Motivation: $A = A^*$ in $\mathcal{H}$ and $V \subset V^*$

Question: Is $A+V$ self adjoint? In general: No

Can we find assumptions for A and V s.t. $A+V$ is self adjoint and what can one say about $\sigma(A+V)$

5 Perturbation theory for self adjoint operators

Throughout this section $\mathcal{H}$ is a complex Hilbert space with inner product $(\cdot, \cdot)$ and induced norm $\|\cdot\|$.

5.1 Relatively bounded perturbations

Definition 5.1. Let A and V be linear operators in $\mathcal{H}$. Then V is called *A-bounded* (or *relatively bounded with respect to A*), if $\operatorname{dom} A \subset \operatorname{dom} V$ and if there exist $a, b \geq 0$ such that

$$\|Vx\| \leq a\|x\| + b\|Ax\|$$

holds for all $x \in \operatorname{dom} A$. The infimum over all b, such that there exists an a so that the above inequality holds, is called *A-bound of V*.

Remark:

- If $V \in \mathcal{L}(\mathcal{H})$, then V is A bounded with A-bound zero.
- If V is a bounded with A-bound b, then there exists for all $\varepsilon > 0$ a number $a_\varepsilon \geq 0$ such that
$$\|Vx\| \leq a_\varepsilon\|x\| + (b+\varepsilon)\|Ax\|$$
holds for all $x \in \operatorname{dom} A$. For $\varepsilon = 0$ this does not have to be the case!
- V is A-bounded if and only if $\operatorname{dom} A \subset \operatorname{dom} V$ and there exist $\alpha, \beta \geq 0$ such that
$$\|Vx\|^2 \leq \alpha\|x\|^2 + \beta\|Ax\|^2$$
holds for all $x \in \operatorname{dom} A$. The infimum over all $\sqrt{\beta}$, such that there exists an α so that the above inequality holds, coincides with the A-bound of V (see exercises).

Proposition 5.2. Let $\boxed{A = A^* \text{ in } \mathcal{H}}$ and let V be a linear operator in $\mathcal{H}$ such that $\boxed{\operatorname{dom} A \subset \operatorname{dom} V}$. Set

$$\boxed{c_\pm := \limsup_{\eta \to \pm\infty} \|V(A - i\eta)^{-1}\|}$$

with $c_\pm = \infty$, if $V(A - i\eta)^{-1}$ is unbounded. Then

$$V \text{ is } A\text{-bounded} \quad \Leftrightarrow \quad c_+ < \infty \quad \Leftrightarrow \quad c_- < \infty.$$

In this case one has $c_+ = c_-$ is the A-bound of V and the limit superior is a limit.

Remark: If A is an operator in $\mathcal{H}$, then the graph norm of A is: $\|x\|_A := \|x\| + \|Ax\|$. Then V is A-bounded iff. $\operatorname{dom} A \subset \operatorname{dom} V$ and $V \restriction \operatorname{dom} A : (\operatorname{dom} A, \|\cdot\|_A) \to \mathcal{H}$ is bounded.

Proof of Prop 5.2:

Step 1: "V A-bounded $\Rightarrow c_\pm < \infty$"

Assume that V is A-bounded, i.e. $\exists a,b \geq 0$ s.t.

$(*)$ $\|Vx\|^2 \leq a^2\|x\|^2 + b^2\|Ax\|^2$ $\quad \forall x \in \operatorname{dom} A$.

Then one has for any $\alpha > 0$ and $x \in \mathcal{H}$:

$$\|V(A\pm i\alpha)^{-1}x\|^2 \leq a^2\|(A\pm i\alpha)^{-1}x\|^2 + b^2\|A(A\pm i\alpha)^{-1}x\|^2$$

$$= b^2\left[\frac{a^2}{b^2}\|(A\pm i\alpha)^{-1}x\|^2 + \|A(A\pm i\alpha)^{-1}x\|^2\right]$$

$$= b^2\left\|\left(A\pm i\frac{a}{b}\right)(A\pm i\alpha)^{-1}x\right\|^2$$

Write $\|\cdot\|^2 = (\cdot,\cdot)$ and use linearity of scalar product and $A = A^*$

Choose $\alpha = \frac{a}{b}$ $\Rightarrow$ $\|V(A\pm i\frac{a}{b})^{-1}x\| \leq b\|x\|$

$\Rightarrow V(A\pm i\frac{a}{b})^{-1}$ is bounded, $\|V(A\pm i\frac{a}{b})^{-1}\| \leq b$

$\Rightarrow c_\pm = \limsup_{a\to\infty} \|V(A\pm i\frac{a}{b})^{-1}\| \leq \boxed{b}$

Considering the infimum over all b s.t. $(*)$ holds, we find that $c_\pm \leq$ A-bound of V $\quad (**)$

Step 2: Note first: if $b, \alpha > 0$ with $\|V(A\pm i\alpha)^{-1}\| \leq b$, then

$(\triangle)$ $\|Vx\| = \|V(A-i\alpha)^{-1}(A-i\alpha)x\| \leq b\|(A-i\alpha)x\| \leq b\alpha\cdot\|x\| + b\|Ax\|$

for all $x \in \operatorname{dom} A$.

So assume $c_+ < \infty$. Then for all $b > c_+$ one has $\|V(A-i\alpha)^{-1}\| \leq b$ for all sufficiently large α.

Then by $(\triangle)$ V is A-bounded with A-bound $\leq b$.

$\Rightarrow$ A-bound of $V \leq c_+$

On the other hand, by $(**)$ $c_+ \leq$ A-bound of V

$\Rightarrow c_+ =$ A-bound of V

Next, we show that the limsup in the def. of c_+ is a limit. For this, choose $b > \liminf\limits_{y\to\infty} \|V(A-iy)^{-1}\|$

$\Rightarrow \exists\, \alpha > 0 : \|V(A-i\alpha)^{-1}\| \leq b$

$\overset{(\triangleright)}{\Rightarrow}$ V is A-bounded with A-bound $\leq b$

$\Rightarrow c_+ = A$-bound of $V \leq b$

$\Rightarrow c_+ \leq \liminf\limits_{y\to\infty} \|V(A-iy)^{-1}\| \leq \underbrace{\limsup\limits_{y\to\infty} \|V(A-iy)^{-1}\|}_{= c_+}$

$\Rightarrow c_+ = \liminf\limits_{y\to\infty} \|V(A-iy)^{-1}\|$

$\Rightarrow \|V(A-iy)^{-1}\|$ is converging for $y\to\infty$

With similar arguments one gets the statements for c_- $\square$

Lemma 5.3. Let $A = A^*$ in $\mathcal{H}$ and let V be a linear operator in $\mathcal{H}$ with $\operatorname{dom} A \subset \operatorname{dom} V$ such that $A + V$ is closed. If $\|V(A-\lambda)^{-1}\| < 1$ for some $\lambda \in \rho(A)$, then $\lambda \in \rho(A+V)$.

Proof:

We proof that $A+V-\lambda$ is bijective.

Surjectivity: Define

$$B_k := \sum_{n=0}^{k} (-1)^n \underline{(A-\lambda)^{-1}} \left(\underline{V(A-\lambda)^{-1}}\right)^n \in \mathcal{L}(\mathcal{H})$$

For $\ell > k$

$$\underline{\|B_\ell - B_k\|} = \left\| \sum_{n=k+1}^{\ell} (-1)^n (A-\lambda)^{-1} \left(V(A-\lambda)^{-1}\right)^n \right\|$$

$$\leq \sum_{n=k+1}^{\ell} \left\| (-1)^n (A-\lambda)^{-1} \left(V(A-\lambda)^{-1}\right)^n \right\|$$

$$\leq \sum_{n=k+1}^{\ell} \|(A-\lambda)^{-1}\| \cdot \|V(A-\lambda)^{-1}\|^n$$

$$= \|(A-\lambda)^{-1}\| \sum_{n=0}^{\ell-(k+1)} \|V(A-\lambda)^{-1}\|^{n+k+1}$$

$$\leq \|(A-\lambda)^{-1}\| \cdot \|V(A-\lambda)^{-1}\|^{k+1} \underbrace{\sum_{n=0}^{\infty} \underbrace{\|V(A-\lambda)^{-1}\|}_{<1}{}^n}_{C}$$

$$\leq C \|(A-\lambda)^{-1}\| \cdot \underbrace{\|V(A-\lambda)^{-1}\|}_{<1}{}^{k+1} \xrightarrow{k\to\infty} 0$$

$\Rightarrow$ (B_k) is a Cauchy sequence in $\mathcal{L}(\mathcal{H})$

$\Rightarrow$ $B_k \xrightarrow{k\to\infty} B$.

We show: $(A+V-\lambda)B = I$

$$(A+V-\lambda)B_k = \sum_{n=0}^{k} \underbrace{(A+V-\lambda)}_{=(A-\lambda)+\underbrace{V}_{-(-1)^{n+1}}} (-1)^n (A-\lambda)^{-1} (V(A-\lambda)^{-1})^n$$

$$= \underline{\sum_{n=0}^{k} (-1)^n (V(A-\lambda)^{-1})^n} + \underbrace{\sum_{n=0}^{k} (-1)^n (V(A-\lambda)^{-1})^{n+1}}_{= -\sum_{n=1}^{k+1} (-1)^n (V(A-\lambda)^{-1})^n}$$

$$= I - (-1)^{k+1} (V(A-\lambda)^{-1})^{k+1} \xrightarrow{k\to\infty} I \text{ in } \mathcal{L}(\mathcal{H})$$

This means: for any fixed $x \in \mathcal{H}$:

$$\left.\begin{aligned} &(B_k x) \in \operatorname{dom}(A+V-\lambda) \\ &B_k x \to Bx, \quad k\to\infty \\ &(A+V-\lambda)B_k x \to x, \quad k\to\infty \end{aligned}\right\} \quad \begin{aligned} &Bx \in \operatorname{dom}(A+V-\lambda) \\ &\Rightarrow \underline{(A+V-\lambda)Bx = x} \end{aligned}$$

[since $A+V$ is closed by assumption]

$\Rightarrow x \in \operatorname{ran}(A+V-\lambda) \quad \forall x \in \mathcal{H}$

$\Rightarrow A+V-\lambda$ is surjective ✓

- $\underline{A+V-\lambda \text{ is injective}}$. Let $0 \neq x \in \operatorname{dom} A$ $\quad \overbrace{\|(A-\lambda)x\| > 0}^{>0,\ \lambda\in\rho(A)}$

$$\|(A+V-\lambda)x\| \geq \|(A-\lambda)x\| - \|Vx\| \geq (\underbrace{1-\|V(A-\lambda)^{-1}\|}_{>0,\ \text{as } \|V(A-\lambda)^{-1}\|<1}) \|(A-\lambda)x\| > 0$$

$$\|V(A-\lambda)^{-1}(A-\lambda)x\| \leq \|(A-\lambda)x\| \cdot \|V(A-\lambda)^{-1}\|$$

$\Rightarrow A+V-\lambda$ is injective

$\Rightarrow A+V-\lambda$ is bijective $\Rightarrow \lambda \in \rho(A+V)$ $\quad \square$

Theorem 5.4 (Kato-Rellich). Let A be a linear operator in $\mathcal{H}$ and let V be a symmetric operator in $\mathcal{H}$ that is A-bounded with A-bound less than one. Then, the following is true:

(i) If $A = A^*$, then $(A+V)^* = A+V$, i.e. $A+V$ is self adjoint.

(ii) If $\overline{A} = A^*$, then $(A+V)^* = \overline{A+V}$, i.e. $A+V$ is essentially self adjoint.

Proof:

(i) Since V is A-bounded with A-bound <1, there exist $a \geq 0$ and $b \in (0,1)$ s.t.

$$\|Vx\| \leq a\|x\| + b\|Ax\| \quad \forall x \in \operatorname{dom} A.$$

We show first that $A+V$ is closed. For that, recall that $A+V$ is closed if and only if $(\underbrace{\operatorname{dom} A}_{\operatorname{dom}(A+V)}, \|\cdot\|_{A+V})$ is complete. For that we show that $\|\cdot\|_{A+V}$ is equivalent $\|\cdot\|_A$. For $x \in \operatorname{dom} A$

$$\|Ax\| \leq \|(A+V)x\| + \|Vx\| \leq \|(A+V)x\| + a\|x\| + b\|Ax\|$$

$$\Rightarrow \underbrace{(1-b)}_{>0}\|Ax\| \leq \|(A+V)x\| + a\|x\|$$

$$\Rightarrow \|x\|_A \leq c_1 \|x\|_{A+V} \qquad (1)$$

$$\|(A+V)x\| \leq \|Ax\| + \|Vx\| \leq \|Ax\| + a\|x\| + b\|Ax\| \leq c_2 \cdot \|x\|_A$$

$$\Rightarrow \|x\|_{A+V} \leq c_3 \|x\|_A \overset{(1)}{\Longrightarrow} \|\cdot\|_A \text{ and } \|\cdot\|_{A+V} \text{ are equivalent.}$$

$\Rightarrow A+V$ is closed ✓

Clearly, $A+V$ is symmetric, as for $x \in \operatorname{dom} A \subset \operatorname{dom} V$:

$$((A+V)x,x) = \underbrace{(Ax,x)}_{\in\mathbb{R}} + \underbrace{(Vx,x)}_{\in\mathbb{R}} \in \mathbb{R} \quad \checkmark$$

In order to show that $A+V$ is self adjoint, we use Theorem 3.10 and show that there are $\lambda_\pm \in \mathbb{C}_\pm \cap \rho(A+V)$.

Indeed, by Prop. 5.2. there exist $y>0$ s.t.

$\|V(A\mp iy)^{-1}\| \leq b < 1$. Then by Lemma 5.3 $\pm iy \in \rho(A+V)$

$\Rightarrow$ $A+V$ is self adjoint $\checkmark$

(ii) V is symmetric by assumption. We show that $\overline{V}$ is $\overline{A}$-bounded with the same bound (i.e. the A-bound of V)

Let $x \in \operatorname{dom} \overline{A}$. Then $\exists (x_n) \subset \operatorname{dom} A$, s.t. $x_n \to x$, $Ax_n \to \overline{A}x$.

$$\Rightarrow \|V(x_n-x_m)\| \leq a\|x_n-x_m\| + b\|A(x_n-x_m)\| \xrightarrow{m,n\to\infty} 0 \qquad (\square)$$

$\Rightarrow (Vx_n)$ is a Cauchy sequence $\Rightarrow x \in \operatorname{dom}\overline{V}$

$\Rightarrow \operatorname{dom}\overline{A} \subset \operatorname{dom}\overline{V}$. As above in $(\square)$ one gets

$$\|\overline{V}x\| \leq a\|x\| + b\|\overline{A}x\|$$

$\Rightarrow$ $\overline{V}$ is $\overline{A}$-bounded with $\overline{A}$-bound <1

By (i) we get that $\overline{A}+\overline{V}$ is self adjoint.

It remains to show that $\overline{A}+\overline{V} = \overline{A+V}$.

$\overline{A+V} \subset \overline{A}+\overline{V}$ clear $\checkmark$

Conversely, for $x \in \operatorname{dom}(\overline{A}+\overline{V}) = \operatorname{dom}\overline{A}$, there exists a sequence $(x_n) \subset \operatorname{dom} A$ s.t. $(A+V)x_n \to (\overline{A}+\overline{V})x$

(consequence of $(\square)$) $\Rightarrow x \in \operatorname{dom}(\overline{A+V})$ and $(A+V)x_n \to (\overline{A+V})x$

$\Rightarrow \operatorname{dom}(\overline{A}+\overline{V}) \subset \operatorname{dom}(\overline{A+V})$ and $(\overline{A+V})x = (\overline{A}+\overline{V})x \quad \forall x \in \operatorname{dom}\overline{A}$

$\Rightarrow \overline{A} + \overline{V} \subset (\overline{A+V}) \Rightarrow \overline{A} + \overline{V} = \overline{A+V}$ $\square$

Theorem 5.5 (Wüst)**.** Let $\overline{A} = A^*$ in $\mathcal{H}$ and let V be a symmetric operator in $\mathcal{H}$ such that $\operatorname{dom} A \subset \operatorname{dom} V$. If there exists an $a \geq 0$ such that $\|Vx\| \leq a\|x\| + \|Ax\|$ holds for all $x \in \operatorname{dom} A$, then $A + V$ is essentially self adjoint.

Caution: The condition in Wüst's theorem is not equivalent to

"V is A-bounded with A-bound 1."

Under the last condition, the statement of the theorem is not true in general!

Definition 5.6. A self adjoint operator A in $\mathcal{H}$ is called *semibounded from below*, if there exists a $\gamma \in \mathbb{R}$ such that

$$(Ax, x) \geq \gamma\|x\|^2$$

holds for all $x \in \operatorname{dom} A$. Each such γ is called *lower bound of* A and we write $A \geq \gamma$ in this case.

Lemma 5.7. Let A be a self adjoint operator in $\mathcal{H}$. Then $A \geq \gamma$ if and only if $\sigma(A) \subset [\gamma, \infty)$.