Recall: E spectral measure and $x \in \mathcal{H}$

$\Rightarrow \Sigma \ni B \mapsto (E_B x, x)$ is Borel measure

4.6 Spectral theorem for unbounded self-adjoint operators

First, we discuss, how $f(A)$ can be constructed, if $A = A^* \in \mathcal{L}(\mathcal{H})$ and $f : \mathbb{R} \to \mathbb{C}$ is measurable, but unbounded.

$\int f\,dE$

Proposition 4.18. Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $f : \mathbb{R} \to \mathbb{C}$ be measurable. Define

$$f_n(\lambda) := f(\lambda)\mathbb{1}_{|f|\le n}(\lambda) = \begin{cases} f(\lambda), & \text{if } |f(\lambda)| \le n, \\ 0, & \text{if } |f(\lambda)| > n. \end{cases}$$

Then f is measurable and bounded for all $n \in \mathbb{N}$. Moreover, $\boxed{\lim_{n\to\infty} f_n(A)x}$ exists, if and only if $x \in D_f := \{x \in \mathcal{H} : \int_{\mathbb{R}} |f(\lambda)|^2 d(E(\lambda)x, x) < \infty\}$. In particular,

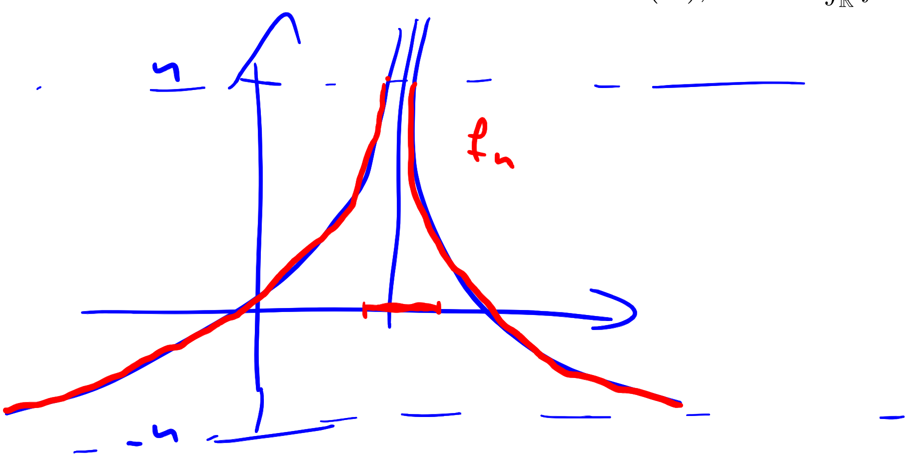
$$f(A)x := \lim_{n\to\infty} f_n(A)x, \quad x \in \operatorname{dom} f(A) := D_f,$$

is a well-defined linear operator in $\mathcal{H}$. If f is real-valued, then $f(A)$ is self adjoint.

Notation: We set

$$\int_{\mathbb{R}} f\,dEx := \lim_{n\to\infty} f_n(A)x = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\,dEx,$$

where the limit is w.r.t. the norm in $\mathcal{L}(\mathcal{H})$, so that $\int_{\mathbb{R}} f\,dEx = f(A)x$ for all $x \in D_f$.



Remark: dom A can be characterised with the help of the associated spectral measure

$$\operatorname{dom} A = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} |t|^2 \, d(E(t)x, x) < \infty \right\}$$

Theorem 4.19. (Spectral theorem for self adjoint operators) Let $A : \mathcal{H} \supset \operatorname{dom} A \to \mathcal{H}$ be self adjoint. Then there exists a spectral measure such that

$$Ax = \int_{\mathbb{R}} t dE(t)x \qquad \forall x \in \operatorname{dom} A.$$

If $h : \mathbb{R} \to \mathbb{R}$ is measurable, then

$$h(A)x := \int_{\mathbb{R}} h dEx, \qquad \boxed{\operatorname{dom} h(A) = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} |h|^2 d(Ex, x) < \infty \right\},}$$

defines a self adjoint operator in $\mathcal{H}$.

The integral in the definition of $h(A)$ has to be understood as in Proposition 4.18, so $\int_{\mathbb{R}} h dEx = \lim_{n\to\infty} \int_{\mathbb{R}} h_n dEx$.

We remark that versions of Theorem 4.16 and Theorem 4.17 hold for unbounded A as well.

Sketch of the proof of Theorem 4.19

We are going to discuss the proof under the additional assumption that there exists $\mu \in \rho(A) \cap \mathbb{R}$.

Then $\tilde{A} := (A-\mu)^{-1}$ is bounded and self adjoint.

The original operator A is related to $\tilde{A}$ by

$$\boxed{A = f(\tilde{A})} \quad \text{with} \quad f(t) := \begin{cases} \frac{1}{t} + \mu, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

(defined as in Prop. 4.18).

[As $f(\tilde{A})$ is the unique self adjoint operator satisfying $\tilde{A}(f(\tilde{A}) - \mu)x = x \quad \forall x \in \operatorname{dom} A$]

$\iff (A-\mu)^{-1}(f(\tilde{A}) - \mu)x = x$

$$\Rightarrow \sigma(\tilde{A}) = \sigma((A-\mu)^{-1}) = \left\{ \frac{1}{t-\mu} : t \in \sigma(A) \right\}$$

[$\lambda \in \rho(\tilde{A}) \iff \tilde{A} - \lambda$ is bijective $\iff (A-\mu)^{-1} - \lambda$ is bijective

$\iff (I - \lambda(A-\mu))(A-\mu)^{-1}$ is bijective $\overset{(A-\mu)^{-1} \text{ is bijective}}{\iff} I - \lambda A + \lambda\mu$ bij.

$\iff \lambda = 0$ or $A - \mu - \frac{1}{\lambda}$ is bijective $\iff \lambda = 0$ or $\mu + \frac{1}{\lambda} \in \rho(A)$]

Define now $E: \Sigma \to \mathcal{L}(\mathcal{H})$, $B \mapsto E_B := \mathbb{1}_{\tilde{B}}(\tilde{A})$

where $\tilde{B} := \left\{ \frac{1}{t-\mu} : t \in B \cap \sigma(A) \right\} = f^{-1}(B)$

Then $E_B = E^{\tilde{A}}_{\tilde{B}}$, where $E^{\tilde{A}}$ is the spectral measure associated to $\tilde{A}$.

We show that E is a spectral measure.

- Since $E_B = E^{\tilde{A}}_{\tilde{B}}$ and $E^{\tilde{A}}_{\tilde{B}}$ is an orthogonal projection, as $E^{\tilde{A}}$ is a spectral measure, E_B is an orthogonal projection for each $B \in \Sigma$.
- $E_\emptyset = E^{\tilde{A}}_{\tilde{B}} = E^{\tilde{A}}_\emptyset = 0$ (as $E^{\tilde{A}}$ is a spectral measure)

  $E_{\mathbb{R}} = E^{\tilde{A}}_{\tilde{B}} = E^{\tilde{A}}_{\sigma(\tilde{A})} = \mathbb{1}_{\sigma(\tilde{A})}(\tilde{A}) = I$ (—— " ——)
- σ-additivity: Let $B_1, B_2, \ldots \in \Sigma$ be pairwise disjoint.

  $\Rightarrow \tilde{B}_1, \tilde{B}_2, \ldots$ are pairwise disjoint

$$\Rightarrow E_{\bigcup_{k=1}^\infty B_k} x = E^{\tilde{A}}_{\bigcup_{k=1}^\infty \tilde{B}_k} x \overset{E^{\tilde{A}} \text{ spectr. measure}}{=} \sum_{k=1}^\infty E^{\tilde{A}}_{\tilde{B}_k} x = \sum_{k=1}^\infty E_{B_k} x$$

$\Rightarrow$ $\boxed{E \text{ is a spectral measure}}$

Let $h: \mathbb{R} \to \mathbb{R}$ be measurable and let $h(A)$ be defined as in the theorem via the integral w.r.t. E.

We prove: $\boxed{\int_{\mathbb{R}} h\,dE = h(A) = (h \circ f)(\tilde{A})}$ $(*)$

First, if h is a step function, $h = \sum_{k=1}^n \alpha_k \mathbb{1}_{B_k}$

$$\Rightarrow h(A) = \int_{\mathbb{R}} h\,dE = \sum_{k=1}^n \alpha_k E_{B_k} = \sum_{k=1}^n \alpha_k E^{\tilde{A}}_{f^{-1}(B_k)}$$

$$= \sum_{k=1}^n \alpha_k \underbrace{\mathbb{1}_{f^{-1}(B_k)}}_{= \mathbb{1}_{B_k} \circ f} (\tilde{A}) = \sum_{k=1}^n \alpha_k (\mathbb{1}_{B_k} \circ f)(\tilde{A})$$

$$= (h \circ f)(\tilde{A}) = \int_{\mathbb{R}} (h \circ f)\, dE^{\tilde{A}}$$

For $h \in B(\mathbb{R})$ identity $(*)$ follows then, by approximation of h by simple functions and for unbounded, but measurable $h : \mathbb{R} \to \mathbb{R}$ identity $(*)$ follows from Proposition 4.18. In a similar way one also gets

$$\operatorname{dom} h(A) = \operatorname{dom}(h \circ f)(\tilde{A}) = \Big\{ x \in \mathcal{H} : \underbrace{\int_{\mathbb{R}} |h \circ f|^2 d(E^{\tilde{A}} x, x)}_{= \int_{\mathbb{R}} |h|^2 d(E x, x)} < \infty \Big\}$$

By Proposition 4.18 $h(A)$ is self-adjoint in $\mathcal{H}$.

Finally, considering $(*)$ for $g(t) = t$:

$$g(A)x = (g \circ f)(\tilde{A})x \underset{g(t)=t}{=} f(\tilde{A})x = Ax$$

$$\| $$

$$\int_{\mathbb{R}} t \, dE(t) x$$

This finishes the proof of the theorem.

Motivation: $A = A^*$ in $\mathcal{H}$ and $V \subset V^*$

Question: Is $A+V$ self adjoint? In general: No

Can we find assumptions for A and V s.t. $A+V$ is self adjoint and what can one say about $\sigma(A+V)$

5 Perturbation theory for self adjoint operators

Throughout this section $\mathcal{H}$ is a complex Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$.

5.1 Relatively bounded perturbations

Definition 5.1. Let A and V be linear operators in $\mathcal{H}$. Then V is called *A-bounded* (or *relatively bounded with respect to A*), if $\operatorname{dom} A \subset \operatorname{dom} V$ and if there exist $a, b \geq 0$ such that

$$\|Vx\| \leq a\|x\| + b\|Ax\|$$

holds for all $x \in \operatorname{dom} A$. The infimum over all b, such that there exists an a so that the above inequality holds, is called *A-bound of V*.

$\operatorname{dom} A \subset \operatorname{dom} V$

Remark:

As: $\|Vx\| \leq \|V\| \cdot \|x\| = \|V\| \cdot \|x\| + 0 \cdot \|Ax\|$

- If $V \in \mathcal{L}(\mathcal{H})$, then V is A-bounded with A-bound zero.
- If V is a A bounded with A-bound b, then there exists for all $\varepsilon > 0$ a number $a_\varepsilon \geq 0$ such that

  $\exists a: \cancel{\|Vx\| \leq a\|x\| + b\|Ax\|}$

  $$\|Vx\| \leq a_\varepsilon\|x\| + (b+\varepsilon)\|Ax\|$$

  holds for all $x \in \operatorname{dom} A$. For $\varepsilon = 0$ this does not have to be the case!
- V is A-bounded if and only if $\operatorname{dom} A \subset \operatorname{dom} V$ and there exist $\alpha, \beta \geq 0$ such that

  $$\|Vx\|^2 \leq \alpha\|x\|^2 + \beta\|Ax\|^2$$

  holds for all $x \in \operatorname{dom} A$. The infimum over all $\sqrt{\beta}$, such that there exists an α so that the above inequality holds, coincides with the A-bound of V (see exercises).

Proposition 5.2. Let $A = A^*$ in $\mathcal{H}$ and let V be a linear operator in $\mathcal{H}$ such that $\operatorname{dom} A \subset \operatorname{dom} V$. Set

$$c_\pm := \limsup_{\eta \to \pm\infty} \left\|V(A - i\eta)^{-1}\right\|$$

with $c_\pm = \infty$, if $V(A-i\eta)^{-1}$ is unbounded. Then

$$V \text{ is } A\text{-bounded} \quad \Leftrightarrow \quad c_+ < \infty \quad \Leftrightarrow \quad c_- < \infty.$$

In this case one has $c_+ = c_-$ is the A-bound of V and the limit superior is a limit.