Calculus of variations

Exercise sheet 1 March 30, 2022

Exercise 1: Let X be a Banach space and $\emptyset \neq U \subseteq X$ be a closed linear subspace with $U \neq X$. Prove that there exists some $x \in X$ with $\|x\| = 1$ and

$$\|x - u\| \geq \frac{1}{2}, \qquad u \in U.$$

Hint: Choose $x_0 \in X \setminus U$ arbitrary and prove in the first step that $d := \inf_{u \in U} \|x_0 - u\| > 0$. As a consequence there exists some $u_0 \in U$ with $\|x_0 - u_0\| \leq 2d$.

Exercise 2: Let X be an infinite dimensional Banach space. Prove that the closed unit ball

$$\overline{B_1(0)} := \{ x \in X \mid \|x\| \leq 1 \} \text{ is not compact.}$$

Hint: Use Exercise 1 to construct a sequence $(x_n)_{n \in \mathbb{N}} \in \overline{B_1(0)}$ with $\|x_n - x_m\| \geq \frac{1}{2}$ for all $n \neq m$.

Exercise 3: Prove that the minimization problem

$$F(u) = \int_{-1}^{1} (xu'(x))^2 dx = \min, \qquad u \in C^1([-1,1]),\ u(-1) = -1,\ u(1) = 1,$$

has no solution.

Hint: The sequence of functions $u_n(x) = \frac{\arctan(nx)}{\arctan(n)}$ may be helpful.

Exercise 4: Prove that the minimization problem

$$F(u) := \int_0^1 u(x)^2 dx, \qquad u \in C([-1,1]),\ u(0) = 0,\ u(1) = 1,$$

has no solution.

Exercise 5: Consider the infinite dimensional Banach space

$$C_0([0,1]) = \{ u : [0,1] \to \mathbb{R} \text{ continuous} \mid u(0) = u(1) = 0 \}$$

equipped with the norm $\|u\| := \sup_{x \in [0,1]} |u(x)|$. Prove that the function $F : C_0([0,1]) \to \mathbb{R}$,

$$F(u) := \int_0^1 |u(x) - 1| dx, \qquad u \in C_0([0,1]),$$

is continuous but does not admit a minimum on the closed unit ball $\overline{B_1(0)}$.

Exercise 6: Prove the following version of the *Fundamental lemma of calculus*: Let $I \subseteq \mathbb{R}$ be an open interval. If $u \in L^1_{\text{loc}}(I)$ satisfies

$$\int_I u(x)\varphi'(x) dx = 0, \qquad \varphi \in \mathcal{C}_0^\infty(I),$$

then u is constant almost everywhere on I.

Hint: For $\eta, \psi \in \mathcal{C}_0^\infty(I)$ with $\int_I \eta(x) dx = 1$ construct $\varphi \in \mathcal{C}_0^\infty(I)$ with $\varphi'(x) = \psi(x) - \eta(x) \int_I \psi(y) dy$.

Exercise 7: Let $a, b \in \mathbb{R}$. A function $f : (a, b) \to \mathbb{C}$ is called *absolute continuous* if there exists $c \in \mathbb{C}$ and $g \in L^1(a, b)$ with

$$f(x) = c + \int_a^x g(y)dy, \qquad x \in [a, b]. \tag{1}$$

Prove the following statements:

a) Every absolute continuous function is continuius.

b) $W_1^2(a, b) = \left\{ f \in L^2(a, b) \;\middle|\; f \text{ is absolute continuous, with (1) satisfied by } c \in \mathbb{C},\, g \in L^2(a, b) \right\}$

Hints: Use Exercise 6.

Exercise 8: For some open and bounded $G \subseteq \mathbb{R}^n$, the Poincaré inequality

$$\sum_{j=1}^{n} \int_G (\partial_j u)^2 dx \geq C \int_G u^2 dx, \qquad u \in \mathring{W}_1^2(G),$$

was proven in the lecture. Show, that this inequality does not hold true for

a) $G = \mathbb{R}^n$.

b) $\mathring{W}_2^1(G)$ replaced by $W_2^1(G)$.

Exercise 9: Let X be a real Banach space, $\mathfrak{a} : X \times X \to \mathbb{R}$ bilinear, $b : X \to \mathbb{R}$ linear with

$$c_1\|u\|^2 \leq \mathfrak{a}(u, u) \leq c_2\|u\|^2, \qquad \mathfrak{a}(u, v) = \mathfrak{a}(v, u) \qquad \text{and} \qquad |b(u)| \leq d\|u\|, \qquad u, v \in X,$$

for some $c_1, c_2, d \geq 0$. Prove, that the minimization problem

$$\frac{1}{2}\mathfrak{a}(u, u) - b(u) = \min!, \qquad u \in X,$$

admits a unique solution. With the notation $F(u) := \frac{1}{2}\mathfrak{a}(u, u) - b(u)$ use the following steps

i) Show that there exists some $C > 0$ such that $\gamma := \inf_{u \in X} F(u) > -\infty$.

ii) Prove that

$$\frac{1}{4}\mathfrak{a}(u - v, u - v) = F(u) + F(v) - 2F\left(\frac{u}{2} + \frac{v}{2}\right), \qquad u, v \in X.$$

iii) Let $(u_n)_n \in X$ with $F(u_n) \to \gamma$. Then $u_0 := \lim_{n \to \infty} u_n$ exists and $F(u_0) = \gamma$.