_n)_n \subset \mathscr{D}(\Omega)$ such that $\|u_n - u\|_{H^k(\Omega)} \to 0$ as $n \to \infty$. Note that $\|\widetilde{u}_n - \widetilde{u}\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$. Then for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and all α with $|\alpha| \leq k$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \widetilde{u}(x) D^\alpha \varphi(x)dx &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \widetilde{u}_n(x) D^\alpha \varphi(x)dx \\
&= (-1)^{|\alpha|} \lim_{n\to\infty} \int_{\mathbb{R}^d} (D^\alpha \widetilde{u}_n)(x)\varphi(x)dx \\
&= (-1)^{|\alpha|} \lim_{n\to\infty} \int_{\Omega} (D^\alpha u_n)(x)\varphi(x)dx \\
&= \int_{\Omega} (D^\alpha u)(x)\varphi(x)dx \\
&= \int_{\mathbb{R}^d} \widetilde{(D^\alpha u)}(x)\varphi(x)dx,
\end{aligned}$$

which implies the assertions. □

Lemma 1.34. *Let $u \in H_0^1(\Omega)$ with $\frac{\partial u}{\partial x_j} = 0$ almost everywhere on Ω for each $j \in \{1, \dots, d\}$. Then $u(x) = 0$ for almost all $x \in \Omega$.*

Proof. By Proposition 1.33 we have $\widetilde{u} \in H^1(\mathbb{R}^d)$ and $\frac{\partial \widetilde{u}}{\partial x_j} = 0$ for $j = 1, \dots, d$. Thus by Lemma 1.26 there exists $c \in \mathbb{C}$ such that $\widetilde{u}(x) = c$ for almost all $x \in \Omega$. Now $\widetilde{u} \in L^2(\mathbb{R}^d)$ implies $c = 0$. □

The following lemma is left for the exercise classes.

Lemma 1.35. *Let $u \in H^1(\Omega)$ and $U \Subset \Omega$. If $u(x) = 0$ for a.e. $x \in \Omega \setminus U$ then $u \in H^1_0(\Omega)$.*

Since $\Omega = \mathbb{R}^d$ has no boundary it seems intuitive that $H^1_0(\mathbb{R}^d) = H^1(\mathbb{R}^d)$. In fact, one has the more general result:

Theorem 1.36. *$H^k_0(\mathbb{R}^d) = H^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}_0$.*

Proof. Let $u \in H^k(\mathbb{R}^d)$. Then

$$D^\alpha(\rho_n * u) = \rho_n * D^\alpha u$$

for all $n \in \mathbb{N}$ and all $\alpha \in \mathbb{N}^d_0$ with $|\alpha| \le k$ by Lemma 1.32. In particular, $D^\alpha(\rho_n * u) \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and $\|D^\alpha(\rho_n * u) - D^\alpha u\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$ by Theorem 1.6. Hence $\rho_n * u \to u$ in $H^k(\mathbb{R}^d)$. Furthermore, by Lemma 1.7 there exists $\eta \in \mathscr{D}(\mathbb{R}^d)$ such that $0 \le \eta \le 1$, $\eta(x) = 1$ for all x with $|x| \le 1$ and $\eta(x) = 0$ for all x with $|x| \ge 2$. For $m \in \mathbb{N}$ let further $\eta_m(x) := \eta(\frac{x}{m})$ for $x \in \mathbb{R}^d$. Then

$$D^\alpha(\eta_m(\rho_n * u)) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta}\eta_m)(D^\beta(\rho_n * u)),$$

where $\beta \le \alpha$ means $\beta_j \le \alpha_j$, $j = 1, \dots, d$, and $\binom{\alpha}{\beta} = \prod_{j=1}^d \binom{\alpha_j}{\beta_j}$. Thus for each $x \in \mathbb{R}^d$

$$\begin{aligned} D^\alpha(\eta_m(\rho_n * u))(x) &= \sum_{\substack{\beta \le \alpha \\ \beta \ne \alpha}} \binom{\alpha}{\beta} (D^{\alpha-\beta}\eta_m)(x)(D^\beta(\rho_n * u))(x) + \eta_m(x) D^\alpha(\rho_n * u)(x) \\ &= \sum_{\substack{\beta \le \alpha \\ \beta \ne \alpha}} \frac{1}{m^{|\alpha-\beta|}} \binom{\alpha}{\beta} (D^{\alpha-\beta}\eta)(x/m)(D^\beta(\rho_n * u)) \\ &\quad + \eta_m(x) D^\alpha(\rho_n * u)(x) \\ &\longrightarrow D^\alpha(\rho_n * u)(x) \end{aligned}$$

as $m \to \infty$. Using $D^\beta(\rho_n * u) \in L^2(\mathbb{R}^d)$ for $|\beta| \le k$ with dominated convergence we obtain

$$\|D^\alpha(\eta_m(\rho_n * u)) - D^\alpha(\rho_n * u)\|_{L^2(\mathbb{R}^d)} \longrightarrow 0 \quad \text{as} \quad m \to \infty$$

for any α with $|\alpha| \le k$, that is, $\eta_m(\rho_n * u) \to \rho_n * u$ in $H^k(\mathbb{R}^d)$. As $\eta_m(\rho_n * u) \in \mathscr{D}(\mathbb{R}^d)$ for all m, n, the claim follows. $\square$

In the following theorem we say that Ω is bounded with respect to one direction if there exist $j \in \{1, \dots, d\}$ and $\delta > 0$ such that $|x_j| < \delta$ for all $x \in \Omega$.

Theorem 1.37 (Poincaré inequality). *Let Ω be bounded with respect to one direction with $\delta > 0$ as above. Then*

$$\|u\|_{L^2(\Omega)} \leq \sqrt{2}\delta \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)} \quad \forall u \in H_0^1(\Omega).$$

Proof. In order to simplify notation we assume $|x_1| < \delta$ for all $x \in \Omega$. Then for any $h \in C^1([-\delta, \delta])$ with $h(-\delta) = 0$ we have

$$\begin{aligned}\int_{-\delta}^{\delta} |h(t)|^2 dt &= \int_{-\delta}^{\delta} \left| \int_{-\delta}^{t} h'(s) \cdot 1 ds \right|^2 dt \leq \int_{-\delta}^{\delta} \int_{-\delta}^{t} |h'(s)|^2 ds (t+\delta) dt \\ &\leq \int_{-\delta}^{\delta} |h'(s)|^2 ds \int_{-\delta}^{\delta} (t+\delta) dt = 2\delta^2 \int_{-\delta}^{\delta} |h'(s)|^2 ds\end{aligned}$$

by the Cauchy–Schwarz inequality. Thus for any $u \in \mathscr{D}(\Omega)$ we obtain

$$\begin{aligned}\int_{\Omega} |u(x)|^2 dx &\leq \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{-\delta}^{\delta} 2\delta^2 \left| \frac{\partial \widetilde{u}}{\partial x_1}(x_1, \dots, x_d) \right|^2 dx_1 dx_2 \cdots dx_d \\ &\leq 2\delta^2 \int_{\Omega} |\nabla u(x)|^2 dx,\end{aligned}$$

which shows the assertion for $u \in \mathscr{D}(\Omega)$. For general $u \in H_0^1(\Omega)$ it follows by approximation. □

Corollary 1.38. *If Ω is bounded with respect to one direction then*

$$|u|_{H^1(\Omega)} := \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)}, \quad u \in H_0^1(\Omega),$$

defines a norm on $H_0^1(\Omega)$, which is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$ on $H_0^1(\Omega)$.

The next theorem is due to Meyers and Serrin; a proof can be found in, e.g. [1, Theorem 3.17]

Theorem 1.39. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Then $C^\infty(\Omega) \cap H^k(\Omega)$ is dense in $H^k(\Omega)$ for all $k \in \mathbb{N}_0$.*

1.4 Sobolev spaces via Fourier transformation

In this section we express the Sobolev space $H^k(\mathbb{R}^d)$ in terms of the Fourier transformation. Recall that for $u \in L^1(\mathbb{R}^d)$ the Fourier transform of u is defined as

$$\hat{u}(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot y} u(y)dy, \quad x \in \mathbb{R}^d.$$

Moreover, recall that by the Riemann–Lebesgue lemma

$$\hat{u} \in C_0(\mathbb{R}^d) := \Big\{ v \in C(\mathbb{R}^d) : \lim_{|x|\to\infty} v(x) = 0 \Big\}$$

holds for each $u \in L^1(\mathbb{R}^d)$. The following theorem is known as Plancherel theorem.

Theorem 1.40. *There exists a unique unitary operator $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $\mathcal{F}u = \hat{u}$ holds for all $u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In particular, the Parseval identity*

$$(\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^d)} = (u, v)_{L^2(\mathbb{R}^d)}, \quad u, v \in L^2(\mathbb{R}^d),$$

holds. Moreover, $(\mathcal{F}^{-1}u)(x) = (\mathcal{F}u)(-x)$, $x \in \mathbb{R}^d$, holds for all $u \in L^2(\mathbb{R}^d)$.

Lemma 1.41. *Let $k \in \mathbb{N}_0$ and $u \in H^k(\mathbb{R}^d)$. Then*

$$(\mathcal{F}(D^\alpha u))(x) = i^{|\alpha|} x^\alpha (\mathcal{F}u)(x) \quad \textit{for almost all } x \in \mathbb{R}^d$$

holds for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$.

Proof. Let first $u \in \mathscr{D}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$ arbitrary. Then integration by parts yields

$$\begin{aligned}(\mathcal{F}(D^\alpha u))(x) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot y} (D^\alpha u)(y)dy \\ &= (-1)^{|\alpha|} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-i)^{|\alpha|} x^\alpha e^{-ix\cdot y} u(y)dy = i^{|\alpha|} x^\alpha (\mathcal{F}u)(x)\end{aligned}$$

for all $x \in \mathbb{R}^d$. Now fix $u \in H^k(\mathbb{R}^d)$ and pick a sequence (u_n) in $\mathscr{D}(\mathbb{R}^d)$ such that $u_n \to u$ as $n \to \infty$ in $H^k(\Omega)$. Since $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is continuous and $D^\alpha u_n \to D^\alpha u$ for $|\alpha| \le k$ it is clear that $\mathcal{F}(D^\alpha u_n) \to \mathcal{F}(D^\alpha u)$ and $\mathcal{F}u_n \to \mathcal{F}u$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Choose a subsequence such that $(\mathcal{F}(D^\alpha u_{n_k}))(x) \to (\mathcal{F}(D^\alpha u))(x)$ and $(\mathcal{F}u_{n_k})(x) \to (\mathcal{F}u)(x)$ for a.e. $x \in \mathbb{R}^d$ as $k \to \infty$ and, in fact it can be assumed that (u_n) was chosen right away such that $(\mathcal{F}(D^\alpha u_n))(x) \to (\mathcal{F}(D^\alpha u))(x)$ and $(\mathcal{F}u_n)(x) \to (\mathcal{F}u)(x)$ for a.e. $x \in \mathbb{R}^d$. It is clear that also $i^{|\alpha|} x^\alpha (\mathcal{F}u_n)(x) \to i^{|\alpha|} x^\alpha (\mathcal{F}u)(x)$ for a.e. $x \in \mathbb{R}^d$ as $n \to \infty$, and hence the assertion follows. □

Note that in the above lemma one has $\mathcal{F}(D^\alpha u) \in L^2(\mathbb{R}^d)$ and hence also $x \to i^{|\alpha|} x^\alpha (\mathcal{F}u)(x) \in L^2(\mathbb{R}^d)$.

Theorem 1.42. *Let $k \in \mathbb{N}_0$. Then*

$$H^k(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |x|^2)^k |(\mathcal{F}u)(x)|^2 dx < \infty \right\}.$$

Moreover, there exist constants $c, C > 0$ such that

$$c\|u\|^2_{H^k(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} (1 + |x|^2)^k |(\mathcal{F}u)(x)|^2 dx \le C\|u\|^2_{H^k(\mathbb{R}^d)} \quad \forall u \in H^k(\mathbb{R}^d).$$

Proof. Note first that there exist constants $c, C > 0$ such that

$$c \sum_{|\alpha| \le k} |x^\alpha|^2 \le (1 + |x|^2)^k \le C \sum_{|\alpha| \le k} |x^\alpha|^2 \quad \forall x \in \mathbb{R}^d \tag{1.11}$$

(exercise). Let $u \in H^k(\mathbb{R}^d)$. Then the Parseval identity yields

$$\begin{aligned} \|u\|^2_{H^k(\mathbb{R}^d)} &= \sum_{|\alpha| \le k} \|D^\alpha u\|^2_{L^2(\mathbb{R}^d)} = \sum_{|\alpha| \le k} \|\mathcal{F}(D^\alpha u)\|^2_{L^2(\mathbb{R}^d)} \\ &\overset{\text{Lemma 1.41}}{=} \sum_{|\alpha| \le k} \int_{\mathbb{R}^d} |x^\alpha (\mathcal{F}u)(x)|^2 dx \overset{(1.11)}{\ge} \frac{1}{C} \int_{\mathbb{R}^d} (1 + |x|^2)^k |(\mathcal{F}u)(x)|^2 dx; \end{aligned} \tag{1.12}$$

in particular the last integral is finite. Let now, conversely, $u \in L^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} (1+|x|^2)^k |(\mathcal{F}u)(x)|^2 dx < \infty$. Then due to (1.11) the functions $x \mapsto x^\alpha (\mathcal{F}u)(x)$ belong to $L^2(\mathbb{R}^d)$ for $|\alpha| \le k$. In particular, for each α with $1 \le |\alpha| \le k$ there exists $u_\alpha \in L^2(\mathbb{R}^d)$ such that

$$i^{|\alpha|} x^\alpha (\mathcal{F}u)(x) = (\mathcal{F}u_\alpha)(x) \quad \text{for almost all } x \in \mathbb{R}^d.$$

Thus for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and any α with $|\alpha| \le k$ with the Parseval identity we have

$$\begin{aligned} (-1)^{|\alpha|} \int_{\mathbb{R}^d} u(x)(D^\alpha \varphi)(x) dx &= (-1)^{|\alpha|} (u, \overline{D^\alpha \varphi})_{L^2(\mathbb{R}^d)} \\ &= (-1)^{|\alpha|} (\mathcal{F}u, \mathcal{F}(\overline{D^\alpha \varphi}))_{L^2(\mathbb{R}^d)} \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^d} (\mathcal{F}u)(x) \overline{i^{|\alpha|} x^\alpha (\mathcal{F}\overline{\varphi})(x)} dx \end{aligned}$$

$$
\begin{aligned}
&= (\mathcal{F}u_\alpha, \mathcal{F}\overline{\varphi})_{L^2(\mathbb{R}^d)} \\
&= (u_\alpha, \overline{\varphi})_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} u_\alpha(x)\varphi(x)dx,
\end{aligned}
$$

which implies $u \in H^k(\mathbb{R}^d)$. The second statement follows from (1.12) and (1.11). □

In general functions in Sobolev spaces are not automatically continuous (cf. exercises). However, this is true if the space dimension is small enough in comparison with the Sobolev index.

Theorem 1.43 (Sobolev embedding theorem)**.** *If $k > d/2$ then $H^k(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$.*

Proof. Let $u \in H^k(\mathbb{R}^d)$. First we show that $\mathcal{F}u \in L^1(\mathbb{R}^d)$. Indeed,

$$
\begin{aligned}
\int_{\mathbb{R}^d} |(\mathcal{F}u)(x)|dx &= \int_{\mathbb{R}^d} (1+|x|^2)^{k/2}|(\mathcal{F}u)(x)|(1+|x|^2)^{-k/2}dx \\
&\le \left(\int_{\mathbb{R}^d} (1+|x|^2)^k|(\mathcal{F}u)(x)|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^d}(1+|x|^2)^{-k}dx\right)^{1/2}
\end{aligned}
$$

by the Hölder inequality. The first integral on the right-hand side is finite by Theorem 1.42. Moreover, the integrand of the second integral is bounded and

$$
\int_{|x|\ge 1}(1+|x|^2)^{-k}dx \le \int_{|x|\ge 1}|x|^{-2k}dx = C_d\int_1^\infty r^{-2k}r^{d-1}dr = C_d\int_1^\infty r^{d-2k-1}dr,
$$

which is finite since $d-2k-1 < -1$ by assumption; here the C_d is the surface area of the unit sphere in $\mathbb{R}^d$. Thus $\mathcal{F}u \in L^1(\mathbb{R}^d)$. As $u(x) = (\mathcal{F}^{-1}(\mathcal{F}u))(x) = (\mathcal{F}(\mathcal{F}u))(-x)$ for almost all $x \in \mathbb{R}^d$ by the Plancherel theorem, the Riemann–Lebesgue lemma implies $u \in C_0(\mathbb{R}^d)$. □

Corollary 1.44. *Let $k \in \mathbb{N}$ with $k > d/2$ and $m \in \mathbb{N}_0$. Then $H^{k+m}(\mathbb{R}^d) \subset C^m(\mathbb{R}^d)$.*

Proof. For $m = 0$ this follows from Theorem 1.43. Now by induction: Assume $H^{k+m}(\mathbb{R}^d) \subset C^m(\mathbb{R}^d)$ for a fixed m and let $u \in H^{k+m+1}(\mathbb{R}^d) \subset H^{k+m}(\mathbb{R}^d)$. Then $u \in C^m(\mathbb{R}^d)$ and $\frac{\partial u}{\partial x_j} \in H^{k+m}(\mathbb{R}^d) \subset C^m(\mathbb{R}^d)$ for $j = 1, \ldots, d$ by assumption. It follows $u \in C^{m+1}(\mathbb{R}^d)$. □

In the following we call the identity map from $H_0^1(\Omega)$ to $L^2(\Omega)$ the *embedding* of $H_0^1(\Omega)$ into $L^2(\Omega)$. Recall that a linear operator $T : U \to V$ between Banach spaces U and V is called *compact* if for each bounded sequence $(u_n)_n \subset U$ there exists a subsequence $(u_{n_k})_k$ such that $(Tu_{n_k})_k$ converges in V. Recall further that a sequence $(u_n)_n$ in a Banach space U *converges weakly* to some $u \in U$ if $F(u_n)$ converges to $F(u)$ in $\mathbb{C}$ for each bounded, linear functional $F : U \to \mathbb{C}$, and that in a Hilbert space each bounded sequence contains a weakly convergent subsequence.

Theorem 1.45 (Rellich embedding theorem)**.** *Let Ω be bounded. Then the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact.*

Proof. Let $(u_n)_n \subset H_0^1(\Omega)$ be bounded in $H_0^1(\Omega)$, that is, there exists $c > 0$ with $\|u_n\|_{H^1(\Omega)} \leq c$ for all n. Then $(u_n)_n$ contains a weakly convergent subsequence; without loss of generality we assume that $(u_n)_n$ itself converges weakly in $H^1(\Omega)$ to some $u \in H^1(\Omega)$. (Note: $\|u\|_{H^1(\Omega)} \leq c$.) Then the sequence $(\widetilde{u}_n)_n$ belongs to $H^1(\mathbb{R}^d)$ (see Proposition 1.33) and converges weakly in $H^1(\mathbb{R}^d)$ to $\widetilde{u}$ (for this write down inner products). Moreover, $\|\widetilde{u}_n\|_{H^1(\mathbb{R}^d)} \leq c$ for all n and $\|\widetilde{u}\|_{H^1(\mathbb{R}^d)} \leq c$. Our aim is to show $\|\widetilde{u}_n - \widetilde{u}\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$. For this let $\varepsilon > 0$ and let $R > 0$ such that $4c^2C(1+R^2)^{-1} < \varepsilon$, where $C > 0$ is as in Theorem 1.42. Note that for any $x \in \mathbb{R}^d$ the linear mapping $F_x : H^1(\mathbb{R}^d) \to \mathbb{C}$,

$$F_x(v) := \frac{1}{(2\pi)^{d/2}} \int_\Omega e^{-ix\cdot y} v(y)dy, \quad v \in H^1(\mathbb{R}^d),$$

is bounded since

$$\begin{aligned} |F_x(v)| &\leq \frac{1}{(2\pi)^{d/2}} \int_\Omega |v(y)|dy \leq \frac{1}{(2\pi)^{d/2}} \left(\int_\Omega 1^2 dx\right)^{1/2} \left(\int_\Omega |v(y)|^2 dy\right)^{1/2} \\ &\leq \frac{1}{(2\pi)^{d/2}} |\Omega|^{1/2} \|v\|_{H^1(\mathbb{R}^d)} \end{aligned} \tag{1.13}$$

for all $v \in H^1(\mathbb{R}^d)$. Thus

$$(\mathcal{F}\widetilde{u}_n)(x) = F_x(\widetilde{u}_n) \longrightarrow F_x(\widetilde{u}) = (\mathcal{F}\widetilde{u})(x) \quad \text{as } n \to \infty$$

for all $x \in \mathbb{R}^d$. Moreover, by (1.13) we have

$$|(\mathcal{F}\widetilde{u}_n - \mathcal{F}\widetilde{u})(x)| \leq \frac{1}{(2\pi)^{d/2}} |\Omega|^{1/2} 2c$$

for all n. Thus by dominated convergence

$$\int_{|x|\le R} |(\mathcal{F}\widetilde{u}_n)(x) - (\mathcal{F}\widetilde{u})(x)|^2 dx \longrightarrow 0 \quad \text{as } n \to \infty. \tag{1.14}$$

On the other hand, by Theorem 1.42

$$\begin{aligned}
&\int_{|x|>R} |(\mathcal{F}\widetilde{u}_n)(x) - (\mathcal{F}\widetilde{u})(x)|^2 dx \\
&\qquad = \int_{|x|>R} (1+|x|^2)^{-1}(1+|x|^2)|(\mathcal{F}\widetilde{u}_n)(x) - (\mathcal{F}\widetilde{u})(x)|^2 dx \\
&\qquad \le \frac{C}{1+R^2}\|\widetilde{u}_n - \widetilde{u}\|^2_{H^1(\mathbb{R}^d)} \\
&\qquad \le \frac{C}{1+R^2}\big(\|\widetilde{u}_n\|_{H^1(\mathbb{R}^d)} + \|\widetilde{u}\|_{H^1(\mathbb{R}^d)}\big)^2 < \varepsilon.
\end{aligned}$$

From this and (1.14) we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |(\mathcal{F}\widetilde{u}_n)(x) - (\mathcal{F}\widetilde{u})(x)|^2 dx < \varepsilon.$$

As $\varepsilon > 0$ was chosen arbitrarily it follows $\|\mathcal{F}(\widetilde{u}_n - \widetilde{u})\|_{L^2(\mathbb{R}^d)} \to 0$ and thus $\|\widetilde{u}_n - \widetilde{u}\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$. It follows $\|u_n - u\|_{L^2(\Omega)} \to 0$. □

Chapter 2

The Poisson equation with Dirichlet boundary conditions

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be open and nonempty. In this chapter we deal with the *Poisson equation*

$$\begin{aligned} -\Delta u - \lambda u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\boldsymbol{f \in L^2(\Omega)}$ and $\boldsymbol{\lambda \leq 0}$ are given. In the setting of Sobolev spaces the equation (2.1) is only formal since in general $\partial\Omega$ is a set of measure zero and thus evaluation of some Sobolev function u on $\partial\Omega$ is not well-defined. As mentioned above, for $u \in H^1(\Omega)$ the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ can be modeled via the requirement $u \in H_0^1(\Omega)$. Therefore the equation under consideration in this chapter is

$$-\Delta u - \lambda u = f, \quad u \in H_0^1(\Omega), \tag{2.2}$$

where $-\Delta u$ has to be understood in the sense of distributional derivatives and the equality is an equality of distributions. Note that $-\Delta u - \lambda u = f$ with $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ implies $-\Delta u \in L^2(\Omega)$.

Lemma 2.1. *Let $u \in H^1(\Omega)$. Then u is a solution of $-\Delta u - \lambda u = f$ (in the distributional sense) if and only if*

$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x)\, dx - \lambda \int_\Omega u(x)\varphi(x)\, dx = \int_\Omega f(x)\varphi(x)\, dx$$

holds for all $\varphi \in H_0^1(\Omega)$.

Proof. If $u \in H^1(\Omega)$ satisfies $-\Delta u - \lambda u = f$ distributionally then

$$\begin{aligned}\int_\Omega f(x)\varphi(x)dx &= (-\Delta T_u)(\varphi) - \lambda \int_\Omega u(x)\varphi(x)dx \\ &= -\sum_{j=1}^d \frac{\partial^2 T_u}{\partial x_j^2}(\varphi) - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ &= \sum_{j=1}^d \frac{\partial T_u}{\partial x_j}\Big(\frac{\partial \varphi}{\partial x_j}\Big) - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ &\overset{u\in H^1(\Omega)}{=} \sum_{j=1}^d \int_\Omega \frac{\partial u}{\partial x_j}(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx - \lambda \int_\Omega u(x)\varphi(x)dx \\ &= \int_\Omega \nabla u(x)\cdot\nabla\varphi(x)dx - \lambda\int_\Omega u(x)\varphi(x)\,dx\end{aligned}$$

for each $\varphi \in \mathscr{D}(\Omega)$. Via approximation this identity extends to all $\varphi \in H_0^1(\Omega)$. The converse implication follows by going the same steps backwards. ☐

Theorem 2.2. *Let $f \in L^2(\Omega)$ and $\lambda \leq 0$. Assume in addition that Ω is bounded with respect to one direction or that $\lambda < 0$. Then* (2.2) *is uniquely solvable.*

In the proof we use the Lax–Milgram theorem. Recall that a symmetric sesquilinear form on a (complex) Hilbert space V is a mapping $\mathfrak{a} : V \times V \to \mathbb{C}$ which satisfies

$$\mathfrak{a}[\alpha u + \beta w, v] = \alpha\mathfrak{a}[u, v] + \beta\mathfrak{a}[w, v], \qquad u, v, w \in V, \alpha, \beta \in \mathbb{C},$$

and

$$\mathfrak{a}[u, v] = \overline{\mathfrak{a}[v, u]}, \qquad u, v \in V.$$

In particular, $\mathfrak{a}[u, \alpha v] = \overline{\alpha}\mathfrak{a}[u, v]$ for all $u, v \in V$, $\alpha \in \mathbb{C}$, and $\mathfrak{a}[u] := \mathfrak{a}[u, u]$ is real for all $u \in V$. We denote by V^* the *anti-dual* of V, i.e. the space of all bounded, antilinear functionals $F : V \to \mathbb{C}$.

Theorem (Lax–Milgram theorem)**.** *Let V be a Hilbert space and let $\mathfrak{a} : V \times V \to \mathbb{C}$ be a symmetric sesquilinear form such that*

(a) $\mathfrak{a}$ is bounded, i.e., there exists $M > 0$ with $|\mathfrak{a}[u, v]| \leq M\|u\|_V\|v\|_V$ for all $u, v \in V$;

(b) $\mathfrak{a}$ *is coercive, i.e., there exists* $\eta > 0$ *with* $\mathfrak{a}[u] \geq \eta \|u\|_V^2$ *for all* $u \in V$.

Then for each $F \in V^*$ *there exists a unique* $u \in V$ *such that*

$$\mathfrak{a}[u, \varphi] = F(\varphi), \qquad \varphi \in V.$$

Proof of Theorem 2.2. Define $\mathfrak{a} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$,

$$\mathfrak{a}[u, v] := \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)}\, dx - \lambda \int_\Omega u(x)\overline{v(x)}\, dx, \quad u, v \in H_0^1(\Omega).$$

Then $\mathfrak{a}$ is sesquilinear and symmetric. Moreover, by the Cauchy–Schwarz inequality

$$\begin{aligned} |\mathfrak{a}[u, v]| &\leq \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)} \|\nabla v\|_{L^2(\Omega;\mathbb{C}^d)} + \lambda \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq (1 + \lambda) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \end{aligned}$$

for all $u, v \in H_0^1(\Omega)$, that is, $\mathfrak{a}$ is bounded. Moreover, $\mathfrak{a}$ is coercive: if $\lambda < 0$ then

$$\mathfrak{a}[u] = \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)}^2 - \lambda \|u\|_{L^2(\Omega)}^2 \geq \min\{1, -\lambda\} \|u\|_{H^1(\Omega)}^2, \qquad u \in H_0^1(\Omega).$$

If Ω is bounded with respect to one direction and $\lambda = 0$ then the Poincaré inequality (Theorem 1.37) yields

$$\begin{aligned} \mathfrak{a}[u] &= \frac{1}{2} \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)}^2 \geq \frac{1}{2} \frac{1}{2\delta^2} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)}^2 \\ &\geq \min \left\{ \frac{1}{4\delta^2}, \frac{1}{2} \right\} \|u\|_{H^1(\Omega)}^2 \end{aligned}$$

for all $u \in H_0^1(\Omega)$. Hence in both cases $\mathfrak{a}$ is coercive. Moreover, the mapping $F : H_0^1(\Omega) \to \mathbb{C}$,

$$F(\varphi) := \int_\Omega f(x)\overline{\varphi(x)}\, dx, \qquad \varphi \in H_0^1(\Omega),$$

is bounded. Thus by the Lax–Milgram theorem there exists a unique $u \in H_0^1(\Omega)$ such that

$$\mathfrak{a}[u, \varphi] = F(\varphi), \qquad \varphi \in H_0^1(\Omega),$$

and Lemma 2.1 yields the desired assertion. □

Let us next consider the case $\Omega = \mathbb{R}^d$. We show that solutions are more regular in this case.

Lemma 2.3. *For $f \in L^2(\mathbb{R}^d)$ the equation $-\Delta u + u = f$ has a unique solution $u \in H^2(\mathbb{R}^d)$. In particular, if $v, f \in L^2(\mathbb{R}^d)$ with $-\Delta v + v = f$ then $v \in H^2(\mathbb{R}^d)$.*

Proof. Since $f \in L^2(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \frac{1}{(1+|x|^2)^2} |(\mathcal{F}f)(x)|^2 dx \leq \int_{\mathbb{R}^d} |(\mathcal{F}f)(x)|^2 dx < \infty.$$

Hence by the Plancherel theorem there exists $u \in L^2(\mathbb{R}^d)$ such that

$$\frac{1}{1+|x|^2}(\mathcal{F}f)(x) = (\mathcal{F}u)(x) \quad \text{for almost all } x \in \mathbb{R}^d.$$

Moreover,

$$\int_{\mathbb{R}^d} (1+|x|^2)^2 |(\mathcal{F}u)(x)|^2 dx = \int_{\mathbb{R}^d} |(\mathcal{F}f)(x)|^2 dx < \infty.$$

With the help of Theorem 1.42 it follows $u \in H^2(\mathbb{R}^d)$. Furthermore, Lemma 1.41 implies

$$\big(\mathcal{F}(-\Delta u + u)\big)(x) = (|x|^2 + 1)(\mathcal{F}u)(x) = (\mathcal{F}f)(x)$$

for almost all $x \in \mathbb{R}^d$ and hence $-\Delta u + u = f$, that is, u is the unique solution of $-\Delta u + u = f$ in $H^1(\mathbb{R}^d)$.

Let now $v \in L^2(\mathbb{R}^d)$ such that $-\Delta v + v = f$ and define $w := u - v$. Then one has $-\Delta w + w = 0$ and hence

$$\int_{\mathbb{R}^d} w(x)\overline{(\varphi(x) - \Delta\varphi(x))}\, dx = \int_{\mathbb{R}^d} (-\Delta w(x) + w(x))\overline{\varphi(x)}\, dx = 0, \quad \varphi \in \mathscr{D}(\mathbb{R}^d),$$

and thus for any $\varphi \in H^2(\mathbb{R}^d)$. By the first part of the lemma we can choose $\varphi \in H^2(\mathbb{R}^d)$ such that $\varphi - \Delta\varphi = w$ and it follows $w = 0$ and thus $v = u \in H^2(\mathbb{R}^d)$. □

The following theorem is sometimes called a regularity shift theorem.

Theorem 2.4. *Let $u \in L^2(\mathbb{R}^d)$ and let $f \in H^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}_0$. If $-\Delta u = f$ then $u \in H^{k+2}(\mathbb{R}^d)$.*

Proof. We use induction over k. For $k = 0$ we have $-\Delta u + u = f + u \in L^2(\mathbb{R}^d)$ and Lemma 2.3 yields $u \in H^2(\mathbb{R}^d)$. Assume now that the assertion of the theorem holds for a fixed k and let $f \in H^{k+1}(\mathbb{R}^d) \subset H^k(\mathbb{R}^d)$. Then $u \in H^{k+2}(\mathbb{R}^d)$ and hence $-\Delta u + u = f + u =: g \in H^{k+1}(\mathbb{R}^d)$. It follows

$$\begin{aligned}\int_{\mathbb{R}^d} (1+|x|^2)^{k+3} |(\mathcal{F}u)(x)|^2 dx &= \int_{\mathbb{R}^d} (1+|x|^2)^{k+1} \big|(1+|x|^2)(\mathcal{F}u)(x)\big|^2 dx \\ &= \int_{\mathbb{R}^d} (1+|x|^2)^{k+1} |(\mathcal{F}g)(x)|^2 dx < \infty\end{aligned}$$

as $g \in H^{k+1}(\mathbb{R}^d)$. Hence $u \in H^{k+3}(\mathbb{R}^d)$. □

For $\Omega \neq \mathbb{R}^d$ the regularity issue is more involved and $\Delta u \in L^2(\Omega)$ does in general **not** imply $u \in H^2(\Omega)$. Our main goal will be to show below local regularity properties making use of the so-called difference quotient method. First we define the spaces

$$H^k_{\mathrm{loc}}(\Omega) := \big\{u \in L^2_{\mathrm{loc}}(\Omega) : D^\alpha u \in L^2_{\mathrm{loc}}(\Omega)\ \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq k\big\}$$

and note that the following properties hold:

- $C^k(\Omega) \subset H^k_{\mathrm{loc}}(\Omega)$.
- $u \in H^k_{\mathrm{loc}}(\Omega)$ and $U \Subset \Omega \Rightarrow u|_U \in H^k(U)$.

The next lemma is useful to relate the local spaces $H^k_{\mathrm{loc}}(\Omega)$ with the spaces $H^k_0(\Omega)$.

Lemma 2.5. *The identity*

$$H^k_{\mathrm{loc}}(\Omega) = \big\{u \in L^2_{\mathrm{loc}}(\Omega) : \eta u \in H^k_0(\Omega) \text{ for all } \eta \in \mathscr{D}(\Omega)\big\} \tag{2.3}$$

holds for each $k \in \mathbb{N}$. Moreover, for $u \in H^k_{\mathrm{loc}}(\Omega)$ and $\eta \in \mathscr{D}(\Omega)$ we have

$$D^\alpha(\eta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta}\eta)(D^\beta u) \tag{2.4}$$

for all $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq k$.

Proof. Let $u \in H^k_{\mathrm{loc}}(\Omega)$. Then for any $\eta \in \mathscr{D}(\Omega)$ we have $\eta u \in L^2(\Omega)$ and $(D^\gamma \eta)(D^\beta u) \in L^2(\Omega)$ for all $\beta, \gamma \in \mathbb{N}_0^d$ with $|\beta| \le k$. A computation shows that for each $\varphi \in \mathscr{D}(\Omega)$ and each α with $|\alpha| \le k$ we have

$$\sum_{\beta \le \alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} D^\beta\big((D^{\alpha-\beta}\eta)\varphi\big) = (-1)^{|\alpha|} \eta D^\alpha \varphi \tag{2.5}$$

and hence

$$\begin{aligned}\int_\Omega \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta}\eta)(x)(D^\beta u)(x)\varphi(x)\,dx \\ = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} \int_\Omega D^\beta\big((D^{\alpha-\beta}\eta)(x)\varphi(x)\big) u(x)\,dx \\ = (-1)^{|\alpha|} \int_\Omega (\eta u)(x) D^\alpha \varphi(x) dx.\end{aligned}$$

Thus $\eta u \in H^k(\Omega)$ and (2.4) holds. As ηu vanishes outside a compact subset of Ω it follows $\eta u \in H^k_0(\Omega)$ (exercise). For the converse inclusion in (2.3) let $u \in L^2_{\mathrm{loc}}(\Omega)$ such that $\eta u \in H^k_0(\Omega)$ holds for each $\eta \in \mathscr{D}(\Omega)$. Let $U \Subset \Omega$ and let $\eta \in \mathscr{D}(\Omega)$ such that $\eta(x) = 1$ for all $x \in U$; cf. Lemma 1.7. Then for each $\varphi \in \mathscr{D}(\Omega)$ with $\operatorname{supp}\varphi \subset U$

$$\begin{aligned}(-1)^{|\alpha|} \int_U u(x) D^\alpha \varphi(x)\,dx = (-1)^{|\alpha|} \int_\Omega (\eta u)(x) D^\alpha \varphi(x)\,dx = \int_\Omega D^\alpha(\eta u)(x)\varphi(x)\,dx \\ = \int_U D^\alpha(\eta u)(x)\varphi(x) dx\end{aligned}$$

for each α with $|\alpha| \le k$, i.e., $D^\alpha u = D^\alpha(\eta u)$ on U. Since $\eta u \in H^k(\Omega)$ by assumption, we obtain $(D^\alpha u)|_U \in L^2(U)$. Hence $u \in H^k_{\mathrm{loc}}(\Omega)$. □

The following theorem is a local version of the Sobolev embedding theorem.

Theorem 2.6. *Let $k \in \mathbb{N}$ with $k > d/2$ and $m \in \mathbb{N}_0$. Then $H^{k+m}_{\mathrm{loc}}(\Omega) \subset C^m(\Omega)$.*

Proof. Let $u \in H^{k+m}_{\mathrm{loc}}(\Omega)$ and $U \Subset \Omega$ and let $\eta \in \mathscr{D}(\Omega)$ such that $\eta(x) = 1$ for all $x \in U$. Then $\eta u \in H^{k+m}_0(\Omega)$ by Lemma 2.5 and thus $\widetilde{\eta u} \in H^{k+m}(\mathbb{R}^d)$ by Proposition 1.33, and Theorem 1.43 yields $\widetilde{\eta u} \in C^m(\mathbb{R}^d)$. Hence $u \in C^m(U)$. □

We come to the local regularity result.

Theorem 2.7. *Let $u \in L^2_{\mathrm{loc}}(\Omega)$ and let $f \in H^k_{\mathrm{loc}}(\Omega)$ for some $k \in \mathbb{N}_0$. If $-\Delta u = f$ then $u \in H^{k+2}_{\mathrm{loc}}(\Omega)$. In particular, $f \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.*

Proof. **Step 1.** We assume that $f \in L^2_{\mathrm{loc}}(\Omega)$ and show $u \in H^2_{\mathrm{loc}}(\Omega)$. Let $\eta \in \mathscr{D}(\Omega)$ be fixed and define

$$F(\varphi) := \int_\Omega (\eta u)(x)\overline{(\varphi - \Delta\varphi)(x)}\, dx, \quad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Then the mapping $\mathscr{D}(\mathbb{R}^d) \ni \varphi \mapsto F(\varphi)$ is antilinear and satisfies

$$F(\varphi) = \int_\Omega \big(\eta u\overline{\varphi} - u\Delta(\overline{\varphi}\eta) + u\overline{\varphi}\Delta\eta + 2u\overline{\nabla\varphi}\cdot\nabla\eta\big)(x)\, dx, \quad \varphi \in \mathscr{D}(\mathbb{R}^d), \tag{2.6}$$

where we have used the product rule $\Delta(\overline{\varphi}\eta) = \eta\Delta\overline{\varphi} + 2\overline{\nabla\varphi}\cdot\nabla\eta + \overline{\varphi}\Delta\eta$ and, as a consequence,

$$-u\Delta(\overline{\varphi}\eta) + u\overline{\varphi}\Delta\eta + 2u\overline{\nabla\varphi}\cdot\nabla\eta = -u\eta\Delta\overline{\varphi}.$$

Note that $\overline{\varphi}|_\Omega\eta \in \mathscr{D}(\Omega)$ and hence

$$\begin{aligned} -\int_\Omega u(x)\Delta(\overline{\varphi}\eta)(x)\ dx &= T_u(-\Delta(\overline{\varphi}\eta)) = -(\Delta T_u)(\overline{\varphi}\eta) = T_f(\overline{\varphi}\eta) \\ &= \int_\Omega f(x)\overline{\varphi(x)}\eta(x)\, dx \end{aligned}$$

by the definition of the distributional derivative. From this and (2.6) it follows

$$F(\varphi) = \int_\Omega (\eta u + f\eta + u\Delta\eta)(x)\overline{\varphi(x)}\, dx + 2\int_\Omega \overline{\nabla\varphi(x)}\cdot\nabla\eta(x)u(x)\, dx, \quad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Thus an application of Cauchy–Schwarz shows that $\varphi \mapsto F(\varphi)$ is bounded with respect to the norm in $H^1(\mathbb{R}^d)$ and, hence, can be extended to a bounded antilinear functional $F : H^1(\mathbb{R}^d) \to \mathbb{C}$. By the Fréchet–Riesz theorem there exists a unique $v \in H^1(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} v(x)\overline{\varphi(x)}\, dx + \int_{\mathbb{R}^d} \nabla v(x)\cdot\overline{\nabla\varphi(x)}\, dx = F(\varphi) = \int_{\mathbb{R}^d} \widetilde{\eta u}(x)\overline{(\varphi - \Delta\varphi)}(x)\, dx$$

for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and thus

$$\int_{\mathbb{R}^d} v(x)\overline{(\varphi - \Delta\varphi)(x)}\, dx = \int_{\mathbb{R}^d} \widetilde{\eta u}(x)\overline{(\varphi - \Delta\varphi)(x)}\, dx, \quad \varphi \in \mathscr{D}(\mathbb{R}^d), \tag{2.7}$$

using the definition of the weak derivative. The last identity extends by continuity to all $\varphi \in H^2(\mathbb{R}^d)$. By Lemma 2.3 there exists a unique $\varphi \in H^2(\mathbb{R}^d)$ such that $\varphi - \Delta\varphi = v - \widetilde{\eta u}$. Plugging this φ into (2.7) leads to

$$\int_{\mathbb{R}^d} |v - \widetilde{\eta u}|^2(x)dx = 0$$

and thus $v = \widetilde{\eta u}$ almost everywhere on $\mathbb{R}^d$. In particular, $\eta u = v|_\Omega \in H^1(\Omega)$ and ηu has compact support. Hence $\eta u \in H^1_0(\Omega)$ and Lemma 2.5 implies $u \in H^1_{\text{loc}}(\Omega)$. In order to show $u \in H^2_{\text{loc}}(\Omega)$ observe that the product rule for the Laplacian and $-\Delta u = f$ imply

$$\Delta(\widetilde{\eta u}) = \big(u\Delta\eta + 2\nabla\eta \cdot \nabla u - \eta f\big)^{\sim} \tag{2.8}$$

on $\mathbb{R}^d$ in the sense of distributional derivatives (exercise). As $u \in H^1_{\text{loc}}(\Omega)$ the right-hand side belongs to $L^2(\mathbb{R}^d)$ and Theorem 2.4 implies $\widetilde{\eta u} \in H^2(\mathbb{R}^d)$. As above it follows $\eta u \in H^2_0(\Omega)$ and hence $u \in H^2_{\text{loc}}(\Omega)$ by Lemma 2.5.

Step 2. We use induction in order to establish the general statement. For $k = 0$ the assertion was proven in Step 1. Assume that for a fixed k the assertion is true and let $f \in H^{k+1}_{\text{loc}}(\Omega) \subset H^k_{\text{loc}}(\Omega)$. Then $u \in H^{k+2}_{\text{loc}}(\Omega)$ and for each $\eta \in \mathscr{D}(\Omega)$ it follows from (2.8) $\Delta(\widetilde{\eta u}) \in H^{k+1}(\mathbb{R}^d)$. Another application of Theorem 2.4 implies $\widetilde{\eta u} \in H^{k+3}(\mathbb{R}^d)$ and hence $u \in H^{k+3}_{\text{loc}}(\Omega)$.

Finally, if $f \in C^\infty(\Omega)$ then $f \in H^k_{\text{loc}}(\Omega)$ for each $k \in \mathbb{N}$ and thus $u \in H^{k+2}_{\text{loc}}(\Omega)$ for each k. With Theorem 2.6 it follows $u \in C^m(\Omega)$ for each m and thus $u \in C^\infty(\Omega)$. □

Remark 2.8. (i) Theorem 2.7 shows that each distributional solution of $-\Delta u = f$ with $f \in L^2(\Omega)$ is in fact a weak solution, i.e., the derivatives are weak derivatives.

(ii) For each open, nonempty $\Omega \subset \mathbb{R}^d$, $d \geq 2$, one can construct functions $u, f \in C(\Omega)$ with compact supports such that $-\Delta u = f$ distributionally but $u \notin C^2(\Omega)$. Thus there always exist weak solutions which are not classical solutions.

The main objective in the following is to prove a result on the regularity of solutions up to the boundary of Ω. For this the so-called difference quotient operators will be defined. For a function $u : \Omega \to \mathbb{C}$, $i = 1, \dots, d$, and $h > 0$ let

$$D_i^{+h}u(x) = \frac{u(x + he_i) - u(x)}{h} \quad \text{and} \quad D_i^{-h}u(x) = \frac{u(x) - u(x - he_i)}{h}. \tag{2.9}$$

The functions $D_i^{\pm h}$ in (2.9) are well-defined for all $x \in \Omega$ such that $x \pm he_i \in \Omega$. These subsets of Ω will sometimes be denoted by $\Omega^{\pm h}$. Occassionally it is also useful to extend u by zero to a neighbourhood of Ω and to regard $D_i^{\pm h} u$ as a function on Ω. The following preparatory lemma provides some elemenentary properties of the difference quotients $D_i^{\pm h}$.

Lemma 2.9. *Let $u, v : \Omega \to \mathbb{C}$ and $i = 1, \dots, d$. Then the following assertions are true.*

(i) *For $h > 0$ and all $x \in \Omega^{\pm h}$ the product rules hold:*

$$(D_i^{+h} uv)(x) = u(x + he_i) D_i^{+h} v(x) + (D_i^{+h} u(x)) v(x),$$
$$(D_i^{-h} uv)(x) = u(x) D_i^{-h} v(x) + (D_i^{-h} u(x)) v(x - he_i).$$

(ii) *If $u, v \in L^2_{\mathrm{loc}}(\Omega)$ and at least one of the functions has compact support in Ω then for $h > 0$ sufficiently small one has*

$$(u, D_i^{+h} v)_{L^2(\Omega)} = -(D_i^{-h} u, v)_{L^2(\Omega)}.$$

Proof. (i) Making use of the definition of D_i^{+h} in (2.9) one has

$$\begin{aligned} &u(x + he_i) D_i^{+h} v(x) + (D_i^{+h} u(x)) v(x) \\ &\qquad = u(x + he_i) \frac{v(x + he_i) - v(x)}{h} + \frac{u(x + he_i) - u(x)}{h} v(x) \\ &\qquad = \frac{(uv)(x + he_i) - (uv)(x)}{h} \\ &\qquad = (D_i^{+h} uv)(x) \end{aligned}$$

for all $x \in \Omega^{+h}$. The product rule for D_i^{-h} is shown in the same way.

(ii) Let $u, v \in L^2_{\mathrm{loc}}(\Omega)$ and assume that $\operatorname{supp} u$ is compact. Choose $h > 0$ such that also $\operatorname{supp} u(\cdot - he_i) \subset \Omega$. Then it follows that

$$\begin{aligned} &(u, D_i^{+h} v)_{L^2(\Omega)} + (D_i^{-h} u, v)_{L^2(\Omega)} \\ &\quad = \frac{1}{h} \int_\Omega u(x) \overline{(v(x + he_i) - v(x))} dx + \frac{1}{h} \int_\Omega (u(x) - u(x - he_i)) \overline{v(x)} dx \\ &\quad = \frac{1}{h} \int_\Omega u(x) \overline{v(x + he_i)} dx - \frac{1}{h} \int_\Omega u(x - he_i) \overline{v(x)} dx \\ &\quad = \frac{1}{h} \int_{\Omega + he_i} u(x - he_i) \overline{v(x)} dx - \frac{1}{h} \int_\Omega u(x - he_i) \overline{v(x)} dx \\ &\quad = 0. \end{aligned}$$

□

Proposition 2.10. *Let $D_i^{\pm h}$ be as in (2.9), $i = 1, \dots, d$. Then the following assertions hold.*

(i) *For $u \in H^1(\Omega)$ and $\Omega' \subset \Omega$ such that $\Omega' + h'e_i \subset \Omega$ for some $h' > 0$ one has*

$$\|D_i^{\pm h} u\|_{L^2(\Omega')} \le \|\partial_i u\|_{L^2(\Omega)}, \qquad 0 < h < h'. \tag{2.10}$$

(ii) *Furthermore, for $\tau \in \mathscr{D}(\Omega)$ and $h > 0$ sufficiently small the functions $\tau D_i^{\pm h} u$ and $D_i^{\pm h}(\tau u)$ are defined on Ω and one has*

$$\begin{aligned} \|\tau D_i^{\pm h} u\|_{L^2(\Omega)} &\le \|\tau\|_{L^\infty(\Omega)} \|\partial_i u\|_{L^2(\Omega)}, \\ \|D_i^{\pm h}(\tau u)\|_{L^2(\Omega)} &\le \|\partial_i (\tau u)\|_{L^2(\Omega)}. \end{aligned} \tag{2.11}$$

Proof. (i) We will show the estimate for D_i^{+h}; the proof for D_i^{-h} is the same. Assume first that $u \in C^\infty(\Omega) \cap H^1(\Omega)$. For $\xi > 0$ we have

$$\frac{d}{d\xi} u(x + \xi e_i) = \lim_{k \to 0} \frac{u(x + (\xi + k)e_i) - u(x + \xi e_i)}{k} = \frac{\partial}{\partial x_i} u(x + \xi e_i)$$

and therefore for $0 < h < h'$

$$D_i^{+h} u(x) = \frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_0^h \frac{d}{d\xi} u(x + \xi e_i)\, d\xi = \frac{1}{h} \int_0^h \frac{\partial}{\partial x_i} u(x + \xi e_i)\, d\xi.$$

For $0 < h < h'$ we obtain with the help of the Cauchy-Schwarz inequality

$$\begin{aligned} \|D_i^{+h} u\|_{L^2(\Omega')}^2 &= \frac{1}{h^2} \int_{\Omega'} \left| \int_0^h \frac{\partial}{\partial x_i} u(x + \xi e_i)\, d\xi \right|^2 dx \\ &\le \frac{1}{h} \int_{\Omega'} \int_0^h \left| \frac{\partial}{\partial x_i} u(x + \xi e_i) \right|^2 d\xi\, dx \\ &= \frac{1}{h} \int_0^h \int_{\Omega'} \left| \frac{\partial}{\partial x_i} u(x + \xi e_i) \right|^2 dx\, d\xi \\ &\le \frac{1}{h} \int_0^h \int_{\Omega} \left| \frac{\partial}{\partial x_i} u(x) \right|^2 dx\, d\xi \\ &= \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)}^2 . \end{aligned}$$

Since $C^\infty(\Omega) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$ by Theorem 1.39 and $u_n \to u$ in $H^1(\Omega)$ yields $D_i^{+h} u_n \to D_i^{+h} u$ in $L^2(\Omega')$ the first assertion follows via approximation.

(ii) For the first estimate in (2.11) note that with $\tau \in \mathscr{D}(\Omega)$ and $h > 0$ sufficiently small we can view $\tau D_i^{+h}u$ as a function defined on Ω which vanishes a.e. on $\Omega\setminus U$, where $U \Subset \Omega$ is suitably chosen. Then one has for $h > 0$ sufficiently small

$$\begin{aligned}
\|\tau D_i^{+h}u\|_{L^2(\Omega)}^2 &= \frac{1}{h^2}\int_U |\tau(x)|^2 \left|\int_0^h \frac{\partial}{\partial x_i}u(x+\xi e_i)\,d\xi\right|^2 dx \\
&\le \frac{1}{h}\|\tau\|_{L^\infty(\Omega)}^2 \int_U \int_0^h \left|\frac{\partial}{\partial x_i}u(x+\xi e_i)\right|^2 d\xi\,dx \\
&= \frac{1}{h}\|\tau\|_{L^\infty(\Omega)}^2 \int_0^h \int_U \left|\frac{\partial}{\partial x_i}u(x+\xi e_i)\right|^2 dx\,d\xi \\
&\le \frac{1}{h}\|\tau\|_{L^\infty(\Omega)}^2 \int_0^h \int_\Omega \left|\frac{\partial}{\partial x_i}u(x)\right|^2 dx\,d\xi \\
&= \|\tau\|_{L^\infty(\Omega)}^2 \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega)}^2 .
\end{aligned}$$

For the second estimate in (2.11) note that $\|D_i^{\pm h}(\tau u)\|_{L^2(\Omega')} \le \|\partial_i(\tau u)\|_{L^2(\Omega)}$ holds for any $\Omega' \subset \Omega$ such that $\Omega' + he_i \subset \Omega$ according to (2.10). Since τu has compact support in Ω for $h > 0$ sufficiently small the support of $D_i^{\pm h}(\tau u)$ is also contained in Ω and hence $\|D_i^{\pm h}(\tau u)\|_{L^2(\Omega)} \le \|\partial_i(\tau u)\|_{L^2(\Omega)}$ follows. □

In the next proposition we show how a uniform bound in h for a difference quotient yields the existence of weak derivatives locally in L^2.

Proposition 2.11. *Let $D_i^{\pm h}$ be as in (2.9), $i = 1, \dots, d$. If for $u \in L^2(\Omega)$ and $U \Subset \Omega$ there exists $C(u) > 0$ such that*

$$\|D_i^{+h}u\|_{L^2(U)} \le C(u)$$

holds for all $h > 0$ sufficiently small then the weak derivative $\partial_i u$ exists in $L^2(U)$ and satisfies

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(U)} \le C(u). \tag{2.12}$$

Proof. Since $\|D_i^{+h}u\|_{L^2(U)}$ is bounded by $C(u)$ for all $h > 0$ sufficiently small there exists a sequence (h_k) in $(0, h)$ with $h_k \to 0$ for $k \to \infty$ such that $D_i^{+h_k}u$ converges weakly to some $u_i \in L^2(U)$ and, in particular, $(D_i^{+h_k}u, \varphi)_{L^2(U)} \to (u_i, \varphi)_{L^2(U)}$ for all $\varphi \in \mathscr{D}(U)$ when $k \to \infty$. On the other hand we have

$$(D_i^{+h_k}u, \varphi)_{L^2(U)} = -(u, D_i^{-h_k}\varphi)_{L^2(U)} = -\int_U u(x)\overline{D_i^{-h_k}\varphi(x)}\,dx$$

by Lemma 2.9 (ii) and dominated convergence shows that the last term tends to

$$-\int_U u(x)\overline{\frac{\partial \varphi}{\partial x_i}(x)}dx$$

when $k \to \infty$. Therefore

$$\int_U u_i(x)\overline{\varphi(x)}\,dx = -\int_U u(x)\overline{\frac{\partial \varphi}{\partial x_i}(x)}dx, \qquad \varphi \in \mathscr{D}(U),$$

and hence $\frac{\partial u}{\partial x_i} = u_i \in L^2(U)$. The bound (2.12) follows from

$$\begin{aligned} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{L^2(U)} = \left(u_i, \frac{\partial u}{\partial x_i} \right)_{L^2(U)} &= \lim_{k\to\infty} \left(D_i^{+h_k}u, \frac{\partial u}{\partial x_i} \right)_{L^2(U)} \\ &\leq \limsup \|D_i^{+h_k}u\|_{L^2(U)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(U)} \end{aligned} \tag{2.13}$$

and the assumption $\|D_i^{+h}u\|_{L^2(U)} \leq C(u)$. □

The next theorem is a local version of the regularity shift Theorem 2.4 for $k = 0$; here no additional assumptions on Ω are imposed, but the solution u is required to be in $H^1(\Omega)$. Instead of the Laplacian we consider here a more general second order uniformly elliptic differential expression.

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be open and nonempty, and consider the differential expression*

$$\mathcal{L} = -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\,\alpha^{jk}\,\frac{\partial}{\partial x_k}$$

on Ω. Assume that $\alpha^{jk} \in C^1(\overline{\Omega})$ are real-valued functions that satisfy $\alpha^{jk} = \alpha^{kj}$, $j, k = 1, \dots, d$, and suppose that $\mathcal{L}$ is uniformly elliptic, that is,

$$\big((\alpha^{jk}(x))_{j,k=1}^d \xi, \xi\big)_{\mathbb{C}^d} \geq E\|\xi\|^2, \qquad \xi \in \mathbb{C}^d,\ x \in \Omega, \tag{2.14}$$

for some $E > 0$. If $\mathcal{L}u = f$ holds for some $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ then $u \in H^2_{\mathrm{loc}}(\Omega)$.

Proof. Fix $U \Subset \Omega$ and $h > 0$ such that $h < \frac{1}{3}\mathrm{dist}\,(U, \Omega)$, and choose $\tau \in \mathscr{D}(\Omega)$ with $0 \le \tau(x) \le 1$, $x \in \Omega$, and

$$\tau(x) = \begin{cases} 1, & x \in U, \\ 0, & x \notin U + \frac{1}{3}\,\mathrm{dist}(U, \Omega) \end{cases}.$$

In order to show the assertion we have to verify $\partial_i u \in H^1(U)$, $i = 1, \dots, d$. For this consider the function

$$\varphi(x) = (D_i^{-h} \tau^2 D_i^{+h} u)(x), \tag{2.15}$$

which has the more explicit form

$$\begin{aligned} \varphi(x) &= D_i^{-h} \left(\tau^2(\cdot) \frac{u(\cdot + he_i) - u(\cdot)}{h} \right)(x) \\ &= \frac{1}{h} \left(\tau^2(x) \frac{u(x + he_i) - u(x)}{h} - \tau^2(x - he_i) \frac{u(x) - u(x - he_i)}{h} \right). \end{aligned}$$

Observe that the function φ is well defined for all $x \in \Omega$ if we extend u by zero into a suitable small neighborhood of Ω. Moreover, for $x \notin U + \frac{2}{3}\mathrm{dist}(U, \Omega)$ the choice of τ shows $\varphi(x) = 0$. Since $u \in H^1(\Omega)$ this implies $\varphi \in H_0^1(\Omega)$; cf. Lemma 1.35. With $\varphi \in H_0^1(\Omega)$ it follows in the same way as in the proof of Lemma 2.1 that

$$(f, \varphi) = (\mathcal{L}u, \varphi) = -\sum_{j,k=1}^{d} (\partial_j \alpha^{jk} \partial_k u, \varphi) = \sum_{j,k=1}^{d} (\alpha^{jk} \partial_k u, \partial_j \varphi)$$

holds. Using the particular form of φ in (2.15) and Lemma 2.9 (ii) we compute (all following inner products and norms are in $L^2(\Omega)$ or $L^2(\Omega; \mathbb{C}^d)$ if not stated otherwise)

$$\begin{aligned} -\big(f, D_i^{-h} \tau^2 D_i^{+h} u\big) &= -\sum_{j,k=1}^{d} \big(\alpha^{jk} \partial_k u, \partial_j (D_i^{-h} \tau^2 D_i^{+h} u)\big) \\ &= -\sum_{j,k=1}^{d} \big(\alpha^{jk} \partial_k u, D_i^{-h} \big(\partial_j (\tau^2 D_i^{+h} u)\big)\big) \\ &= \sum_{j,k=1}^{d} \big(D_i^{+h} (\alpha^{jk} \partial_k u), \partial_j (\tau^2 D_i^{+h} u)\big), \end{aligned}$$

where we have also used $\partial_j D_i^{\pm h}\psi = D_i^{\pm h}\partial_j\psi$ for $\psi \in H^1(\Omega)$ and $j = 1,\dots,d$. With the help of the product rule in Lemma 2.9 (i) we further conclude

$$\begin{aligned}
&= \sum_{j,k=1}^{d} \big(\alpha^{jk}(\cdot + he_i)\partial_k D_i^{+h}u + (D_i^{+h}\alpha^{jk})\partial_k u, \partial_j(\tau^2 D_i^{+h}u)\big) \\
&= 2\sum_{j,k=1}^{d} \big(\alpha^{jk}(\cdot + he_i)\partial_k D_i^{+h}u, \tau(\partial_j\tau)D_i^{+h}u)\big) \\
&\quad + \sum_{j,k=1}^{d} \big(\alpha^{jk}(\cdot + he_i)\partial_k D_i^{+h}u, \tau^2\partial_j D_i^{+h}u\big) + \sum_{j,k=1}^{d} \big((D_i^{+h}\alpha^{jk})\partial_k u, \partial_j(\tau^2 D_i^{+h}u)\big).
\end{aligned}$$

Rearranging the terms leads to the identity

$$\begin{aligned}
&\sum_{j,k=1}^{d} \big(\alpha^{jk}(\cdot + he_i)\tau\partial_k D_i^{+h}u, \tau\partial_j D_i^{+h}u\big) \\
&\qquad = -\big(f, D_i^{-h}\tau^2 D_i^{+h}u\big) - 2\sum_{j,k=1}^{d} \big(\alpha^{jk}(\cdot + he_i)\partial_k D_i^{+h}u, \tau(\partial_j\tau)D_i^{+h}u)\big) \\
&\qquad\quad - \sum_{j,k=1}^{d} \big((D_i^{+h}\alpha^{jk})\partial_k u, \partial_j(\tau^2 D_i^{+h}u)\big).
\end{aligned} \tag{2.16}$$

The ellipticity condition (2.14) implies for the term on the left hand side

$$\begin{aligned}
&\sum_{j,k=1}^{d} \big(\alpha^{jk}(\cdot + he_i)\tau\partial_k D_i^{+h}u, \tau\partial_j D_i^{+h}u\big) \\
&\qquad = \int_\Omega \big((\alpha^{jk}(x + he_i))_{j,k=1}^d \tau(x)\nabla D_i^{+h}u(x), \tau(x)\nabla D_i^{+h}(x)\big)_{\mathbb{C}^d}\,dx \\
&\qquad \geq \int_\Omega E\big\|\tau(x)\nabla D_i^{+h}u(x)\big\|_{\mathbb{C}^d}^2\,dx = E\|\tau\nabla D_i^{+h}u\|^2.
\end{aligned}$$

Using Proposition 2.10 (ii) and $\tau^2 \leq \tau \leq 1$ the first term on the right hand side of (2.16) can be estimated as follows

$$\begin{aligned}
-(f, D_i^{-h}\tau^2 D_i^{+h}u) &\leq \|f\|\|D_i^{-h}\tau^2 D_i^{+h}u\| \\
&\leq \|f\|\|\partial_i(\tau^2 D_i^{+h}u)\| \\
&\leq \|f\|\big(\|2(\partial_i\tau)\tau D_i^{+h}u\| + \|\tau^2\partial_i D_i^{+h}u\|\big) \\
&\leq \|f\|\big(C_1\|\partial_i u\| + \|\tau\partial_i D_i^{+h}u\|\big) \\
&\leq C_2\big(\|\nabla u\| + \|\tau\nabla D_i^{+h}u\|\big),
\end{aligned}$$

and for the second term on the right hand side of (2.16) we observe

$$\begin{aligned}
-2\sum_{j,k=1}^{d}\big(\alpha^{jk}(\cdot + he_i)\tau\partial_k D_i^{+h}u, (\partial_j\tau)D_i^{+h}u\big) &\leq C_3\sum_{j,k=1}^{d}\|\tau\partial_k D_i^{+h}u\|\|(\partial_j\tau)D_i^{+h}u\| \\
&\leq C_4\sum_{k=1}^{d}\|\tau\partial_k D_i^{+h}u\|\|\partial_i u\| \\
&\leq C_5\|\tau\nabla D_i^{+h}u\|\|\nabla u\|.
\end{aligned}$$

For the third term on the right hand side of (2.16) we compute and estimate

$$\begin{aligned}
&-\sum_{j,k=1}^{d}\big((D_i^{+h}\alpha^{jk})\partial_k u, \partial_j(\tau^2 D_i^{+h}u)\big) \\
&= -2\sum_{j,k=1}^{d}\big((D_i^{+h}\alpha^{jk})\partial_k u, \tau(\partial_j\tau)D_i^{+h}u\big) - \sum_{j,k=1}^{d}\big((D_i^{+h}\alpha^{jk})\partial_k u, \tau^2\partial_j D_i^{+h}u\big) \\
&\leq C_6\sum_{j,k=1}^{d}\|\partial_k u\|\|\partial_i u\| + C_7\sum_{j,k=1}^{d}\|\partial_k u\|\|\tau\partial_j D_i^{+h}u\| \\
&\leq C_8\|\nabla u\|^2 + C_9\|\nabla u\|\|\tau\nabla D_i^{+h}u\|.
\end{aligned}$$

Summing up we have the following inequality

$$E\|\tau\nabla D_i^{+h}u\|^2 \leq C'(u)\|\nabla u\| + C''(u)\|\tau\nabla D_i^{+h}u\|,$$

where $C'(u) = C_2 + C_8\|\nabla u\|$ and $C''(u) = C_2 + (C_5 + C_9)\|\nabla u\|$. Making use of the inequality $ab \leq \frac{1}{2E}a^2 + \frac{E}{2}b^2$ for $a, b \geq 0$ we find

$$E\|\tau\nabla D_i^{+h}u\|^2 \leq C'(u)\|\nabla u\| + \frac{1}{2E}C''(u)^2 + \frac{E}{2}\|\tau\nabla D_i^{+h}u\|^2,$$

and hence
$$\frac{E}{2}\|\tau\nabla D_i^{+h}u\|^2 \le C'(u)\|\nabla u\| + \frac{1}{2E}C''(u)^2 =: \widetilde{C}(u).$$
As $\tau = 1$ on U it follows that $\|\nabla D_i^{+h}u\|^2_{L^2(U;\mathbb{C}^d)} \le \frac{2}{E}\widetilde{C}(u)$ and this yields for $k = 1, \dots, d$

$$\|D_i^{+h}\partial_k u\|_{L^2(U)} = \|\partial_k D_i^{+h}u\|_{L^2(U)} \le \|\nabla D_i^{+h}u\|_{L^2(U;\mathbb{C}^d)} \le \sqrt{\frac{2}{E}\widetilde{C}(u)}.$$

Now Proposition 2.11 implies that the weak derivative $\partial_i\partial_k u$ of $\partial_k u$ exists in $L^2(U)$. Since this is true for all $i = 1, \dots, d$ and all $U \Subset \Omega$ we finally conclude $u \in H^2_{\mathrm{loc}}(\Omega)$. □

Observe that the constant $\widetilde{C}(u)$ in the proof of Theorem 2.12 depends on the distance of the subset $U \Subset \Omega$ from the boundary $\partial\Omega$ (as e.g. $\|\partial_k\tau\|_{L^\infty(\Omega)}$ enters in the estimates). Therefore a global regularity result can not be proved along the same lines without additional assumptions. In the next theorem it is assumed that Ω is a bounded C^2-domain (see Definition 3.2 and the following Remark 3.3) and the solution satisfies Dirichlet boundary conditions.

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^d$, $d \ge 1$, be open, nonempty, bounded and of class C^2, and assume that $-\Delta u = f$ holds for some $f \in L^2(\Omega)$ and $u \in H^1_0(\Omega)$. Then $u \in H^2(\Omega)$.*

Proof. Let us choose open bounded sets $U_1, \dots, U_r \subset \mathbb{R}^d$ such that

$$\partial\Omega \subset \bigcup_{l=1}^{r} U_l$$

and C^2-mappings $\Phi_l : U_l \to B(0,1)$ with inverses $\Psi_l : \Phi(U_l) \to U_l$, $l = 1, \dots, r$, such that

$$\Phi_l(U_l\cap\Omega) \subset B_+(0,1), \quad \Phi_l(U_l\cap\partial\Omega) \subset \mathbb{R}^{d-1}\times\{0\}, \quad \Phi_l(U_l\cap(\mathbb{R}^d\setminus\overline{\Omega})) \subset B_-(0,1),$$

for all $l = 1, \dots, r$; here and in the following we use the notation

$$B_\pm(0,\gamma) = B(0,\gamma)\cap\{x\in\mathbb{R}^d : \pm x_d > 0\} \quad \text{with } \gamma > 0.$$

In addition, choose $\vartheta \in (0, 1/3)$ such that $B_+(0, 3\vartheta) \subset \Phi_l(U_l \cap \Omega)$, $l = 1, \dots, r$, and it is no restriction to assume that U_l were chosen such that

$$\partial\Omega \subset \bigcup_{l=1}^{r} \Psi_l\big(B(0,\vartheta)\big).$$

Furthermore, let $\Omega_0 \Subset \Omega$ be such that

$$\Omega \subset \left(\Omega_0 \cup \bigcup_{l=1}^{r} \Psi_l\big(B(0,\vartheta)\big) \right). \tag{2.17}$$

We note that from Theorem 2.12 and its proof it is clear that

$$\partial_i \partial_k u \in L^2(\Omega_0), \qquad i,k = 1,\dots,d. \tag{2.18}$$

For the following considerations fix some $s \in \{1,\dots,r\}$ and consider the differential expression

$$\mathcal{L}_s = -\sum_{j,k=1}^{d} \frac{\partial}{\partial y_j} \alpha_s^{jk} \frac{\partial}{\partial y_k}$$

on $B_+(0,3\vartheta)$, where

$$\alpha_s^{jk} = \big(\nabla \Phi_s^j \circ \Psi_s, \nabla \Phi_s^k \circ \Psi_s\big)_{\mathbb{C}^d} \in C^1\big(\overline{B_+(0,3\vartheta)}\big).$$

Note that the component functions Φ_s^j are real valued so that $\alpha_s^{jk} = \alpha_s^{kj}$. With $f_s(y) := f(\Psi_s(y))$ and $u_s(y) := u(\Psi_s(y))$, $y \in B_+(0,3\vartheta)$, one has[1]

$$(\mathcal{L}_s u_s, \varphi)_{L^2(B_+(0,3\vartheta))} = (f_s, \varphi)_{L^2(B_+(0,3\vartheta))}, \qquad \varphi \in H_0^1(B_+(0,3\vartheta)),$$

[1] Note that $(\nabla u_s)(y) = (\nabla u)(\Psi_s(y))(D\Psi_s)(y)$ and hence

$$\frac{\partial u_s}{\partial y_k}(y) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(\Psi_s(y)) \frac{\partial \Psi_s^i}{\partial y_k}(y).$$

It follows for $\omega_s \in H_0^1(B_+(0,3\vartheta))$ and $\omega \in H_0^1(\Psi_s(B_+(0,3\vartheta)))$ with $\omega_s(y) = \omega(\Psi_s(y))$ that

$$\begin{aligned}
\big(\mathcal{L}_s u_s, \overline{\omega}_s\big)_{L^2(B_+(0,3\vartheta))} &= \int \sum_{j,k=1}^{d} \alpha_s^{jk}(y) \frac{\partial u_s}{\partial y_k}(y) \frac{\partial \omega_s}{\partial y_j}(y)\, dy \\
&= \int \sum_{j,k,t=1}^{d} \frac{\partial \Phi_s^j}{\partial x_t}(\Psi_s(y)) \frac{\partial \Phi_s^k}{\partial x_t}(\Psi_s(y)) \left(\sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(\Psi_s(y)) \frac{\partial \Psi_s^i}{\partial y_k}(y) \right) \left(\sum_{p=1}^{d} \frac{\partial \omega}{\partial x_p}(\Psi_s(y)) \frac{\partial \Psi_s^p}{\partial y_j}(y) \right) dy \\
&= \int \sum_{i,p,t=1}^{d} \frac{\partial u}{\partial x_i}(\Psi_s(y)) \frac{\partial \omega}{\partial x_p}(\Psi_s(y)) \left(\sum_{k=1}^{d} \frac{\partial \Psi_s^i}{\partial y_k}(y) \frac{\partial \Phi_s^k}{\partial x_t}(\Psi_s(y)) \right) \left(\sum_{j=1}^{d} \frac{\partial \Psi_s^p}{\partial y_j}(y) \frac{\partial \Phi_s^j}{\partial x_t}(\Psi_s(y)) \right) dy
\end{aligned}$$

and $\mathcal{L}_s$ is uniformly elliptic since

$$\begin{aligned}\left((\alpha_s^{jk}(y))_{j,k=1}^d \xi, \xi\right)_{\mathbb{C}^d} &= \sum_{j,k=1}^d \left(\nabla\Phi_s^j(\Psi_s(y)), \nabla\Phi_s^k(\Psi_s(y))\right)_{\mathbb{C}^d} \xi_k \overline{\xi}_j \\ &= \sum_{j,k=1}^d \left(\xi_k \nabla\Phi_s^k(\Psi_s(y)), \xi_j \nabla\Phi_s^j(\Psi_s(y))\right)_{\mathbb{C}^d} \\ &= \left\| \sum_{k=1}^d \xi_k \nabla\Phi_s^k(\Psi_s(y)) \right\|_{\mathbb{C}^d}^2 \\ &= \left\| \left(\nabla\Phi_s^1(\Psi_s(y)), \dots, \nabla\Phi_s^d(\Psi_s(y))\right) \begin{pmatrix}\xi_1 \\ \vdots \\ \xi_d\end{pmatrix} \right\|_{\mathbb{C}^d}^2 \\ &= \left\| \left(D\Phi_s(\Psi_s(y))\right)^\top \xi \right\|_{\mathbb{C}^d}^2 \\ &\geq E\|\xi\|_{\mathbb{C}^d}^2,\end{aligned} \tag{2.19}$$

where we have used in the last estimate that $D\Phi_s(\Psi_s(y))$ is an invertible matrix. Now let $\tau \in \mathscr{D}(B(0,1))$ such that $0 \leq \tau(y) \leq 1$ and $\tau = 1$ on $B(0,\vartheta)$ and $\tau = 0$ in $B(0,1) \setminus B(0,2\vartheta)$. Since $u \in H_0^1(\Omega)$ it follows that $u_s(y_1, \dots, y_{d-1}, 0) = 0$ and

$$\varphi = D_i^{-h} \tau^2 D_i^{+h} u_s \in H_0^1(B_+(0,3\vartheta)) \quad \text{for} \quad i = 1, \dots, d-1.$$

As in the proof of Theorem 2.12 it follows that

$$\left\| D_i^{+h} \frac{\partial}{\partial y_k} u_s \right\|_{L^2(B_+(0,\vartheta))} \leq K_s, \quad k = 1, \dots, d, \; i = 1, \dots, d-1, \tag{2.20}$$

and therefore

$$\frac{\partial^2}{\partial y_i \partial y_k} u_s \in L^2(B_+(0,\vartheta)), \quad k = 1, \dots, d, \; i = 1, \dots, d-1, \tag{2.21}$$

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and since $(D\Psi_s) \cdot (D\Phi_s) = I$ we have $\sum_{k=1}^d \frac{\partial \Psi_s^l}{\partial y_k}(y) \frac{\partial \Phi_s^k}{\partial x_m}(\Psi_s(y)) = \delta_{lm}$. Therefore

$$\begin{aligned}\left(\mathcal{L}_s u_s, \overline{\omega}_s\right)_{L^2(B_+(0,3\vartheta))} &= \int \sum_{t=1}^d \frac{\partial u}{\partial x_t}(\Psi_s(y)) \frac{\partial \omega}{\partial x_t}(\Psi_s(y))\, dy \\ &= \left(\nabla u(\Psi_s(\cdot)), \nabla \overline{\omega}(\Psi_s(\cdot)\right)_{L^2(B_+(0,3\vartheta))} \\ &= \left(f(\Psi_s(\cdot)), \overline{\omega}(\Psi_s(\cdot)\right)_{L^2(B_+(0,3\vartheta))} \\ &= (f_s, \overline{\omega}_s)_{L^2(B_+(0,3\vartheta))}.\end{aligned}$$

and when viewing $\frac{\partial^2}{\partial y_i \partial y_k} u_s$ as a distribution we have $\frac{\partial^2}{\partial y_i \partial y_k} u_s = \frac{\partial^2}{\partial y_k \partial y_i} u_s$ and hence

$$\frac{\partial^2}{\partial y_k \partial y_i} u_s \in L^2(B_+(0,\vartheta)), \quad k = 1,\ldots,d, \; i = 1,\ldots,d-1. \tag{2.22}$$

Since $\mathcal{L}_s u_s = f_s$ we also have

$$-\frac{\partial}{\partial y_d}\, \alpha_s^{dd}(y)\, \frac{\partial}{\partial y_d} u_s(y) = f_s(y) + \sum_{\substack{j,k=1 \\ (j,k)\neq(d,d)}}^{d} \frac{\partial}{\partial y_j}\, \alpha_s^{jk}(y)\, \frac{\partial}{\partial y_k} u_s(y)$$

and since $\alpha_s^{dd}(y) = \left((\alpha_s^{jk}(y))_{j,k=1}^d e_d, e_d\right)_{\mathbb{C}^d} \geq E\|e_d\|_{\mathbb{C}^d}^2 = E > 0$ by (2.19) we obtain

$$\frac{\partial^2 u_s}{\partial y_d^2}(y) = \frac{-1}{\alpha_s^{dd}(y)} \left[f_s(y) + \sum_{\substack{j,k=1 \\ (j,k)\neq(d,d)}}^{d} \frac{\partial}{\partial y_j}\, \alpha_s^{jk}(y)\, \frac{\partial}{\partial y_k} u_s(y) + \left(\frac{\partial \alpha_s^{dd}}{\partial y_d}(y)\right) \frac{\partial}{\partial y_d} u_s(y) \right].$$

Now it follows from (2.21) and (2.22) that the right hand side belongs to $L^2(B_+(0,\vartheta))$, so that

$$\frac{\partial^2 u_s}{\partial y_d^2} \in L^2(B_+(0,\vartheta)). \tag{2.23}$$

Since Φ_s is a C^2-mapping it follows from (2.21), (2.22), and (2.23) that

$$\partial_i \partial_k u \in L^2(\Psi_s(B_+(0,\vartheta))), \qquad i,k = 1,\ldots d.$$

Since this is true for all $s = 1,\ldots,j$ it follows together with (2.17) and (2.18) that $u \in H^2(\Omega)$. □

For completeness we state (but do not proof) one more result on the global regularity of solutions up to the boundary. Recall that Ω is called *convex* if $x, y \in \Omega$ implies $tx + (1-t)y \in \Omega$ for all $t \in (0,1)$.

Theorem 2.14. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex open set and let $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$ with $-\Delta u = f$. Then $u \in H^2(\Omega)$.*

Remark 2.15. The techniques of this (and the following) chapter apply not only to the Poisson equation but to more general elliptic differential expressions. Recall that a differential equation of the form

$$\mathcal{L}u := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{j=1}^{d} b_j \frac{\partial u}{\partial x_j} + cu = f$$

with bounded, measurable coefficient functions a_{ij}, b_j, c is called *uniformly elliptic* if there exists a constant $E > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq E|\xi|^2 \quad \forall x \in \overline{\Omega}, \xi \in \mathbb{R}^d$$

holds.

Chapter 3

Neumann and Robin boundary conditions

In order to treat more general boundary conditions it is necessary to impose a regularity assumption on the boundary of Ω.

3.1 Lipschitz domains

Recall that a function $g : \mathbb{R}^m \to \mathbb{R}$ is *Lipschitz continuous* if there exists $L > 0$ with

$$|g(x) - g(y)| \leq L|x - y|, \qquad x, y \in \mathbb{R}^m.$$

In this case, L is called a *Lipschitz constant* for g. Lipschitz continuous functions admit derivatives in L^∞ as the following theorem due to Rademacher (see, e.g. REFERENCE) shows.

Theorem 3.1. *Let $g : \mathbb{R}^m \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant $L > 0$. Then g is differentiable almost everywhere and*

$$\Big|\frac{\partial g}{\partial x_j}(x)\Big| \leq L, \quad j = 1, \dots, m,$$

holds for almost all $x \in \mathbb{R}^m$.

Definition 3.2. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be open and nonempty.

(i) Ω is called *Lipschitz hypograph* if there exists a Lipschitz continuous function $g : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega = \left\{(x_1, \ldots, x_d)^\top \in \mathbb{R}^d : x_d < g(x_1, \ldots, x_{d-1})\right\}.$$

(ii) Ω is called *Lipschitz domain* if the boundary $\partial\Omega$ is compact and for each $x \in \partial\Omega$ there exists an open neighborhood $U_x \subset \mathbb{R}^d$ of x, a Lipschitz hypograph Ω_x, and a rotation R_x (an orthogonal matrix with determinant one) such that $U_x \cap \Omega = U_x \cap R_x(\Omega_x)$.

Remark 3.3. (i) Due to compactness the boundary of a Lipschitz domain can be described by the graphs of finitely many Lipschitz continuous functions.

(ii) A C^k-domain is defined analogously with Lipschitz continuous functions replaced by C^k-functions, $k \in \mathbb{N}$.

Example 3.4. (i) Each circle or ball is a Lipschitz (in fact C^∞) boundary.

(ii) Each cube is a Lipschitz domain but not a C^k-domain for any $k \geq 1$.

(More details in the exercises.)

Proposition 3.5 (Partition of unity)**.** *Let $K \subset \mathbb{R}^d$ be compact and let $U_1, \ldots, U_m$ be open sets with $K \subset \bigcup_{j=1}^m U_j$. Then there exist $\eta_1, \ldots, \eta_m \in \mathscr{D}(\mathbb{R}^d)$ such that*

(a) $0 \leq \eta_j \leq 1$, $j = 1, \ldots, m$,

(b) $\operatorname{supp} \eta_j \subset U_j$, $j = 1, \ldots, m$,

(c) $\sum_{j=1}^m \eta_j(x) = 1$ *for each* $x \in K$.

The collection of the functions $\eta_1, \ldots, \eta_m$ is called a *partition of unity* on K associated with $U_1, \ldots, U_m$.

Proof. Each $x \in K$ belongs to some U_j and in particular for each $x \in K$ there exists $r > 0$ with $\overline{B(x,r)} \subset U_j$. As K is compact, finitely many such balls cover K; we call these balls $B_1, \ldots, B_k$. For $j = 1, \ldots, m$ define $K_j := \bigcup_{\overline{B_l} \subset U_j} \overline{B_l}$. Then each K_j is compact and $K_j \subset U_j$, $j = 1, \ldots, m$. Moreover, $K \subset \bigcup_{j=1}^m K_j$. By Lemma 1.7 for $j = 1, \ldots, m$ there exists $\psi_j \in \mathscr{D}(\mathbb{R}^d)$ such that $0 \leq \psi_j \leq 1$, $\operatorname{supp} \psi_j \subset U_j$ and $\psi_j(x) = 1$ for all $x \in K_j$. Define

$$\begin{aligned}
\eta_1 &:= \psi_1, \\
\eta_2 &:= (1 - \psi_1)\psi_2,
\end{aligned}$$

$$\eta_3 := (1-\psi_1)(1-\psi_2)\psi_3,$$
$$\vdots$$
$$\eta_m := (1-\psi_1)\cdots(1-\psi_{m-1})\psi_m.$$

Then $\operatorname{supp}\eta_j \subset \operatorname{supp}\psi_j \subset U_j$ for $j = 1,\dots,m$. Moreover, by induction $\sum_{j=1}^n \eta_j + (1-\psi_1)\cdots(1-\psi_n) = 1$ for all $n \in \{1,\dots,m\}$. In particular,

$$\sum_{j=1}^m \eta_j + (1-\psi_1)\cdots(1-\psi_m) = 1$$

and thus $0 \le \sum_{j=1}^m \eta_j \le 1$. Finally, each $x \in K$ belongs to one of the K_j and hence $\psi_j(x) = 1$ or, equivalently, $1-\psi_j(x) = 0$, which implies $\sum_{j=1}^m \eta_j(x) = 1$. $\square$

Let Ω be a Lipschitz domain and let $x_1,\dots,x_m \in \partial\Omega$ such that the sets $U_j := U_{x_j}$, $j = 1,\dots,m$, in Definition 3.2 form an open cover of $\partial\Omega$. Moreover, let $g_1,\dots,g_m$ be corresponding Lipschitz functions and R_j the corresponding rotations as in Definition 3.2 and let $\eta_1,\dots,\eta_m$ be a partition of unity on $\partial\Omega$ associated with $U_1,\dots,U_m$. For any bounded, measurable function $f : \partial\Omega \to \mathbb{C}$ we define

$$\int_{\partial\Omega} f d\sigma := \sum_{j=1}^m \int_{\mathbb{R}^{d-1}} \eta_j\big(R_j(x', g_j(x'))\big) f\big(R_j(x', g_j(x'))\big)\sqrt{1+|\nabla g_j(x')|^2}dx'.$$

Due to the Rademacher theorem this integral is finite. By plugging in $f = \mathbb{1}_B$ for any Borel set $B \subset \partial\Omega$ this defines a surface measure σ on $\partial\Omega$ by

$$\sigma(B) := \int_{\partial\Omega} \mathbb{1}_B d\sigma.$$

In particular this gives rise to the Hilbert space $L^2(\partial\Omega)$ (with respect to the measure σ) and its corresponding inner product $(\cdot,\cdot)_{L^2(\partial\Omega)}$. By Rademacher's theorem the *outer unit normal vector*

$$\nu(x) := \frac{(\nabla g_j(x'), -1)^\top}{\sqrt{|\nabla g_j(x')|^2+1}}$$

exists for almost all $x = (x', g_j(x'))^\top \in \partial\Omega$, where j is chosen such that $x \in U_j$. It is orthogonal to the tangential space at x, which is spanned by $(e_1, \frac{\partial g_j}{\partial x_1}(x'))^\top, \dots, (e_{d-1}, \frac{\partial g_j}{\partial x_{d-1}}(x'))^\top$, where $e_1,\dots,e_{d-1}$ are the unit vectors in $\mathbb{R}^{d-1}$. Moreover, $|\nu(x)| = 1$ for almost all $x \in \partial\Omega$.

As in the Analysis lecture one proves the divergence theorem ("Satz von Gauß").

Theorem 3.6 (Divergence theorem)**.** *Let Ω be a bounded Lipschitz domain. Then*

$$\int_\Omega \operatorname{div} u(x)\,dx = \int_{\partial\Omega} u\cdot\nu\,d\sigma$$

holds for all $u \in C^1(\overline{\Omega}, \mathbb{C}^d)$.

As a corollary one obtains Green's identities. (See exercises.) Here we write $\frac{\partial u}{\partial\nu} := \nabla u\cdot\nu$ for the *normal derivative* of some $u \in C^2(\overline{\Omega})$ on $\partial\Omega$.

Corollary 3.7 (Green's identities)**.** *For any $u \in C^2(\overline{\Omega})$*

(i) $\int_\Omega (\Delta u)(x)v(x)\,dx + \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}v\,d\sigma$ *for all $v \in C^1(\overline{\Omega})$;*

(ii) $\int_\Omega ((\Delta u)(x)v(x) - u(x)(\Delta v)(x))\,dx = \int_{\partial\Omega}(\frac{\partial u}{\partial\nu}v - u\frac{\partial v}{\partial\nu})\,d\sigma$ *for all $v \in C^2(\overline{\Omega})$.*

In the following our aim is to extend Green's identities (and thus the boundary evaluation of u and the normal derivative $\frac{\partial u}{\partial\nu}$) to $u \in H^1(\Omega)$. This requires some preparation.

Theorem 3.8 (Extension property)**.** *Let Ω be a bounded Lipschitz domain. Then for each $u \in H^1(\Omega)$ there exists $w \in H^1(\mathbb{R}^d)$ with $w|_\Omega = u$.*

Sketch of proof. Let $u \in H^1(\Omega)$. Let g be a Lipschitz function whose graph describes locally the boundary of Ω within an open ball U. Then $u(x', g(x') + s)$ is well-defined for appropriate $x' \in \mathbb{R}^{d-1}$ and $s < 0$. For appropriate x' and $|s|$ small define

$$w(x', g(x') + s) := \begin{cases} u(x', g(x') + s), & s < 0,\\ u(x', g(x') - s), & s \geq 0.\end{cases}$$

Using the definition of a Lipschitz domain and a corresponding partition of unity construct w on some smooth domain $\widetilde{\Omega}$ with $\Omega \Subset \widetilde{\Omega}$. Then one can show $w \in H^1(\widetilde{\Omega})$. A further extension leads to a function in $H^1(\mathbb{R}^d)$. □

Corollary 3.9. *Let Ω be a bounded Lipschitz domain. Then*

$$\{\varphi|_\Omega : \varphi \in \mathscr{D}(\mathbb{R}^d)\}$$

is dense in $H^1(\Omega)$.

Proof. Let E be the extension operator in Theorem 3.10 for $\widetilde{\Omega} = \mathbb{R}^d$ and let $u \in H^1(\Omega)$. Then $Eu \in H_0^1(\mathbb{R}^d)$ and by Theorem 1.36 there exists a sequence $(\varphi_n)_n \subset \mathscr{D}(\mathbb{R}^d)$ such that $\varphi_n \to Eu$ in $H^1(\mathbb{R}^d)$. Since the restriction operator $v \mapsto v|_\Omega$ from $H^1(\mathbb{R}^d)$ to $H^1(\Omega)$ is bounded the assertion follows. □

Theorem 3.10 (Extension operator)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $\widetilde{\Omega} \subset \mathbb{R}^d$ be open such that $\Omega \Subset \widetilde{\Omega}$. Then there exists a bounded linear operator $E : H^1(\Omega) \to H_0^1(\widetilde{\Omega})$ with $(Eu)|_\Omega = u$ for all $u \in H^1(\Omega)$.*

Proof. Let $T : H^1(\mathbb{R}^d) \to H^1(\Omega)$ be the restriction operator, i.e., $Tu = u|_\Omega$ for all $u \in H^1(\mathbb{R}^d)$. Then T is linear and bounded and by Theorem 3.8 T is surjective. Thus $T|_{(\ker T)^\perp}$ is bounded and bijective and, hence, has a bounded inverse $S : H^1(\Omega) \to (\ker T)^\perp \subset H^1(\mathbb{R}^d)$. For any $u \in H^1(\Omega)$ one has $(Su)|_\Omega = TSu = u$. Let now $\widetilde{\Omega}$ be as in the theorem. By Lemma 1.7 there exists $\eta \in \mathscr{D}(\mathbb{R}^d)$ such that $\operatorname{supp} \eta \subset \widetilde{\Omega}$ and $\eta(x) = 1$ for all $x \in \Omega$. With $Eu := (\eta S u)|_{\widetilde{\Omega}}$ we get the required extension operator. □

Theorem 3.11. *Let Ω be a bounded Lipschitz domain. Then the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact.*

Proof. Let $\widetilde{\Omega}$ be a bounded, open set with $\Omega \Subset \widetilde{\Omega}$. By Theorem 1.45 the embedding ι of $H_0^1(\widetilde{\Omega})$ into $L^2(\widetilde{\Omega})$ is compact. Let $E : H^1(\Omega) \to H_0^1(\widetilde{\Omega})$ be a bounded extension operator as in Theorem 3.10 and let $R : L^2(\widetilde{\Omega}) \to L^2(\Omega)$ be the restriction operator, which is bounded. Then the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ equals $R\iota E$ and, hence, is compact. □

Note that the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ may be noncompact for general, non-Lipschitz domains; see exercises.

In the next theorem the *Dirichlet trace operator* τ_D is introduced.

Theorem 3.12 (Trace theorem)**.** *Let Ω be a bounded Lipschitz domain. Then there exists a unique bounded linear operator $\tau_D : H^1(\Omega) \to L^2(\partial\Omega)$ such that $\tau_D u = u|_{\partial\Omega}$ holds for all $u \in C(\overline{\Omega}) \cap H^1(\Omega)$.*

Proof. We show that there exists $C > 0$ such that

$$\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)} \tag{3.1}$$

holds for all $u \in C^1(\overline{\Omega})$. Let first $u \in C^1(\overline{\Omega})$ with $\operatorname{supp} u \subset U_j$ for one j and assume that $R_j = I$. Without loss of generality assume that u is real-valued;

for the general case do the following estimate for the real and imaginary parts separately. Then with $c_j := \sup \sqrt{1 + |\nabla g_j(x')|^2}$ for each sufficiently large $h > 0$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \big|u(x', g_j(x'))\big|^2 \sqrt{1 + |\nabla g_j(x')|^2} dx' \\
&\quad \leq c_j \int_{\mathbb{R}^{d-1}} \big|u(x', g_j(x'))\big|^2 dx' \\
&\quad = -c_j \int_{\mathbb{R}^{d-1}} \int_0^h \frac{d}{ds}\Big[u\big(x', g_j(x') - s\big)^2\Big] ds dx' \\
&\quad = -c_j \int_{\mathbb{R}^{d-1}} \int_0^h 2u\big(x', g_j(x') - s\big) \frac{d}{ds} u\big(x', g_j(x') - s\big) ds dx' \\
&\quad \leq c_j \int_{\mathbb{R}^{d-1}} \int_0^h u\big(x', g_j(x') - s\big)^2 + \big((\nabla u)(x', g_j(x') - s) \cdot (-e_d)\big)^2 ds dx' \\
&\quad \leq c_j \int_\Omega (u(x))^2 + ((\nabla u)(x))^2 dx = c_j \|u\|^2_{H^1(\Omega)},
\end{aligned}$$

where we have used $-2\alpha\beta \leq \alpha^2 + \beta^2$ for real α, β. From this for arbitrary $u \in C^1(\overline{\Omega})$ it follows

$$\begin{aligned}
\|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} &= \sum_{j=1}^m \int_{\mathbb{R}^{d-1}} (\sqrt{\eta_j} u)^2 \big(R_j(x', g_j(x'))\big) \sqrt{1 + |\nabla g_j(x')|^2} dx' \\
&\leq \sum_{j=1}^m c_j \|\sqrt{\eta_j} u\|^2_{H^1(\Omega)} \leq \sum_{j=1}^m \widetilde{c}_j \|u\|^2_{H^1(\Omega)}
\end{aligned}$$

for certain constants $\widetilde{c}_j$, where we have used that $\sqrt{\eta_j}$ and its derivatives of first order are bounded. With $C = (\sum_{j=1}^m \widetilde{c}_j)^{1/2}$ this leads to (3.1). Thus the linear mapping $C^1(\overline{\Omega}) \ni u \mapsto u|_{\partial\Omega}$ is bounded from $H^1(\Omega)$ to $L^2(\partial\Omega)$. By Proposition 3.9 $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ and hence there exists a unique bounded, linear operator $\tau_D : H^1(\Omega) \to L^2(\partial\Omega)$ such that $\tau_D u = u|_{\partial\Omega}$ holds for all $u \in C^1(\overline{\Omega})$. For $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ the latter property follows via approximation (exercise). □

Theorem 3.13. *Let Ω be a bounded Lipschitz domain with trace operator τ_D. Then*

$$\ker \tau_D = H^1_0(\Omega).$$

Proof. For $u \in H_0^1(\Omega)$ by definition there exist $u_n \in \mathscr{D}(\Omega)$ such that $u_n \to u$ in $H^1(\Omega)$. As $u_n \in C(\overline{\Omega}) \cap H^1(\Omega)$ it follows

$$\tau_D u = \lim_{n\to\infty} \tau_D u_n = \lim_{n\to\infty} (u_n|_{\partial\Omega}) = 0.$$

The inclusion $\ker \tau_D \subset H_0^1(\Omega)$ is more difficult... □

3.2 Neumann boundary conditions

The following definition is motivated by the first Green identity.

Definition 3.14. Let Ω be a bounded Lipschitz domain with trace operator τ_D. Moreover, let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$, where Δu is formed in the distributional sense first. If there exists $b \in L^2(\partial\Omega)$ such that

$$\int_\Omega \Delta u(x) v(x)\, dx + \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_{\partial\Omega} b(\tau_D v) d\sigma$$

holds for all $v \in H^1(\Omega)$ then we call b *normal derivative of* u. We shall often use the notation $\tau_N u = \frac{\partial u}{\partial \nu} := b$.

Remark 3.15. (i) The normal derivative is unique; this follows from the fact that the traces $\tau_D v$ for $v \in H^1(\Omega)$ form a dense subspace of $L^2(\partial\Omega)$ (here without proof).

(ii) For $u \in C^2(\overline{\Omega})$ it follows from Corollary 3.7 (i) that b coincides with the (classical) normal derivative. Therefore we write $\frac{\partial u}{\partial \nu}$ or $\tau_N u$ instead of b for any $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$.

(iii) Definition 3.14 implies the validity of Green's first identity for $u, v \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$. In a similar manner one also obtains Green's second identity, that is,

$$\int_\Omega \Delta u(x) v(x)\, dx - \int_\Omega u(x) \Delta v(x)\, dx = \int_{\partial\Omega} (\tau_N u)(\tau_D v) d\sigma - \int_{\partial\Omega} (\tau_D u)(\tau_N v) d\sigma$$

This section is devoted to the Neumann boundary value problem for the Poisson equation: for fixed $\lambda \le 0$ and $f \in L^2(\Omega)$ we are interested in solutions $u \in H^1(\Omega)$ of the problem

$$\begin{aligned} -\Delta u - \lambda u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Note that the first condition implies $\Delta u \in L^2(\Omega)$ so that $\frac{\partial u}{\partial \nu}$ is understood in the sense of Definition 3.14.

Lemma 3.16. *A function $u \in H^1(\Omega)$ is a (distributional) solution of* (3.2) *if and only if*

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx - \lambda \int_\Omega u(x)v(x)\, dx = \int_\Omega f(x)v(x)\, dx, \quad v \in H^1(\Omega). \tag{3.3}$$

Proof. Let first $u \in H^1(\Omega)$ be a solution of (3.2). Then for any $v \in H^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega f(x)v(x)\, dx &= -\int_\Omega \Delta u(x)v(x)\, dx - \lambda \int_\Omega u(x)v(x)\, dx \\ &= \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx - \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}}_{=0} (\tau_D v) d\sigma - \lambda \int_\Omega u(x)v(x)\, dx,\end{aligned}$$

where we have used Definition 3.14. If, conversely, u satisfies (3.3) then for any $\varphi \in \mathscr{D}(\Omega)$ we have

$$\begin{aligned}(-\Delta T_u)\varphi = \sum_{j=1}^{d} \Big(\frac{\partial T_u}{\partial x_j}\Big)\Big(\frac{\partial \varphi}{\partial x_j}\Big) &= \int_\Omega \nabla u(x) \cdot \nabla \varphi(x)\, dx \\ &= \int_\Omega (f(x) + \lambda u(x))\varphi(x)\, dx = T_{(f+\lambda u)}\varphi\end{aligned}$$

distributionally, which implies $-\Delta u = f + \lambda u \in L^2(\Omega)$. Moreover, for any $v \in H^1(\Omega)$ it follows from (3.3) that

$$\begin{aligned}0 &= \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx - \int_\Omega (f(x) + \lambda u(x))v(x)\, dx \\ &= \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \int_\Omega \Delta u(x)v(x)\, dx \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(\tau_D v) d\sigma\end{aligned}$$

holds with $\frac{\partial u}{\partial \nu} = 0$. Hence u is a solution of (3.2). □

Remark 3.17. Sometimes the Neumann problem (3.2) is considered on very irregular (non-Lipschitz) domains. Then the problem is directly interpreted in the sense of Lemma 3.16, which requires no assumptions on Ω.

Theorem 3.18. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, let $\lambda < 0$ and $f \in L^2(\Omega)$. Then* (3.2) *has a unique solution $u \in H^1(\Omega)$.*

Proof. The mapping $F : H^1(\Omega) \to \mathbb{C}$, $F(v) := \int_\Omega f(x)\overline{v(x)}\,dx$ is a bounded, antilinear functional. As in the proof of Theorem 2.2 one shows that the mapping $\mathfrak{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$,

$$\mathfrak{a}[u,v] := \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)}\,dx - \lambda \int_\Omega u(x)\overline{v(x)}\,dx, \quad u, v \in H^1(\Omega),$$

is a symmetric sesquilinear form which is bounded and coercive. By the Lax–Milgram theorem there exists a unique $u \in H^1(\Omega)$ such that

$$\mathfrak{a}[u,v] = F(v), \qquad v \in H^1(\Omega),$$

and Lemma 3.16 leads to the assertion. □

Remark 3.19. For $\lambda = 0$ uniqueness of a solution of (3.2) cannot be guaranteed. In fact, each constant function u satisfies $-\Delta u = 0$ and $\frac{\partial u}{\partial \nu} = 0$. Thus constants can be added to any solution. Therefore an additional condition is required in order to obtain uniqueness. Moreover, if u is a solution of (3.2) then plugging the constant function $v = 1$ into (3.3) for $\lambda = 0$ yields $\int_\Omega f(x)dx = 0$. Therefore a solution can only exist if the integral of f vanishes.

In the proof of Theorem 3.21 we shall use the following result; its proof is postponed after the proof of Theorem 3.21.

Theorem 3.20 (Second Poincaré inequality)**.** *Let Ω be a bounded, connected, nonempty Lipschitz domain. Then there exists a constant $c > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)}$$

holds for all $u \in H^1(\Omega)$ with the property $\int_\Omega u(x)dx = 0$. In particular, on the closed subspace

$$H^1_{\mathrm{m}}(\Omega) := \left\{ u \in H^1(\Omega) : \int_\Omega u(x)dx = 0 \right\}$$

of $H^1(\Omega)$ the norm

$$|u|_{H^1(\Omega)} := \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)}, \quad u \in H^1_{\mathrm{m}}(\Omega),$$

is equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$ on $H^1_{\mathrm{m}}(\Omega)$.

Theorem 3.21. *Let Ω be a bounded, nonempty, connected Lipschitz domain and $f \in L^2(\Omega)$ with $\int_\Omega f(x)dx = 0$. Then (3.2) with $\lambda = 0$ has a unique solution $u \in H^1(\Omega)$ such that $\int_\Omega u(x)dx = 0$.*

Proof. Consider the Hilbert space

$$H^1_{\mathrm{m}}(\Omega) = \left\{ u \in H^1(\Omega) : \int_\Omega u(x)dx = 0 \right\},$$

equipped with the norm

$$|u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega;\mathbb{C}^d)}, \quad u \in H^1_{\mathrm{m}}(\Omega)$$

cf. Theorem 3.20. Next we define the antilinear functional $F : H^1_{\mathrm{m}}(\Omega) \to \mathbb{C}$, $F(v) := \int_\Omega f(x)\overline{v(x)}dx$. Then F is bounded since

$$|F(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}|v|_{H^1(\Omega)}, \quad v \in H^1_{\mathrm{m}}(\Omega),$$

where c is the constant from the second Poincaré inequality Theorem 3.20. By the Fréchet–Riesz theorem there exists a unique $u \in H^1_{\mathrm{m}}(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)}\, dx = F(v) = \int_\Omega f(x)\overline{v(x)}\, dx, \quad v \in H^1_{\mathrm{m}}(\Omega). \tag{3.4}$$

Let now $v \in H^1(\Omega)$ be arbitrary and let $w = |\Omega|^{-1/2}$ identically. Then

$$v = v - (v,w)_{L^2(\Omega)}w + (v,w)_{L^2(\Omega)}w$$

and since $v - (v,w)_{L^2(\Omega)}w \in H^1_{\mathrm{m}}(\Omega)$, (3.4) yields

$$\begin{aligned}\int_\Omega f(x)\overline{v(x)}\, dx &= \int_\Omega f(x)\overline{\big(v(x) - (v,w)_{L^2(\Omega)}w(x)\big)}\, dx + (v,w)_{L^2(\Omega)}\underbrace{(f,w)_{L^2(\Omega)}}_{=0} \\ &= \int_\Omega \nabla u(x) \cdot \overline{\nabla\big(v - (v,w)_{L^2(\Omega)}w\big)(x)}\, dx \\ &= \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)}\, dx.\end{aligned}$$

Hence by Lemma 3.16 u is a solution of (3.2) with $\lambda = 0$. The uniqueness of the solution follows from the uniqueness of u with the property (3.4). □

It remains to prove Theorem 3.20.

Proof of Theorem 3.20. Assume the converse. Then there exists a sequence $(u_n)_n \subset H^1(\Omega)$ such that $\int_\Omega u_n(x)dx = 0$ and $\|u_n\|_{L^2(\Omega)} = 1$ for all $n \in \mathbb{N}$, but $\|\nabla u_n\|_{L^2(\Omega;\mathbb{C}^d)} \to 0$ as $n \to \infty$. As the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact by Theorem 3.11, there exists a subsequence (without loss of generality again $(u_n)_n$) which converges in $L^2(\Omega)$ to some $u \in L^2(\Omega)$; in particular $\|u\|_{L^2(\Omega)} = 1$. Then for any $\varphi \in \mathscr{D}(\Omega)$ and $j = 1, \dots, d$

$$\int_\Omega u(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx = \lim_{n\to\infty}\int_\Omega u_n(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx = -\lim_{n\to\infty}\int_\Omega \frac{\partial u_n}{\partial x_j}(x)\varphi(x)\,dx = 0$$

since $\frac{\partial u_n}{\partial x_j} \to 0$ in $L^2(\Omega)$ as $n \to \infty$. Hence $u \in H^1(\Omega)$ and $\frac{\partial u}{\partial x_j} = 0$ for $j = 1, \dots, d$. Thus by Lemma 1.26 there exists $C \in \mathbb{C}$ such that $u(x) = C$ for almost all $x \in \Omega$ since Ω is connected. Moreover,

$$|\Omega|C = \int_\Omega u(x)dx = \lim_{n\to\infty}\int_\Omega u_n(x)dx = 0$$

(for the convergence interpret the integrals as L^2-inner products with the constant function 1), which implies $C = 0$ and contradicts $\|u\|_{L^2(\Omega)} = 1$. □

3.3 Robin boundary conditions

For $f \in L^2(\Omega)$ and measurable, $\lambda \le 0$, and bounded $\vartheta : \partial\Omega \to [0, \infty)$ we consider the Robin boundary value problem

$$\begin{aligned} -\Delta u - \lambda u &= f \quad \text{in } \Omega, \\ \tau_N u + \vartheta \tau_D u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.5}$$

where τ_D denotes the Dirichlet trace operator on the bounded Lipschitz domain Ω from Theorem 3.12 and $\tau_N u$ is the normal derivative from Definition 3.14.

Lemma 3.22. *A function $u \in H^1(\Omega)$ is a (distributional) solution of* (3.5) *if and only if*

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx - \lambda\int_\Omega u(x)v(x)\,dx + \int_{\partial\Omega}\vartheta(\tau_D u)(\tau_D v)\,d\sigma = \int_\Omega f(x)v(x)\,dx$$

holds for all $v \in H^1(\Omega)$.

Proof. Exercise. □

Theorem 3.23. *Let Ω be a bounded, connected, nonempty Lipschitz domain and let $\vartheta : \partial\Omega \to [0, \infty)$ be measurable and bounded. In the case $\lambda = 0$ assume in addition that ϑ is positive on a set of positive measure. Then for each $f \in L^2(\Omega)$ the problem* (3.5) *has a unique solution $u \in H^1(\Omega)$.*

Proof. Define the sesquilinear form $\mathfrak{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$,

$$\mathfrak{a}[u, v] := \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)}\, dx - \lambda \int_\Omega u(x)v(x)\, dx + \int_{\partial\Omega} \vartheta(\tau_D u)\overline{(\tau_D v)}$$

for $u, v \in H^1(\Omega)$. Then $\mathfrak{a}$ is a symmetric sesquilinear form and it follows from the continuity of the trace operator τ_D that $\mathfrak{a}$ is bounded. In the case $\lambda < 0$ it is clear that $\mathfrak{a}$ is coercive since $\vartheta \geq 0$ (see the proof of Theorem 2.2). We show that $\mathfrak{a}$ is coercive in the case $\lambda = 0$. Assume the converse. Then there exists $(u_n)_n \subset H^1(\Omega)$ with $\|u_n\|_{H^1(\Omega)} = 1$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \mathfrak{a}[u_n] = 0$. Since $(u_n)_n$ is bounded we can assume without loss of generality that $(u_n)_n$ converges weakly in $H^1(\Omega)$ to some $u \in H^1(\Omega)$. Since the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact by Theorem 3.11, it follows $u_n \to u$ in $L^2(\Omega)$ as $n \to \infty$; moreover, the condition $\mathfrak{a}[u_n] \to 0$ implies $\|\nabla u_n\|_{L^2(\Omega;\mathbb{C}^d)} \to 0$ since $\vartheta \geq 0$. In particular,

$$\|u\|^2_{L^2(\Omega)} = \lim_{n\to\infty} \|u_n\|^2_{L^2(\Omega)} + \lim_{n\to\infty} \|\nabla u_n\|^2_{L^2(\Omega;\mathbb{C}^d)} = \lim_{n\to\infty} \|u_n\|^2_{H^1(\Omega)} = 1. \tag{3.6}$$

Furthermore, for any $\varphi \in \mathscr{D}(\Omega)$

$$\int_\Omega u(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx = \lim_{n\to\infty} \int_\Omega u_n(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx = - \lim_{n\to\infty} \int_\Omega \frac{\partial u_n}{\partial x_j}(x)\varphi(x)\, dx = 0$$

for $j = 1, \dots, d$, and it follows $\frac{\partial u}{\partial x_j} = 0$ for $j = 1, \dots, d$. By Lemma 1.26 there exists $c \in \mathbb{C}$ with $u(x) = c$ for almost all $x \in \Omega$. As $u_n \to u$ in $L^2(\Omega)$ and $\frac{\partial u_n}{\partial x_j} \to 0$ in $L^2(\Omega)$ for $j = 1, \dots, d$ we conclude $u_n \to u$ in $H^1(\Omega)$. In particular, $\tau_D u_n \to \tau_D u = c$ in $L^2(\partial\Omega)$. Thus

$$|c|^2 \int_{\partial\Omega} \vartheta d\sigma = \lim_{n\to\infty} \int_{\partial\Omega} \vartheta |\tau_D u_n|^2 d\sigma = \lim_{n\to\infty} \mathfrak{a}[u_n] = 0$$

and since ϑ is positive on a set of positive measure we obtain $u(x) = c = 0$ for almost all $x \in \Omega$, which contradicts (3.6). This shows coercivity. Applying the Lax–Milgram theorem (as, e.g., in the proof of Theorem 3.18) and Lemma 3.22 the assertion follows. □

Chapter 4

Laplace operators on bounded domains

4.1 Symmetric and selfadjoint operators

Let $\mathcal{H}$ be a Hilbert space and let S be a linear operator in $\mathcal{H}$ defined on the linear subspace $\operatorname{dom} S \subset \mathcal{H}$. The linear operators that will be considered in this chapter are typically unbounded in $\mathcal{H}$ and are not defined on the whole space $\mathcal{H}$. If the domain $\operatorname{dom} S$ of S is dense in $\mathcal{H}$ then the *adjoint operator* S^* is defined as follows:

$$\begin{aligned} S^* g &= g', \\ \operatorname{dom} S^* &= \bigl\{ g \in \mathcal{H} : \text{exists } g' \in \mathcal{H} \text{ such that } (Sf, g) = (f, g'),\ f \in \operatorname{dom} S \bigr\}. \end{aligned}$$

Observe first that S^* is well defined since $\operatorname{dom} S$ is dense by assumption, that is, the element $g' \in \mathcal{H}$ is unique. It is also easy to check that S^* is a closed operator and that the identities

$$\ker S^* = (\operatorname{ran} S)^\perp \quad \text{and} \quad \ker(S^* - \lambda) = \bigl(\operatorname{ran}(S - \overline{\lambda})\bigr)^\perp, \quad \lambda \in \mathbb{C}, \tag{4.1}$$

hold. It is left as an exercise to check that S is closable if and only if $\operatorname{dom} S^*$ is dense in $\mathcal{H}$ in which case $S^{**} = \overline{S}$. Furthermore, if S is closable one has $S^* = (\overline{S})^*$, and if $S \subset T$ then $T^* \subset S^*$. Another useful observation is the property

$$(S^{-1})^* = (S^*)^{-1}$$

whenever the operator S is densely defined and invertible (that is, $\ker S = \{0\}$) and $\operatorname{dom} S^{-1} = \operatorname{ran} S$ is dense in $\mathcal{H}$; note that $\operatorname{ran} S$ is dense if and only if

$\ker S^* = \{0\}$ by (4.1), i.e. S^* is invertible. Furthermore, in the special case that S is bounded and defined on the space $\mathcal{H}$ (we shall use $\mathcal{L}(\mathcal{H})$ to denote this class of operators) the definition of S^* above reduces to the *standard* definition $(Sf, g) = (f, S^*g)$, $f, g \in \mathcal{H}$, in the bounded case; clearly one has $S^* \in \mathcal{L}(\mathcal{H})$.

In the next definition we consider operators S that are contained in (or even equal to) their adjoints S^*.

Definition 4.1. Let S be a densely defined operator in $\mathcal{H}$. Then S is said to be

(i) *symmetric* if $S \subset S^*$ (i.e. $\operatorname{dom} S \subset \operatorname{dom} S^*$ and $Sf = S^*f$ for $f \in \operatorname{dom} S$);

(ii) *self-adjoint* if $S = S^*$;

(iii) *essentially self-adjoint* if $\overline{S} = S^*$.

It follows from the definition that a densely defined operator S is symmetric if and only if one has

$$(Sf, g) = (f, Sg), \qquad f, g \in \operatorname{dom} S.$$

One can even show the stronger statement that S is symmetric if and only if $(Sf, f) \in \mathbb{R}$ for all $f \in \operatorname{dom} S$. One also verifies that S is symmetric if and only if $\overline{S}$ is symmetric. Note in this context that a symmetric operator is always closable and that its closure satisfies $\overline{S} \subset S^*$. Finally, observe that for $S \in \mathcal{L}(\mathcal{H})$ the concepts of symmetry and self-adjointness coincide.

Lemma 4.2. *Let S be a densely defined symmetric operator in $\mathcal{H}$. Then for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ one has $\ker(S - \lambda) = \{0\}$ and for all $g \in \operatorname{ran}(S - \lambda)$ the estimate*

$$\|(S - \lambda)^{-1} g\| \leq \frac{1}{|\operatorname{Im} \lambda|} \|g\| \tag{4.2}$$

is valid. In particular, if S is closed then $\operatorname{ran}(S - \lambda)$ is closed for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $f \in \operatorname{dom} S$ one has

$$0 \leq |\operatorname{Im} \lambda|(f, f) = |\operatorname{Im}((S - \lambda)f, f)| \leq \|(S - \lambda)f\| \|f\|$$

and hence for $f \neq 0$ it follows that $|\operatorname{Im} \lambda| \|f\| \leq \|(S - \lambda)f\|$. This implies (4.2). In order to see that $\operatorname{ran}(S - \lambda)$ is closed consider a sequence $g_n = (S - \lambda)f_n$ that converges to $g \in \mathcal{H}$. By (4.2) one has

$$\|f_n\| \leq \frac{1}{|\operatorname{Im} \lambda|} \|(S - \lambda)f_n\|$$

and thus (f_n) is a Cauchy sequence in $\mathcal{H}$ which converges to some $f \in \mathcal{H}$. As $S-\lambda$ is closed we conclude $f \in \operatorname{dom} S$ and $(S-\lambda)f = g$. This shows that $\operatorname{ran}(S-\lambda)$ is closed for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. □

Next we recall the notion of spectrum and resolvent set of a closed linear operator and the subdivision of the spectral points in eigenvalues, continuous spectrum, and residual spectrum.

Definition 4.3. Let T be a closed linear operator in a Hilbert (or Banach space) $\mathcal{H}$ and let $\lambda \in \mathbb{C}$. Then we say that

(i) $\lambda \in \rho(T)$ if $(T-\lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ (resolvent set);

(ii) $\lambda \in \sigma(T)$ if $\lambda \in \mathbb{C} \setminus \rho(T)$ (spectrum);

(iii) $\lambda \in \sigma_p(T)$ if $\ker(T-\lambda) \neq \{0\}$ (eigenvalue);

(iv) $\lambda \in \sigma_c(T)$ if $\ker(T-\lambda) = \{0\}$, $\operatorname{ran}(T-\lambda) \neq \mathcal{H}$ dense (continuous spectrum);

(v) $\lambda \in \sigma_r(T)$ if $\ker(T-\lambda) = \{0\}$, $\operatorname{ran}(T-\lambda)$ not dense (residual spectrum).

It is clear that for a closed operator T one has

$$\mathbb{C} = \sigma(T) \mathbin{\dot\cup} \rho(T) \quad \text{and} \quad \sigma(T) = \sigma_p(T) \mathbin{\dot\cup} \sigma_c(T) \mathbin{\dot\cup} \sigma_r(T).$$

Furthermore, if $\lambda \in \sigma_c(T)$ then $(T-\lambda)^{-1}$ is necessarily unbounded (since $(T-\lambda)^{-1}$ is closed it would have a closed domain if it would be bounded).

In the context of closed symmetric operators the following observation on the spectral points follow from Lemma 4.2.

Corollary 4.4. *Let S be a densely defined closed symmetric operator in $\mathcal{H}$. Then*

$$\big(\sigma_p(S) \cup \sigma_c(S)\big) \subset \mathbb{R} \quad \textit{and} \quad \mathbb{C} \setminus \mathbb{R} \subset \big(\sigma_r(S) \cup \rho(S)\big).$$

In the next theorem we provide a useful criterion to check that a given symmetric operator is self-adjoint. For pratical purposes it is an essential advantage that the symmetric operator is not assumed to be closed here.

Theorem 4.5. *Let S be a densely defined symmetric operator in $\mathcal{H}$ and assume that*

$$\operatorname{ran}(S-\mu) = \mathcal{H} = \operatorname{ran}(S-\overline{\mu}) \quad \textit{for some } \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{4.3}$$

Then S is self-adjoint in $\mathcal{H}$ and $\operatorname{ran}(S-\lambda) = \mathcal{H}$ *for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Furthermore,*

$$\sigma(S) = \big(\sigma_p(S) \cup \sigma_c(S)\big) \subset \mathbb{R} \quad \textit{and} \quad \mathbb{C} \setminus \mathbb{R} \subset \rho(S). \tag{4.4}$$

Proof. In order to see that S is self-adjoint we have to check that $S^* \subset S$. For this consider $g \in \operatorname{dom} S^*$ and choose $f \in \operatorname{dom} S$ such that

$$(S^* - \mu)g = (S - \mu)f,$$

which is possible due to (4.3). Since $S \subset S^*$ we conclude that $(S^* - \mu)(f - g) = 0$, that is,

$$f - g \in \ker(S^* - \mu) = \big(\operatorname{ran}(S - \overline{\mu})\big)^{\perp} = \{0\},$$

where we have again used (4.3). Hence $f = g \in \operatorname{dom} S$ and from $S \subset S^*$ it is clear that $Sg = S^* g$. This shows the inclusion $S^* \subset S$ and therefore S is self-adjoint in $\mathcal{H}$. In particular, S is a closed operator in $\mathcal{H}$.

Next we check that $\operatorname{ran}(S - \lambda) = \mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since $S - \mu$ is bijective by Lemma 4.2 and (4.3), and S is closed it follows that $(S - \mu)^{-1} \in \mathcal{L}(\mathcal{H})$ and $\mu \in \rho(S)$. Now assume that $\lambda \in \mathbb{C}$ is in the same complex half-plane as μ and that $|\mu - \lambda| < |\operatorname{Im} \mu|$. From

$$S - \lambda = (S - \mu)\big[I + (\mu - \lambda)(S - \mu)^{-1}\big] \quad \text{and} \quad |\mu - \lambda| \|(S - \mu)^{-1}\| \leq \frac{|\mu - \lambda|}{|\operatorname{Im} \mu|} < 1$$

we conclude

$$(S - \lambda)^{-1} = \big[I + (\mu - \lambda)(S - \mu)^{-1}\big]^{-1}(S - \mu)^{-1} \in \mathcal{L}(\mathcal{H}),$$

that is, all λ in the same half-plane as μ such that $|\mu - \lambda| \leq |\operatorname{Im} \mu|$ belong to $\rho(S)$. Now the same argument repeatedly applied to points in the complex plane with larger imaginary parts finally yields $\mathbb{C} \setminus \mathbb{R} \subset \rho(S)$ and, in particular, $\operatorname{ran}(S - \lambda) = \mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

It remains to show that $\sigma_r(S) = \emptyset$. From the above it is clear that $\sigma_r(S) \subset \mathbb{R}$. Suppose that $\lambda \in \sigma_r(S) \cap \mathbb{R}$. Then one obtains

$$\{0\} \neq \big(\operatorname{ran}(S - \lambda)\big)^{\perp} = \ker(S^* - \overline{\lambda}) = \ker(S - \lambda), \tag{4.5}$$

which implies that $\lambda \in \sigma_p(S)$; a contradiction. □

We remark that the asumption (4.3) can be replaced by the weaker assumption

$$\operatorname{ran}(S - \lambda_+) = \mathcal{H} = \operatorname{ran}(S - \lambda_-) \quad \text{for some } \lambda_\pm \in \mathbb{C}^\pm.$$

In some cases it is also useful to have a variant of the above theorem for real points. We formulate this next and leave the simple modifications of the proof as an exercise.

Theorem 4.6. *Let S be a densely defined symmetric operator in $\mathcal{H}$ and assume that*

$$\operatorname{ran}(S-\mu)=\mathcal{H} \quad \text{for some } \mu\in\mathbb{R}. \tag{4.6}$$

Then S is self-adjoint in $\mathcal{H}$ and $\mu\in\rho(S)$. Furthermore, (4.4) *holds.*

Later we shall often make use of the following lemma, which is formulated in a slightly more general context.

Lemma 4.7. *Let A and T be operators in $\mathcal{H}$ such that $A\subset T$ and $\rho(A)\neq\emptyset$. Then the direct sum decomposition*

$$\operatorname{dom}T=\operatorname{dom}A \dotplus \ker(T-\lambda), \qquad \lambda\in\rho(A), \tag{4.7}$$

is valid.

Proof. Consider $g\in\operatorname{dom}T$ and choose $f\in\operatorname{dom}A$ such that $(T-\lambda)g=(A-\lambda)f$, which is possible whenever $\lambda\in\rho(A)$. As $A\subset T$ this implies $(T-\lambda)(g-f)=0$ and hence $g-f\in\ker(T-\lambda)$. As $g=f+(g-f)$ with $f\in\operatorname{dom}A$ we conclude (4.7). The fact that the decomposition of $\operatorname{dom}T$ in (4.7) is direct is a consequence of the assumption $\lambda\in\rho(A)$. □

The aim in the following is on a description of self-adjoint extensions of symmetric operators. More precisely, assume that S is a densely defined closed symmetric operator in $\mathcal{H}$. The goal is to find self-adjoint extensions A of S in $\mathcal{H}$, that is, $S\subset A=A^*\subset S^*$. There is a well known necessary and sufficient criterion on the existence of self-adjoint extensions and the parametrization of these extensions. The next theorem is known as von Neumanns first and second formula; for a proof we refer to ???

Theorem 4.8. *Let S be a densely defined closed symmetric operator in $\mathcal{H}$. Then the direct sum decomposition (von Neumanns first formula)*

$$\operatorname{dom}S^*=\operatorname{dom}S \dotplus \ker(S^*-i)\dotplus \ker(S^*+i)$$

holds. The operator S admits selfadjoint extensions in $\mathcal{H}$ if and only if

$$\dim\bigl(\ker(S^*-i)\bigr)=\dim\bigl(\ker(S^*+i)\bigr). \tag{4.8}$$

In this case an operator A in $\mathcal{H}$ is a selfadjoint extension of S if and only if there exists a unitary operator $U:\ker(S^*-i)\to\ker(S^*+i)$ such that (von Neumanns second formula)

$$\begin{aligned} Af&=Sf_S+if_i-iUf_i,\\ \operatorname{dom}A&=\bigl\{f=f_S+f_i+f_{-i}\in\operatorname{dom}S^* : f_{-i}=Uf_i\bigr\}. \end{aligned}$$

The quantities in (4.8) are typically called defect numbers or deficiency indices of S; roughly speaking these numbers from $\mathbb{N} \cup \{\infty\}$ (all Hilbert spaces are separable here for simplicity) indicate how many dimensions are missing for the symmetric operator S to be self-adjoint. Note that

$$\ker(S^* \mp i) = \big(\operatorname{ran}(S \pm i)\big)^\perp$$

and that for a self-adjoint operator the latter orthogonal complements are $\{0\}$ according to Theorem 4.5.

4.2 Quasi boundary triples and their Weyl functions

Throughout this section we assume that S is a densely defined, closed, symmetric operator in a Hilbert space $\mathcal{H}$. We start by recalling the notion of quasi boundary triples.

In the following we denote all appearing inner products by $(\cdot\,,\cdot)$; the respective Hilbert space will be clear from the context.

Definition 4.9. Let $T \subset S^*$ be a linear operator in $\mathcal{H}$ such that $\overline{T} = S^*$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *quasi boundary triple* for $T \subset S^*$ if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are linear mappings such that

(i) the abstract Green identity

$$(Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) \tag{4.9}$$

holds for all $f, g \in \operatorname{dom} T$;

(ii) the map $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ has dense range;

(iii) $A_0 := T \restriction \ker \Gamma_0$ is a self-adjoint operator in $\mathcal{H}$.

Before we list some properties of quasi boundary triples let us consider two standard examples first.

Example 4.10. Let Ω be a bounded C^2-domain and consider the operators

$$\begin{aligned} Sf &= -\Delta f, \qquad \operatorname{dom} S = H^2_0(\Omega),\\ Tf &= -\Delta f, \qquad \operatorname{dom} T = H^2(\Omega). \end{aligned}$$

Then it can be shown that $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f = \tau_D f \quad \text{and} \quad \Gamma_1 f = -\tau_N f,$$

is a quasi boundary triple for $T \subset S^*$ such that

$$A_0 f = -\Delta f, \qquad \operatorname{dom} A_0 = H^2(\Omega) \cap H_0^1(\Omega). \tag{4.10}$$

In fact, we shall sketch some of the essential arguments for this observation. First of all it follows from Green's second identity in Corollary 3.7 using Remark 3.15 (iii) that

$$(Tf, g) - (f, Tg) = (-\Delta f, g) - (f, -\Delta g) = (\tau_D f, \tau_N g) - (\tau_N f, \tau_D g)$$

holds for all $f, g \in \operatorname{dom} T = H^2(\Omega)$. The density condition (ii) in Definition 4.9 is well known and will not be proved here. It is also clear that the restriction $A_0 = T \upharpoonright \ker \Gamma_0$ is given by the Dirichlet operator in (4.10). Now observe first that

$$(A_0 f, g) - (f, A_0 g) = (-\Delta f, g) - (f, -\Delta g) = (\tau_N f, \tau_D g) - (\tau_D f, \tau_N g) = 0$$

for $f, g \in \operatorname{dom} A_0$ as $\tau_D f = \tau_D g = 0$. Hence A_0 is a symmetric operator in $L^2(\Omega)$ and it remains to check that A_0 is indeed self-adjoint in $L^2(\Omega)$. Recall that the Dirichlet problem (2.2) for $\lambda = 0$ admits a unique solution in $H_0^1(\Omega)$ for any right hand side $f \in L^2(\Omega)$ by Theorem 2.2 and that in fact this solution is in $H^2(\Omega)$ due to Theorem 2.13, that is, the solution belongs to $\operatorname{dom} A_0$. Hence we can apply Theorem 4.6 with $\mu = 0$ and conclude that $A_0 = A_0^*$. It still remains to show that $\overline{T} = S^*$; which will not be done here. However, we at least remark that

$$S^* f = -\Delta f, \qquad \operatorname{dom} S^* = \big\{ f \in L^2(\Omega) : -\Delta f \in L^2(\Omega) \big\},$$

where $-\Delta f$ is understood in the sense of distributions.

It is already clear from Example 4.10 that a quasi boundary triple (if it exists and is nontrivial) is not unique. Namely, one can argue as above to show the following.

Example 4.11. Let S and T be as in Example 4.10 above. Then the triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f = \tau_N f \quad \text{and} \quad \Gamma_1 f = \tau_D f,$$

is a quasi boundary triple for $T \subset S^*$ such that

$$A_0 f = -\Delta f, \qquad \operatorname{dom} A_0 = \big\{ f \in H^2(\Omega) : \tau_N f = 0 \big\}. \tag{4.11}$$

We note that a quasi boundary triple exists if and only if S admits self-adjoint extensions in $\mathcal{H}$, that is, the deficiency indices of S are equal; cf. Theorem 4.8. Moreover, if $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $T \subset S^*$, then one has $T = S^*$ if and only if $\operatorname{ran}\Gamma = \mathcal{G}\times\mathcal{G}$, in which case $\Gamma = (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} S^* \to \mathcal{G}\times\mathcal{G}$ is onto and continuous with respect to the graph norm of S^*, the abstract Green identity holds for all $f, g \in \operatorname{dom} S^*$, and the restriction $A_0 = S^* \upharpoonright \ker\Gamma_0$ is automatically self-adjoint. In this situation the notion of quasi boundary triples coincides with the notion of so-called ordinary boundary triples. In particular, this is the case when the deficiency indices of S are finite (and equal). For later use let us also introduce the notation $A_1 := T \upharpoonright \ker\Gamma_1$. This operator is always symmetric, which follows from the abstract Green identity.

With each quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ one associates a so-called γ-field and a Weyl function. Before we recall their definitions, note that for each $\lambda \in \rho(A_0)$ one has the direct sum decomposition

$$\operatorname{dom} T = \operatorname{dom} A_0 \dotplus \ker(T-\lambda) = \ker\Gamma_0 \dotplus \ker(T-\lambda)$$

by Lemma 4.7. Thus the restriction of the boundary map Γ_0 to $\ker(T-\lambda)$ is injective, and its range coincides with $\operatorname{ran}\Gamma_0$. The definitions of the γ-field and the Weyl function are providde next.

Definition 4.12. The *γ-field* γ and the *Weyl function* M corresponding to the quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are defined by

$$\lambda \mapsto \gamma(\lambda) := \big(\Gamma_0 \upharpoonright \ker(T-\lambda)\big)^{-1}, \qquad \lambda \in \rho(A_0),$$

and

$$\lambda \mapsto M(\lambda) := \Gamma_1\gamma(\lambda), \qquad \lambda \in \rho(A_0),$$

respectively.

Observe that $\gamma(\lambda)$ is a mapping from $\operatorname{ran}\Gamma_0 \subset \mathcal{G}$ onto $\ker(T-\lambda) \subset \mathcal{H}$ and that the values $M(\lambda)$ of the Weyl function are operators in $\mathcal{G}$ mapping $\operatorname{ran}\Gamma_0$ into $\operatorname{ran}\Gamma_1$. Note that $\operatorname{ran}\Gamma_0$ and $\operatorname{ran}\Gamma_1$ are both dense subspaces of $\mathcal{G}$; this is a consequence of the density of the range of $\Gamma = (\Gamma_0, \Gamma_1)^\top$. Various useful and important properties of the γ-field and the Weyl function can be found in [3, Proposition 2.6] or [4, Propositions 6.13 and 6.14]. For later purposes we recall that the adjoint $\gamma(\lambda)^*$ is a bounded, everywhere defined operator from $\mathcal{H}$ to $\mathcal{G}$, which satisfies

$$\gamma(\lambda)^* = \Gamma_1(A_0 - \overline{\lambda})^{-1}, \qquad \lambda \in \rho(A_0). \tag{4.12}$$

In fact, let $\varphi \in \operatorname{ran}\Gamma_0$, $h \in \mathcal{H}$ and choose $k \in \operatorname{dom} A_0$ such that $(A_0 - \overline{\lambda})k = h$. The one computes

$$\begin{aligned}(\gamma(\lambda)\varphi, h) &= \big(\gamma(\lambda)\varphi, (A_0 - \overline{\lambda})k\big) \\ &= (\gamma(\lambda)\varphi, A_0 k) - (\lambda\gamma(\lambda)\varphi, k) \\ &= (\gamma(\lambda)\varphi, Tk) - (T\gamma(\lambda)\varphi, k) \\ &= (\Gamma_0\gamma(\lambda)\varphi, \Gamma_1 k) - (\Gamma_1\gamma(\lambda)\varphi, \Gamma_0 k) \\ &= \big(\varphi, \Gamma_1(A_0 - \overline{\lambda})^{-1}h\big);\end{aligned}$$

this implies (4.12) and it also follows that $\gamma(\lambda)^* \in \mathcal{L}(\mathcal{H}, \mathcal{G})$. Hence also $\gamma(\lambda) \subset \overline{\gamma(\lambda)} = \gamma(\lambda)^{**} \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ for $\lambda \in \rho(A_0)$.

The Weyl function can be equivalently defined by

$$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda, \qquad f_\lambda \in \ker(T - \lambda),\ \lambda \in \rho(A_0). \tag{4.13}$$

The values of the Weyl function have the property $M(\lambda) \subset M(\overline{\lambda})^*$, $\lambda \in \rho(A_0)$, and, in particular, the operators $M(\lambda)$ are closable. We point out that the operators $M(\lambda)$ and their closures $\overline{M(\lambda)}$ are in general not bounded. However, if $M(\lambda_0)$ is bounded for one $\lambda_0 \in \rho(A_0)$, then $M(\lambda)$ is bounded for all $\lambda \in \rho(A_0)$; see [5, Proposition 3.3 (viii)].

Example 4.13. Let us consider the quasi boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ with $\Gamma_0 f = \tau_D f$ and $\Gamma_1 f = -\tau_N f$ from Example 4.10. The selfadjoint operator $A_0 = T \upharpoonright \ker\Gamma_0$ is the Dirichlet realization A_D of $-\Delta$ in $L^2(\Omega)$. In this situation one has $\operatorname{ran}\Gamma_0 = H^{3/2}(\partial\Omega)$ and for $\lambda \in \rho(A_D)$

$$\gamma(\lambda) : L^2(\partial\Omega) \to L^2(\Omega), \qquad \operatorname{dom}\gamma(\lambda) = H^{3/2}(\partial\Omega),$$

maps $\varphi \in H^{3/2}(\partial\Omega)$ onto $\gamma(\lambda)\varphi = f_\lambda(\varphi) \in H^2(\Omega)$, where $f_\lambda(\varphi)$ is the unique solution of the Dirichlet boundary value problem

$$(-\Delta - \lambda)f_\lambda(\varphi) = 0, \quad \tau_D f_\lambda(\varphi) = \varphi. \tag{4.14}$$

Furthermore, in this situation one has for $\lambda \in \rho(A_0)$

$$M(\lambda) : L^2(\partial\Omega) \to L^2(\partial\Omega), \quad \operatorname{dom} M(\lambda) = H^{3/2}(\partial\Omega), \quad \operatorname{ran} M(\lambda) \subset H^{1/2}(\partial\Omega).$$

If $f_\lambda(\varphi)$ is the unique solution of the Dirichlet boundary value problem (4.14) then $M(\lambda)\varphi = -\tau_N f_\lambda(\varphi)$ is the (minus) Dirichlet-to-Neumann map.

In the next example we provide the γ-field and Weyl function corresponding to the quasi boundary triple in Example 4.11. In the next section we shall make use of the particular quasi boundary triple.

Example 4.14. Let us consider the quasi boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ with $\Gamma_0 f = \tau_N f$ and $\Gamma_1 f = \tau_D f$ from Example 4.11. Here $A_0 = T \upharpoonright \ker \Gamma_0$ is the Neumann realization A_N of $-\Delta$ in $L^2(\Omega)$. One has $\operatorname{ran} \Gamma_0 = H^{1/2}(\partial\Omega)$ and for $\lambda \in \rho(A_N)$

$$\gamma(\lambda) : L^2(\partial\Omega) \to L^2(\Omega), \qquad \operatorname{dom} \gamma(\lambda) = H^{1/2}(\partial\Omega),$$

maps $\varphi \in H^{1/2}(\partial\Omega)$ onto $\gamma(\lambda)\varphi = f_\lambda(\varphi) \in H^2(\Omega)$, where $f_\lambda(\varphi)$ is the unique solution of the Neumann boundary value problem

$$(-\Delta - \lambda) f_\lambda(\varphi) = 0, \quad \tau_N f_\lambda(\varphi) = \varphi. \tag{4.15}$$

Furthermore, in this situation one has for $\lambda \in \rho(A_N)$

$$M(\lambda) : L^2(\partial\Omega) \to L^2(\partial\Omega), \quad \operatorname{dom} M(\lambda) = H^{1/2}(\partial\Omega), \quad \operatorname{ran} M(\lambda) \subset H^{3/2}(\partial\Omega).$$

If $f_\lambda(\varphi)$ is the unique solution of the Neumann boundary value problem (4.14) then $M(\lambda)\varphi = \tau_D f_\lambda(\varphi)$ is the Neumann-to-Dirichlet map.

In the following we are interested in operators of the form

$$A_{[B]} = S^* \upharpoonright \ker(\Gamma_0 - B\Gamma_1),$$

where B is some operator in $\mathcal{G}$ that determines an abstract boundary condition for the functions in $\operatorname{dom} S^*$. Typically, the aim is to derive properties of $A_{[B]}$ from the properties of B. It turns out below (and is easy to see making use of the abstract Greens identity) that a symmetric operator B in $\mathcal{G}$ leads to a symmetric extension $A_{[B]}$ of S in $\mathcal{H}$. However, a selfadjoint B does not automatically lead to a selfadjoint $A_{[B]}$. In the next theorem some more additional conditions are imposed that lead to the desired conclusion.

Theorem 4.15. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M. Let $B = B^* \in \mathcal{L}(\mathcal{G})$ and assume that there exist $\lambda_\pm \in \mathbb{C}^\pm$ such that the following conditions are satisfied:*

(i) $1 \in \rho(B\overline{M(\lambda_\pm)})$;

(ii) $B\big(\operatorname{ran} \overline{M(\lambda_\pm)}\big) \subset \operatorname{ran} \Gamma_0$;

(iii) $B(\operatorname{ran}\Gamma_1) \subset \operatorname{ran}\Gamma_0$ *or* A_1 *is self-adjoint.*

Then the operator

$$A_{[B]}f = Tf, \qquad \operatorname{dom} A_{[B]} = \big\{f \in \operatorname{dom} T : \Gamma_0 f = B\Gamma_1 f\big\}, \tag{4.16}$$

is a self-adjoint extension of S*, and*

$$(A_{[B]} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1}B\gamma(\overline{\lambda})^* \tag{4.17}$$

holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.

In the case $\operatorname{ran}\Gamma_0 = \mathcal{G}$ conditions (ii) and (iii) are automatically satisfied.

Proof of Theorem 4.15. The proof of Theorem 4.15 consists of several steps. In the first four steps we assume that the first condition in (iii) is satisfied.

Step 1. First we show that $A_{[B]}$ is symmetric, which is essentially a simple consequence of the abstract Green identity (4.9) and $B = B^*$. In fact, for $f, g \in \operatorname{dom} A_{[B]}$ we have

$$B\Gamma_1 f = \Gamma_0 f, \quad \text{and} \quad B\Gamma_1 g = \Gamma_0 g,$$

which implies that

$$\begin{aligned}(A_{[B]}f, g) - (f, A_{[B]}g) &= (Tf, g) - (f, Tg) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)\\ &= (\Gamma_1 f, B\Gamma_1 g) - (B\Gamma_1 f, \Gamma_1 g) = 0,\end{aligned}$$

where $B = B^*$ was used in the last step. This shows that $A_{[B]}$ is a symmetric operator in $\mathcal{H}$.

Step 2. In this step we show the inclusions

$$\operatorname{ran}\big(B\gamma(\overline{\lambda_\pm})^*\big) \subset \operatorname{ran}\big(I - BM(\lambda_\pm)\big). \tag{4.18}$$

We consider only $\lambda_+ \in \mathbb{C}^+$; the proof for $\lambda_- \in \mathbb{C}^-$ is the same. Note first that $\operatorname{dom} B = \mathcal{G}$ and hence the product $B\gamma(\overline{\lambda_\pm})^*$ is everywhere defined. Let $g \in \operatorname{ran}(B\gamma(\overline{\lambda_+})^*)$. Then there exists an $f \in \mathcal{H}$ such that $g = B\gamma(\overline{\lambda_+})^* f$. By (4.12) we have $\gamma(\overline{\lambda_+})^* f = \Gamma_1(A_0 - \lambda_+)^{-1} f \in \operatorname{ran}\Gamma_1$, and hence assumption (iii) implies that

$$B\gamma(\overline{\lambda_+})^* f \in \operatorname{ran}\Gamma_0. \tag{4.19}$$

We set

$$\varphi := \big(I - B\overline{M(\lambda_+)}\,\big)^{-1} B\gamma(\overline{\lambda}_+)^* f, \tag{4.20}$$

which is well defined by assumption (i). We can rewrite (4.20) in the form

$$\varphi = B\overline{M(\lambda_+)}\varphi + B\gamma(\overline{\lambda}_+)^* f. \tag{4.21}$$

By assumption (ii) we have $B\overline{M(\lambda_+)}\varphi \in \operatorname{ran}\Gamma_0$ and hence relations (4.19) and (4.21) imply $\varphi \in \operatorname{ran}\Gamma_0 = \operatorname{dom} M(\lambda_+)$. Together with (4.21) this yields

$$\big(I - BM(\lambda_+)\big)\varphi = B\gamma(\overline{\lambda}_+)^* f = g,$$

and hence $g \in \operatorname{ran}(I - BM(\lambda_+))$, i.e. the inclusion (4.18) is shown for $\lambda_+ \in \mathbb{C}^+$.

Step 3. We claim that $\operatorname{ran}(A_{[B]} - \lambda_\pm) = \mathcal{H}$ holds. Again we show the assertion only for $\lambda_+ \in \mathbb{C}^+$; the arguments for $\lambda_- \in \mathbb{C}^-$ are the same. Let $f \in \mathcal{H}$ and consider the element

$$h := (A_0 - \lambda_+)^{-1} f + \gamma(\lambda_+)\big(I - BM(\lambda_+)\big)^{-1} B\gamma(\overline{\lambda}_+)^* f. \tag{4.22}$$

Note that by assumption (i) the inverse $(I - BM(\lambda_+))^{-1}$ exists. It maps into $\operatorname{dom} M(\lambda_+) = \operatorname{ran}\Gamma_0$, so the product with $\gamma(\lambda_+)$ is well defined. Observe also that the product of $(I - BM(\lambda_+))^{-1}$ and $B\gamma(\overline{\lambda}_+)^*$ is well defined by (4.18). We now show that $h \in \operatorname{dom} A_{[B]}$. Clearly, $h \in \operatorname{dom} T$ since

$$(A_0 - \lambda_+)^{-1} f \in \operatorname{dom} A_0 \subset \operatorname{dom} T$$

and

$$\operatorname{ran}\gamma(\lambda_+) = \ker(T - \lambda_+) \subset \operatorname{dom} T.$$

Furthermore, using (4.12) and the definition of $M(\lambda_+)$ we have

$$\begin{aligned} B\Gamma_1 h &= B\Gamma_1(A_0 - \lambda_+)^{-1} f + B\Gamma_1\gamma(\lambda_+)\big(I - BM(\lambda_+)\big)^{-1} B\gamma(\overline{\lambda}_+)^* f \\ &= B\gamma(\overline{\lambda}_+)^* f + BM(\lambda_+)\big(I - BM(\lambda_+)\big)^{-1} B\gamma(\overline{\lambda}_+)^* f \\ &= \big[(I - BM(\lambda_+)) + BM(\lambda_+)\big]\big(I - BM(\lambda_+)\big)^{-1} B\gamma(\overline{\lambda}_+)^* f \\ &= \big(I - BM(\lambda_+)\big)^{-1} B\gamma(\overline{\lambda}_+)^* f; \end{aligned}$$

the relation $\operatorname{dom} A_0 = \ker\Gamma_0$ and the definition of $\gamma(\lambda_+)$ yield

$$\Gamma_0 h = \Gamma_0(A_0 - \lambda_+)^{-1} f + \Gamma_0\gamma(\lambda_+)\big(I - BM(\lambda_+)\big)^{-1} B\gamma(\overline{\lambda}_+)^* f$$

$$= \big(I - BM(\lambda_+)\big)^{-1} B\gamma(\overline{\lambda}_+)^* f.$$

Hence the element h in (4.22) satisfies the boundary condition $\Gamma_0 h = B\Gamma_1 h$. This shows that $h \in \operatorname{dom} A_{[B]}$. Finally, we obtain from (4.22) that

$$(A_{[B]} - \lambda_+)h = (T - \lambda_+)h = (T - \lambda_+)(A_0 - \lambda_+)^{-1} f = f, \tag{4.23}$$

where again $\operatorname{ran}\gamma(\lambda_+) = \ker(T - \lambda_+)$ was used. Hence $\operatorname{ran}(A_{[B]} - \lambda_+) = \mathcal{H}$ holds.

Step 4. It follows from the symmetry of $A_{[B]}$ shown in Step 1, the range condition in Step 3, and Theorem 4.5 that the operator $A_{[B]}$ is self-adjoint in $\mathcal{H}$. The resolvent formula follows for $\lambda = \lambda_\pm$ immediately from the identities (4.22) and (4.23) in Step 3. Assume now that $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$ is arbitrary. We claim that the operator $I - BM(\lambda)$ is injective. Indeed, if $\varphi \in \ker(I - BM(\lambda))$ then $\varphi \in \operatorname{dom} M(\lambda) = \operatorname{ran}\Gamma_0$ and hence $f := \gamma(\lambda)\varphi \in \ker(T - \lambda)$, so that $\Gamma_0 f = \varphi$. From

$$B\Gamma_1 f = BM(\lambda)\Gamma_0 f = BM(\lambda)\varphi = \varphi = \Gamma_0 f$$

we conclude that $f \in \operatorname{dom} A_{[B]}$ and hence $f \in \ker(A_{[B]} - \lambda)$. Since $\lambda \in \rho(A_{[B]})$, we obtain $f = 0$ and $\varphi = \Gamma_0 f = 0$. Thus $I - BM(\lambda)$ is injective.

Next we show the inclusion

$$\operatorname{ran}\big(B\gamma(\overline{\lambda})^*\big) \subset \operatorname{ran}\big(I - BM(\lambda)\big). \tag{4.24}$$

To this end, let $\psi \in \operatorname{ran}(B\gamma(\overline{\lambda})^*)$. Then there exists an $f \in \mathcal{H}$ such that $\psi = B\gamma(\overline{\lambda})^* f$. Set

$$\begin{aligned} g &:= (A_{[B]} - \lambda)^{-1} f - (A_0 - \lambda)^{-1} f \ \in \ker(T - \lambda), \\ k &:= (A_{[B]} - \lambda)^{-1} f \ \in \operatorname{dom} A_{[B]}. \end{aligned}$$

From

$$\begin{aligned} &\Gamma_0 g = \Gamma_0 k, \\ &\Gamma_1 g = \Gamma_1 k - \Gamma_1 (A_0 - \lambda)^{-1} f = \Gamma_1 k - \gamma(\overline{\lambda})^* f \end{aligned}$$

we conclude that

$$\big(I - BM(\lambda)\big)\Gamma_0 k = \Gamma_0 k - BM(\lambda)\Gamma_0 g = B\Gamma_1 k - B\Gamma_1 g = B\gamma(\overline{\lambda})^* f = \psi.$$

This shows the inclusion in (4.24). Now it follows in exactly the same way as in Step 3 that for $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$ the resolvent $(A_{[B]} - \lambda)^{-1}$ is given by the right-hand side of (4.17).

Step 5. Finally, assume that the second condition in (iii) is satisfied, i.e. that A_1 is self-adjoint. Then $\operatorname{ran} M(\lambda_\pm) = \operatorname{ran} \Gamma_1$ by [3, Proposition 2.6 (iii)]. Hence, if $g \in \operatorname{ran} \Gamma_1$ then (ii) implies $Bg \in \operatorname{ran} \Gamma_0$. This shows that the first condition in (iii) is satisfied, and we can apply Steps 1–4 of the proof. □

For the case when the spectrum of the self-adjoint operator A_0 does not cover the whole real line a useful variant of Theorem 4.15 is formulated below. Its proof is almost the same as the proof of Theorem 4.15; here the range condition in Step 3 of the proof needs only to be verified for some real point in $\rho(A_0)$, which then automatically belongs to $\rho(A_{[B]})$.

Theorem 4.16. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M. Let $B = B^* \in \mathcal{L}(\mathcal{G})$ and assume that there exists a $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ such that the following conditions are satisfied:*

(i) $1 \in \rho(B\overline{M(\lambda_0)})$;

(ii) $B\big(\operatorname{ran} \overline{M(\lambda_0)}\big) \subset \operatorname{ran} \Gamma_0$;

(iii) $B(\operatorname{ran} \Gamma_1) \subset \operatorname{ran} \Gamma_0$ *or* $\lambda_0 \in \rho(A_1)$.

Then the operator

$$A_{[B]} f = Tf, \qquad \operatorname{dom} A_{[B]} = \big\{ f \in \operatorname{dom} T : \Gamma_0 f = B\Gamma_1 f \big\}, \tag{4.25}$$

is a self-adjoint extension of S such that $\lambda_0 \in \rho(A_{[B]})$, and

$$(A_{[B]} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\gamma(\overline{\lambda})^* \tag{4.26}$$

holds for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$.

4.3 Laplace operators with Robin boundary conditions

In this section we apply the technique of quasi boundary triples and their Weyl functions to boundary value problems involving the Laplacian and the corresponding selfadjoint Laplace operators with Robin boundary conditions; cf. Section 3.3. Let us again assume that Ω is a bounded C^2-domain and consider the operators

$$\begin{aligned} Sf &= -\Delta f, \qquad \operatorname{dom} S = H_0^2(\Omega), \\ Tf &= -\Delta f, \qquad \operatorname{dom} T = H^2(\Omega); \end{aligned}$$

cf. Examples 4.10 and 4.11. In the following we shall use the quasi boundary triple $\{L^2(\partial\Omega), \tau_N, \tau_D\}$ from Example 4.11 with the corresponding γ-field and Weyl function M discussed in Example 4.14. In this situation the selfadjoint reference extension $A_0 = T \upharpoonright \ker \Gamma_0$ is given by the Neumann operator

$$A_N f = -\Delta f, \qquad \operatorname{dom} A_N = \big\{ f \in H^2(\Omega) : \tau_N f = 0 \big\}, \tag{4.27}$$

and the Weyl function is a Neumann-to-Dirichlet map. As a consequence of the main theorems in the previous section we obtain the result below. We mention that the conditions (ii) and (iii) in Theorem 4.15 can now be interpreted as regularity assumptions on the parameter in the boundary condition; here we assume for simplicity C^2-smoothness of the multiplication operator on $\partial\Omega$.

Theorem 4.17. *Consider the quasi boundary triple $\{L^2(\partial\Omega), \tau_N, \tau_D\}$ for $T \subset S^*$ with corresponding γ-field γ and Weyl function M. Let $\beta \in C^2(\partial\Omega)$ be a real function. Then the Robin realization of the Laplacian,*

$$A_\beta f = -\Delta f, \qquad \operatorname{dom} A_\beta = \big\{ f \in H^2(\Omega) : \tau_N f = \beta \tau_D f \big\}, \tag{4.28}$$

is a self-adjoint extension of S and the resolvent formula

$$(A_\beta - \lambda)^{-1} = (A_N - \lambda)^{-1} + \gamma(\lambda)\big(I - \beta M(\lambda)\big)^{-1} \beta \gamma(\overline{\lambda})^* \tag{4.29}$$

is valid for all $\lambda \in \rho(A_\beta) \cap \rho(A_N)$. Furthermore, the following variant of the Birman-Schwinger principle holds: $\lambda \in \rho(A_N)$ is an eigenvalue of A_β if and only if $\ker(I - \beta M(\lambda)) \neq \{0\}$.

Proof. Recall from Example 4.14 that the Weyl function M corresponding to the quasi boundary triple $\{L^2(\partial\Omega), \tau_N, \tau_D\}$ has the mapping property

$$M(\lambda) : L^2(\partial\Omega) \supset H^{1/2}(\partial\Omega) \to H^{3/2}(\partial\Omega) \subset L^2(\partial\Omega)$$

for $\lambda \in \rho(A_N)$. One can show that $M(\lambda)$ admits a bounded continuation in $L^2(\partial\Omega)$, which coincides with the closure in $L^2(\partial\Omega)$ and maps

$$\overline{M(\lambda)} : L^2(\partial\Omega) \to H^1(\partial\Omega) \subset L^2(\partial\Omega).$$

This is also implies that $\overline{M(\lambda)}$ is closed as an operator from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$, and hence bounded. As $\partial\Omega$ is compact a version of the Rellich embedding theorem (cf. Theorem 3.11) on $\partial\Omega$ implies that $\overline{M(\lambda)}$ is a compact operator in $L^2(\partial\Omega)$. The same remains true for the operator $\beta\overline{M(\lambda)}$, $\lambda \in \rho(A_N)$, since β as a multiplication in $L^2(\partial\Omega)$ is bounded.

We claim that

$$\ker\bigl(I - \beta\overline{M(\lambda)}\bigr) = \{0\}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.30}$$

In fact, since the closure $\overline{M(\lambda)}$ is an extension of the Dirichlet-to-Neumann map (this has to be verified) for $\varphi = \beta\overline{M(\lambda)}\varphi$ and $\varphi = \tau_N f_\lambda(\varphi)$ for some solution $f_\lambda(\varphi)$ (with $H^{3/2}(\Omega)$-regularity) of $(-\Delta - \lambda)u = 0$ one obtains

$$\tau_N f_\lambda(\varphi) = \varphi = \beta\overline{M(\lambda)}\varphi = \beta\overline{M(\lambda)}\tau_N f_\lambda(\varphi) = \beta\tau_D f_\lambda(\varphi). \tag{4.31}$$

Since $\beta \in C^2(\partial\Omega)$ it follows that $\tau_N f_\lambda(\varphi) \in H^1(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ and elliptic regularity then implies $f_\lambda(\varphi) \in H^2(\Omega)$. Together with the boudnayr condition (4.31) this shows $f_\lambda(\varphi) \in \ker(A_\beta - \lambda)$, but as β is real it is clear that A_β is a symmetric operator in $L^2(\Omega)$. Hence $\sigma_p(A_\beta) \cap \mathbb{C} \setminus \mathbb{R} = \emptyset$ by Corollary 4.4 and therefore $f_\lambda(\varphi) = 0$ and $\varphi = \tau_N f_\lambda(\varphi) = 0$. This proves (4.30) and as $\beta\overline{M(\lambda)}$ is compact now the Fredholm alternative implies that

$$\bigl(I - \beta\overline{M(\lambda)}\bigr)^{-1} \in \mathcal{L}(L^2(\partial\Omega)).$$

In other words, $1 \in \rho(\beta\overline{M(\lambda)})$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and hence condition (i) in Theorem 4.15 is satisfied. Conditions (ii) and (iii) in Theorem 4.15 in the present setting translate into

$$\beta\varphi \in H^{1/2}(\partial\Omega)$$

for all $\varphi \in H^1(\partial\Omega)$ and

$$\beta\psi \in H^{1/2}(\partial\Omega)$$

for all $\psi \in H^{3/2}(\partial\Omega)$, which are both valid by our assumption $\beta \in C^2(\partial\Omega)$. Now Theorem 4.15 implies the assertions.

The simple proof of the Birman-Schwinger principle is left to the reader. □

Remark 4.18. For $\lambda \in \rho(A_\beta)$ and $f \in L^2(\Omega)$ the function $u = (A_\beta - \lambda)^{-1} f$ is a $H^2(\Omega)$- solution of the Robin boundary value problem

$$-\Delta u - \lambda u = f \quad \text{and} \quad \tau_N u = \beta\tau_D f;$$

cf. Section 3.3. For $\beta = 0$ one arrives at the Neumann problem discussed in Section 3.2; the formal case $\frac{1}{\beta} = 0$ – which corresponds to Dirichlet boundary conditions – was excluded above, but can be treated with the quasi boundary triple $\{L^2(\partial\Omega), \tau_D, -\tau_N\}$ in Example 4.10.

Bibliography

[1] R.A. Adams and J.F. Fournier, Sobolev Spaces, Pure and Applied Mathematics 140, Elsevier, 2003.

[2] W. Arendt and K. Urban, Partielle Differenzialgleichungen, Spektrum Akademischer Verlag Heidelberg, 2010.

[3] J. Behrndt and M. Langer, *Boundary value problems for elliptic partial differential operators on bounded domains*, J. Funct. Anal., 243(2): 536–565, 2007.

[4] J. Behrndt and M. Langer, *Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples*, London Math. Soc. Lecture Note Series, 404: 121–160, 2012.

[5] J. Behrndt, M. Langer, and V. Lotoreichik, *Schrödinger operators with δ and δ'-potentials supported on hypersurfaces*, Ann. Henri Poincaré, 14(2): 385–423, 2013.

[6] D. Werner, Funktionalanalysis, Springer, 2007.