Exercise 32: Let X be a separable reflexive Banach space over $\mathbb{R}$ and $A : X \to X'$ a monotone operator. Prove that A is locally bounded. I.e., for every $u \in X$ there exists $\delta, r > 0$, such that

$$\|A(v)\| < r, \qquad \forall v \in B_\delta(u).$$

Hint: Do a proof by contradiction and use the numbers $a_n = (1 + \|A(u_n)\| \, \|u_n - u\|)^{-1}$ and the Theorem of Banach-Steinhaus for the operators $a_n A(u_n)$.

Exercise 33: Let X be a separable reflexive Banach space over $\mathbb{R}$ and $A : X \to X'$ a monotone operator which is continuous on finite dimensional subspaces. Moreover, let $u, (u_n)_n \subseteq X$, $b \in X'$ with

$$u = \operatorname*{w-lim}_{n\to\infty} u_n \qquad \text{and} \qquad b = \lim_{n\to\infty} A(u_n).$$

Prove that $Au = b$. *Hint*: Use Lemma 4.7 in the lecture notes.

Exercise 34: Let X be a separable reflexive Banach space over $\mathbb{R}$ and $A : X \to X'$ a monotone and coercive operator which is continuous on finite dimensional subspaces. For any $b \in X'$ consider the set of solutions

$$S_b := \{\, u \in X \mid Au = b \,\}.$$

Verify the following properties:

a) $S_b \neq \emptyset$;

b) S_b is bounded;

c) S_b is closed;

d) S_b is convex.

Exercise 35: Let X be a separable reflexive Banach space over $\mathbb{R}$ and $A : X \to X'$ a strict monotone and coercive operator which is continuous on finite dimensional subspaces. Prove that:

a) A is bijective and A^{-1} is strict monotone.

b) For any $b = \lim\limits_{n\to\infty} b_n$ convergent in X', also $A^{-1}(b) = \operatorname*{w-lim}\limits_{n\to\infty} A^{-1}(b_n)$ weakly converges in X.

Exercise 36: Let $y \in \mathbb{R}^3$. Verify the solvability of the equation

$$x(1 + |x|^2)e^{|x|^2} = y, \qquad x \in \mathbb{R}^3.$$

Exercise 37: Let X be a separable reflexive Banach space over $\mathbb{R}$ and $A : X \to X'$ an operator, which is *Lipschitz continuous*, i.e., there exists some $L \geq 0$ such that

$$\|A(u) - A(v)\| \leq L\|u - v\|, \qquad u, v \in X,$$

as well as *strong monotone*, i.e., there exists some $c > 0$ such that

$$\langle A(u) - A(v), u - v\rangle \geq c\|u - v\|^2, \qquad u, v \in X.$$

Prove that A is bijective and that the inverse operator A^{-1} is Lipschitz continuous and strong monotone as well.

Exercise 38: Let $\Omega \subseteq \mathbb{R}^2$ open, bounded and connected. For the space $X = H_0^1(\Omega, \mathbb{R})$ consider the operator $A : X \to X'$ given by

$$\langle A(u), v\rangle := \int_\Omega f\big(|\nabla v(x)|\big)(\nabla u(x), \nabla v(x))dx, \qquad u, v \in X,$$

where $f : [0, \infty) \to \mathbb{R}$ is a function which satisfies

$$\begin{aligned} f(s)s - f(t)t &\geq c(s-t), && s \geq t \geq 0, \\ |f(s)s - f(t)t| &\leq L|s-t|, && s, t \geq 0, \end{aligned}$$

for some $c > 0$ and $L \geq 0$. Prove that the equation $Au = b$ is uniquely solvable for any $b \in X'$.

Exercise 39: Let $\rho : [a, b] \times \mathbb{R} \to \mathbb{R}$ be continuous, $\rho(x, \cdot)$ monotone increasing and there exists some $L \geq 0$ such that

$$|\rho(x, s) - \rho(x, t)| \leq L|s-t|, \qquad x \in [a, b],\ s, t \in \mathbb{R}.$$

Consider the space $X = H_0^1((a, b), \mathbb{R})$ and the operator $A : X \to X'$ given by

$$\langle A(u), v\rangle := \int_a^b \big(u'(x)v'(x) + \rho(x, u(x))v(x)\big)dx, \qquad u, v \in X.$$

Show that the equation $Au = b$ is uniquely solvable for any $b \in X'$.