Before Christmas: $A = A^*$ ... unperturbed operator
V ... perturbation
Can we find assumptions on V s.t. certain properties of A are carried over to $A+V$ $\rightsquigarrow$ relative boundedness
Goal now: Find assumptions for perturbation s.t. certain parts of spectrum are stable.

5.2 Compact and finite dimensional perturbations

Definition 5.9. Let $\boxed{A = A^* \text{ in } \mathcal{H}}$. The *discrete spectrum* of A is defined by

$$\sigma_d(A) := \{\lambda \in \sigma_p(A) : \dim\ker(A - \lambda) < \infty \text{ and } \exists \varepsilon > 0 : (\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(A) = \{\lambda\}\}.$$

The *essential spectrum* of A is

$$\boxed{\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A).}$$

$\Rightarrow \sigma(A) = \sigma_d(A) \cup \sigma_{ess}(A)$

The discrete spectrum of A consists of all isolated eigenvalues with finite multiplicity and the essential spectrum of all eigenvalues with infinite multiplicity and all accumulation points of $\sigma(A)$. In particular, we have $\sigma_c(A) \subset \sigma_{ess}(A)$.

In the following we characterize points in the essential spectrum. For that we repeat two facts from basic functional analysis:

(i) A sequence $(x_n) \subset \mathcal{H}$ is called *weakly convergent* to $x \in \mathcal{H}$ (notation: $x_n \rightharpoonup x$), if for all $y \in \mathcal{H}$ the relation $(x_n, y) \to (x, y)$ holds for $n \to \infty$. E.g. by the Bessel inequality each infinite orthonormal system converges weakly to zero.

(e_n) ONS $\Rightarrow \sum_n |(x, e_n)|^2 \le \|x\|^2 \quad \forall x \in \mathcal{H}$

(ii) An operator $K \in \mathcal{L}(\mathcal{H})$ is called *compact* (notation $K \in \mathfrak{S}_\infty$), if it maps bounded sets onto relativly compact sets. This is equivalent to the fact that for any bounded sequence $(x_n) \subset \mathcal{H}$ there exists a subsequence (x_{n_k}) such that (Kx_{n_k}) is convergent in $\mathcal{H}$. Another equivalent condition is that $x_n \rightharpoonup x$ implies $Kx_n \to Kx$ in $\mathcal{H}$.

Recall that any operator with $\dim \operatorname{ran} K < \infty$ is compact. Moreover, if $K \in \mathfrak{S}_\infty$ and $A \in \mathcal{L}(\mathcal{H})$, then $AK \in \mathfrak{S}_\infty$ and $KA \in \mathfrak{S}_\infty$.

E.g. $Kx = \sum_{k=1}^{N} (x, e_k) f_k$

Proposition 5.10. Let $A = A^*$ in $\mathcal{H}$ and let $\lambda \in \mathbb{R}$. Then the following is equivalent:

(i) $\lambda \in \sigma_{ess}(A)$;

(ii) $\exists (x_n) \subset \operatorname{dom} A$ with $\|x_n\| = 1$, $x_n \rightharpoonup 0$ and $(A - \lambda)x_n \to 0$ (such a sequence (x_n) is called singular sequence);

(iii) $\dim \operatorname{ran} E_{(\lambda-\varepsilon,\lambda+\varepsilon)} = \infty$ for all $\varepsilon > 0$.

Proof. (i) $\Leftrightarrow$ (iii) exercises

"(i) $\Rightarrow$ (ii)"

If λ is an infinite eigenvalue of A, then $\dim\ker(A-\lambda) = \infty$. We can choose an infinite ONS (x_n) in $\ker(A-\lambda)$. This sequence (x_n) has all desired properties.

If λ is an accumulation point in $\sigma(A)$, then there exist $\lambda_n \in \sigma(A)$, $\lambda_n \ne \lambda_m$ for $n \ne m$, s.t. $\lambda_n \to \lambda$.

Choose $\varepsilon_n > 0$ s.t. $(\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n) \cap (\lambda_m - \varepsilon_m, \lambda_m + \varepsilon_m) = \emptyset$ for $n \ne m$.

Since $\lambda_n \in \sigma(A)$ we have $E_{(\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n)} \neq 0$ and hence $\exists\ x_n \in \mathcal{H}$: $\|x_n\|=1$ and $E_{(\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n)}\, x_n = x_n$.

Note that $x_n \perp x_m$ for $n\neq m$, as

$$(x_n,x_m) = \Big(\underbrace{E_{(\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n)}}_{\mathbb{1}_{(\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n)}(A)} x_n,\ E_{(\lambda_m-\varepsilon_m,\lambda_m+\varepsilon_m)}\, x_m\Big)$$

$$= \Big(\underbrace{\mathbb{1}_{(\lambda_m-\varepsilon_m,\lambda_m+\varepsilon_m)}(A)\, \mathbb{1}_{(\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n)}}_{\mathbb{1}_{(\lambda_m-\varepsilon_m,\lambda_m+\varepsilon_m)\cap(\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n)}(A)=0}(A)\, x_n, x_m\Big) = 0$$

$\Rightarrow (x_n)$ is an ONS $\Rightarrow x_n \rightharpoonup 0$

Next, consider

$$\int_{\mathbb{R}} |t|^2\, d(E(t)x_n,x_n) \overset{x_n \in \operatorname{ran} E_{(\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n)} \perp \operatorname{ran} E_{\mathbb{R}\setminus(\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n)}}{=} \int_{\lambda_n-\varepsilon_n}^{\lambda_n+\varepsilon_n} \underbrace{|t|^2}_{\leq C} d(E(t)x_n,x_n) < \infty .$$

$\Rightarrow x_n \in \operatorname{dom} A$ by the spectral theorem. Finally,

$$\|(A-\lambda)x_n\|^2 = \int_{\mathbb{R}} |t-\lambda|^2\, d(E(t)x_n,x_n) = \int_{\lambda_n-\varepsilon_n}^{\lambda_n+\varepsilon_n} |t-\lambda|^2\, d(E(t)x_n,x_n)$$

$$\leq \int_{\lambda_n-\varepsilon_n}^{\lambda_n+\varepsilon_n} \big(\underbrace{|t-\lambda_n|}_{\leq \varepsilon_n} + |\lambda_n-\lambda|\big)^2\, d(E(t)x_n,x_n)$$

$$\leq (\varepsilon_n + |\lambda_n-\lambda|)^2 \underbrace{\int_{\lambda_n-\varepsilon_n}^{\lambda_n+\varepsilon_n} 1\, d(E(t)x_n,x_n)}_{=\|x_n\|^2 = 1} \xrightarrow{n\to\infty} 0$$

$\Rightarrow (x_n)$ is a singular sequence as in (ii)

(ii) $\Rightarrow$ (iii)

Assume that $\exists (x_n) \subset \operatorname{dom} A$ s.t. $\|x_n\| = 1 \ \forall n$, $x_n \rightharpoonup 0$, $\|(A-\lambda)x_n\| \to 0$ for $n \to 0$ and assume that (iii) is not true, i.e. $\exists \varepsilon > 0$

$$\dim \operatorname{ran} E_{(\lambda-\varepsilon, \lambda+\varepsilon)} < \infty.$$

$\Rightarrow E_{(\lambda-\varepsilon,\lambda+\varepsilon)}$ is compact

$\Rightarrow E_{(\lambda-\varepsilon,\lambda+\varepsilon)} x_n \to 0$ in $\mathcal{H}$.

Consider

$$\underbrace{\|(A-\lambda)x_n\|^2}_{\to 0} = \int_{\mathbb{R}} |t-\lambda|^2 \, d(E(t)x_n, x_n)$$

$$\geq \int_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon)} \underbrace{|t-\lambda|^2}_{\geq \varepsilon^2} \, d(E(t)x_n, x_n)$$

$$\geq \varepsilon^2 \int_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon)} d(E(t)x_n, x_n)$$

$$= \varepsilon^2 \Big[\int_{\mathbb{R}} d(E(t)x_n, x_n) - \underbrace{\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} d(E(t)x_n, x_n)}_{= \int_{\mathbb{R}} \mathbb{1}_{(\lambda-\varepsilon,\lambda+\varepsilon)}(t) \, d(E(t)x_n, x_n)} \Big]$$

$$= \varepsilon^2 \Big[\underbrace{\|x_n\|^2}_{=1} - \underbrace{\|\overbrace{\mathbb{1}_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) x_n}^{E_{(\lambda-\varepsilon,\lambda+\varepsilon)}x_n}\|^2}_{\xrightarrow{n\to\infty} 0} \Big] \xrightarrow{n\to\infty} \varepsilon^2 > 0 \quad \lightning$$

$\Rightarrow \dim \operatorname{ran} E_{(\lambda-\varepsilon,\lambda+\varepsilon)} = \infty \quad \forall \varepsilon > 0$ ✓

$\square$

Lemma 5.11. Let $A = A^*$ in $\mathcal{H}$ and $\mu \in \rho(A)$. Then one has for $\lambda \neq \mu$ that $\lambda \in \sigma_{ess}(A)$ if and only if there exists a sequence $(x_n) \subset \mathcal{H}$ with $\|x_n\| = 1$, $x_n \rightharpoonup 0$ and

$$((A - \mu)^{-1} - (\lambda - \mu)^{-1})x_n \to 0.$$

Proof:

"$\Rightarrow$" Let $\lambda \in \sigma_{ess}(A)$. As in the proof of prop. 5.10 there exists a sequence (x_n) with $x_n \in \operatorname{ran} E_{(\lambda-\frac{1}{n},\lambda+\frac{1}{n})}$, $\|x_n\| = 1$, $x_n \perp x_m$ for $n \neq m$ and $x_n \rightharpoonup 0$, $n \to \infty$.

Set $c := \inf\{|t-\mu|\cdot|\lambda-\mu| : t \in \sigma(A)\} > 0$

Then

$$\left\|\left((A-\mu)^{-1} - \frac{1}{\lambda-\mu}\right)x_n\right\|^2 = \int_{\mathbb{R}} \left|\frac{1}{t-\mu} - \frac{1}{\lambda-\mu}\right|^2 d(E(t)x_n, x_n)$$

$$= \int_{\lambda-\frac{1}{n}}^{\lambda+\frac{1}{n}} \underbrace{\left|\frac{\overbrace{\lambda - t}^{|\cdot| < \frac{1}{n}}}{\underbrace{(t-\mu)(\lambda-\mu)}_{\geq c}}\right|^2}_{\leq \frac{1}{c^2 n^2}} d(E(t)x_n, x_n) \leq \frac{1}{c^2 n^2} \overbrace{\|x_n\|^2}^{=1} \xrightarrow[n\to\infty]{} 0 \quad \checkmark$$

"$\Leftarrow$" Assume that $\lambda \notin \sigma_{ess}(A)$. Then there exists $\varepsilon > 0$ s.t. $\dim \operatorname{ran} E_{(\lambda-\varepsilon,\lambda+\varepsilon)} < \infty$ [Prop. 5.10 (iii)]

$\Rightarrow E_{(\lambda-\varepsilon,\lambda+\varepsilon)}$ is compact.

Let (x_n) be an arbitrary sequence with $\|x_n\| = 1$ and $x_n \rightharpoonup 0$.

$\Rightarrow E_{(\lambda-\varepsilon,\lambda+\varepsilon)} x_n \to 0$ for $n \to \infty$

$$\left\| \left((A-\mu)^{-1} - \frac{1}{\lambda-\mu} \right) x_n \right\|^2$$

$$= \int_{\mathbb{R}} \left| \frac{1}{t-\mu} - \frac{1}{\lambda-\mu} \right|^2 d(E(t)x_n, x_n)$$

$$\geq \int_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon)} \Big| \underbrace{\frac{1}{t-\mu} - \frac{1}{\lambda-\mu}}_{\frac{\lambda-t}{(t-\mu)(\lambda-\mu)}} \Big|^2 d(E(t)x_n, x_n)$$

$$\geq \underbrace{\inf\left\{ \left| \frac{\lambda-t}{(t-\mu)(\lambda-\mu)} \right|^2 : t \in \mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon) \cap \sigma(A) \right\}}_{>0} \cdot \underbrace{\int_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon)} d(E(t)x_n, x_n)}_{\|x_n\|^2 - \|E_{(\lambda-\varepsilon,\lambda+\varepsilon)} x_n\|^2 \xrightarrow{n\to\infty} 1}$$

$$\geq c > 0 \qquad \square$$

Theorem 5.12 (Stability of the essential spectrum under compact perturbations)**.** Let $A = A^*$ and $B = B^*$ in $\mathcal{H}$. If

$$\boxed{(A-\mu)^{-1} - (B-\mu)^{-1} \in \mathfrak{S}_\infty}$$

holds for one (and hence for all) $\mu \in \rho(A) \cap \rho(B)$, then $\sigma_{ess}(A) = \sigma_{ess}(B)$.

Remark 5.13. In the above theorem B is the perturbed operator (in Section 5.1 $B = H + V$). Under our assumptions, one can not find an answer to the question, if $\boxed{V := B - A}$ is compact (or the restriction of a compact operator), as $\mathrm{dom}(B - A) = \mathrm{dom}\, A \cap \mathrm{dom}\, B$ can be an arbitrarily small set for unbounded operators A and B. Hence, one investigates the bounded operators $(B-\mu)^{-1}$ and $(A-\mu)^{-1}$ instead.

Proof:

It suffices to show $\sigma_{ess}(A) \subset \sigma_{ess}(B)$

Let μ be as in the theorem and let $\lambda \in \sigma_{ess}(A)$.

Then, by Lemma 5.11. there exists a sequence (x_n) with $\|x_n\| = 1$, $x \rightharpoonup 0$, $\left((A-\mu)^{-1} - \frac{1}{\lambda-\mu}\right) x_n \to 0$.

Then we have

$$\left((B-\mu)^{-1} - \frac{1}{\lambda-\mu}\right) x_n$$

$$= \underbrace{\left(\underbrace{(B-\mu)^{-1} - (A-\mu)^{-1}}_{\in \mathfrak{S}_\infty}\right) x_n}_{\to 0} + \underbrace{\left((A-\mu)^{-1} - \frac{1}{\lambda-\mu}\right) x_n}_{\to 0} \xrightarrow{n\to\infty} 0$$

Hence, by Lemma 5.11 $\lambda \in \sigma_{ess}(B)$ ✓

The statement that $(A-\mu)^{-1} - (B-\mu)^{-1} \in \mathfrak{S}_\infty$ for one $\mu \in \rho(A) \cap \rho(B)$ implies that $(A-\tilde{\mu})^{-1} - (B-\tilde{\mu})^{-1} \in \mathfrak{S}_\infty$ for all $\tilde{\mu} \in \rho(A) \cap \rho(B)$ is shown in the exercises. □

Theorem 5.14 (without proof)**.** Let $A = A^*$ and $B = B^*$ in $\mathcal{H}$, denote the corresponding spectral measures by E^A and E^B, respectively, and assume

$$\dim \operatorname{ran} \big((A-\mu)^{-1} - (B-\mu)^{-1}\big) = n < \infty$$

for one (and hence for all) $\mu \in \rho(A)\cap\rho(B)$. Let (α, β) be an interval such that $\dim \operatorname{ran} E^A_{(\alpha,\beta)} < \infty$. Then

$$\left|\dim \operatorname{ran} E^A_{(\alpha,\beta)} - \dim \operatorname{ran} E^B_{(\alpha,\beta)}\right| \leq n.$$

If $(\alpha, \beta) \subset \rho(A)$, then $(\alpha, \beta) \cap \sigma(B)$ consists of at most n eigenvalues counted with multiplicities.