Theorem 6.6. $\boxed{\overline{T_0} = T = T^*}$ and T is semibounded from below with lower bound zero.

Proof.

First we show $T_0^* = T$. Consider first for $g \in \operatorname{dom} T = H^2(\mathbb{R})$ and for $f \in \operatorname{dom} T_0 = C_0^\infty(\mathbb{R})$

$$(T_0 f, g) = \int_{-\infty}^{\infty} (f'') \cdot \overline{g}\, dx \overset{\text{part. int.}}{=} \int_{-\infty}^{\infty} f \cdot \overline{(-g'')}\, dx = (f, Tg)$$

Since this is true for all $f \in \operatorname{dom} T_0$, we get $g \in \operatorname{dom} T_0^*$ and $T_0^* g = Tg \Rightarrow \boxed{T \subset T_0^*}$

To verify $T_0^* \subset T$ let $g \in \operatorname{dom} T_0^*$ and define

$$h(x) := -\int_c^x \int_c^t (T_0^* g)(s)\, ds\, dt$$

where $c \in \mathbb{R}$. We are going to show:

$$\boxed{g = h + c_0 + c_1 x \qquad \text{for constants } c_0, c_1 \in \mathbb{C} \quad (*)}$$

If $(*)$ holds, g would be a L^2-function with g, g' being absolutely continuous (as h, h' are absolutely continuous) and $g'' = (h + c_0 + c_1 x)'' = h'' = \underbrace{-T_0^* g}_{\in L^2(\mathbb{R})}$

i.e. $g \in H^2(\mathbb{R}) = \operatorname{dom} T$, $T_0^* g = -g'' = Tg$, i.e. $T_0^* \subset T$

To show (*) consider first for $f \in \operatorname{dom} T_0$

$$\int_{\mathbb{R}} g \overline{T_0 f}\, dx = (g, T_0 f) = (T_0^* g, f) = \int_{\mathbb{R}} (-h'') \cdot \overline{f}\, dx$$

$$\overset{\text{int. by parts}}{=} \int_{\mathbb{R}} h \,\overline{(-f'')}\, dx = \int_{\mathbb{R}} h \cdot \overline{T_0 f}\, dx$$

$$\Rightarrow \int_{\mathbb{R}} (g-h)\, \overline{(T_0 f)}\, dx = 0 \qquad \forall f \in \operatorname{dom} T_0$$

$$\Rightarrow \int_{\mathbb{R}} (g-h)\, \overline{k}\, dx = 0 \qquad \forall k \in \operatorname{ran} T_0$$

$\Rightarrow \underline{\operatorname{ran} T_0 \subset \ker F}$, where $F : C_0^\infty(\mathbb{R}) \to \mathbb{C}$,

$$F(k) = \int_{\mathbb{R}} (g-h)\, \overline{k}\, dx$$

On the other hand, by Prop. 6.4 $\underline{\operatorname{ran} T_0 = \ker F_0 \cap \ker F_1}$

where $F_j : C_0^\infty(\mathbb{R}) \to \mathbb{C}$, $F_j(k) = \int_{\mathbb{R}} x^j \overline{k}\, dx$, $j = 0, 1$

By Lemma 6.5 there exist $c_0, c_1 \in \mathbb{C}$ s.t.

$$F(k) = c_0 F_0(k) + c_1 F_1(k)$$

$$\Leftrightarrow \int_{\mathbb{R}} (g - h - c_0 - c_1 x)\, \overline{k}\, dx = 0 \qquad \forall k \in C_0^\infty(\mathbb{R})$$

Since this holds for all $k \in C_0^\infty(\mathbb{R})$, the fundamental lemma of variational calculus implies $g - h - c_0 - c_1 x = 0$ a.e. in $\mathbb{R}$

Hence (*) is true and $\underline{T_0^* \subset T}$

$$\Rightarrow \boxed{T_0^* = T}$$

We have shown $T_0^* = T \supset T_0$

Hence it suffices to show that T is symmetric to conclude the self adjointness of T, as then $T \subset T^* = T_0^{**} = \overline{T_0} \subset \overline{T}$

Consider for $f \in \operatorname{dom} T = H^2(\mathbb{R})$

$$
\begin{aligned}
(Tf, f) &= \int_{\mathbb{R}} (-f'') \cdot \overline{f} \, dx = \lim_{c\to\infty} \int_{-c}^{c} (-f'') \cdot \overline{f} \, dx \\
&= \lim_{c\to\infty} \Big(\underbrace{-f'(c) \overline{f(c)}}_{\to 0} + \underbrace{f'(-c) \overline{f(-c)}}_{\to 0} + \int_{-c}^{c} |f'(x)|^2 dx \\
&= \int_{\mathbb{R}} |f'(x)|^2 dx \in \mathbb{R} \qquad (**)
\end{aligned}
$$

$\Rightarrow T$ is symmetric $\Rightarrow T = T^* = \overline{T_0}$

By $(**)$ we have $(Tf, f) \geq 0 \quad \forall f \in \operatorname{dom} T$ and hence T is semibounded from below.

$\square$

In the following we add to $-\frac{d^2}{dx^2}$ a real valued function (a potential) $V \in L^2(\mathbb{R})$ and consider in $L^2(\mathbb{R})$ the differential operator $-\frac{d^2}{dx^2} + V$.

Second goal: Show that $-\frac{d^2}{dx^2} + V$ defined on $H^2(\mathbb{R})$ is a self adjoint operator in $L^2(\mathbb{R})$.

Definition 6.7. Let $V \in L^2(\mathbb{R})$ be real valued. The operator

$$T_{0,V} f = -f'' + Vf, \qquad \operatorname{dom} T_{0,V} = C_0^\infty(\mathbb{R}),$$

in $L^2(\mathbb{R})$ is called *minimal Schrödinger operator with potential* V. Moreover, the operator

$$T_V f = -f'' + Vf, \qquad \operatorname{dom} T_V = H^2(\mathbb{R}),$$

in $L^2(\mathbb{R})$ is called *maximal Schrödinger operator with potential* V

Theorem 6.8. $\overline{T_{0,V}} = T_V = T_V^*$ and T_V is semibounded from below.

Proof:

We apply the Kato-Rellich theorem to deduce the result about (essential) self adjointness.

Consider in $L^2(\mathbb{R})$ the multiplication operator M_V

$\operatorname{dom} M_V = \{ f \in L^2(\mathbb{R}) : V \cdot f \in L^2(\mathbb{R}) \}$, $M_V f = V \cdot f$

We show that M_V is T-bounded with T-bound zero.

- $\operatorname{dom} T = H^2(\mathbb{R}) \subset \operatorname{dom} M_V$

For an arbitrary $f \in H^2(\mathbb{R})$, $x \in \mathbb{R}$:

$$|f(x)|^2 = \lim_{c \searrow -\infty} \left(f(x) \cdot \overline{f(x)} - f(c)\overline{f(c)} \right)$$

$$= \int_{-\infty}^{x} \left(f'(t)\overline{f(t)} + f(t)\overline{f'(t)} \right) dt \leq 2 \int_{\mathbb{R}} |f(t)| \cdot |f'(t)| \, dt$$

$$\overset{CS}{\leq} 2 \|f\| \cdot \|f'\| \leq \|f\|^2 + \|f'\|^2$$

$$\overset{\text{Lemma 6.1}}{\leq} (1 + C_\varepsilon) \|f\|^2 + \varepsilon \|f''\|^2$$

$$\Rightarrow \|Vf\|^2 = \int_{\mathbb{R}} |V(x)|^2 \cdot \underbrace{|f(x)|^2}\, dx \leq \|V\|^2 (1 + C_\varepsilon) \|f\|^2 + \|V\|^2 \varepsilon \|Tf\|^2$$

$\Rightarrow$ $\operatorname{dom} T \subset \operatorname{dom} M_V$ and M_V is T-bounded with T-bound zero.

- M_V is symmetric, as for $f \in \operatorname{dom} M_V$

$$(M_V f, f) = \int_{\mathbb{R}} V(x)\, |f(x)|^2 dx \in \mathbb{R} \quad \text{(as } V \text{ real valued)}$$

By Kato-Rellich we conclude that $T_V := T + M_V$ is self adjoint, that $T_{0,V} := T_0 + M_V$ is essentially self adjoint. Moreover, by Theorem 5.8 T_V is semibounded from below.

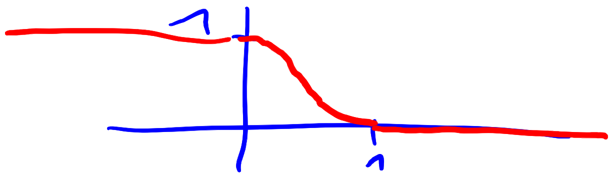
Third Goal: Find information about the spectra of T and T_V.

Theorem 6.9. $\boxed{\sigma(T) = \sigma_c(T) = \sigma_{ess}(T) = [0,\infty)}$ and $\underline{\sigma_p(T) = \emptyset}$.

Proof.

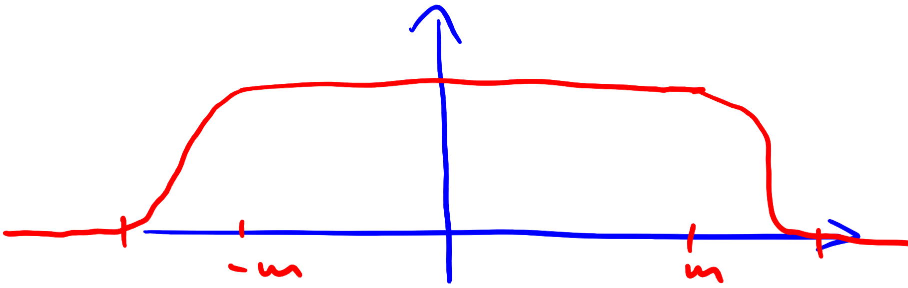
- $T \geq 0$ by Theorem 6.6 $\Rightarrow \sigma(T) \subset [0,\infty)$.
- $[0,\infty) \subset \sigma(T)$. Let $\lambda \in [0,\infty)$. Choose an auxiliary function $\varphi \in C^\infty(\mathbb{R})$, $0 \leq \varphi(x) \leq 1 \ \forall x \in \mathbb{R}$

and
$$\varphi(x) = \begin{cases} 1, & x \leq 0 \\ 0, & x \geq 1 \end{cases}$$



For $m \in \mathbb{N}$ define $\varphi_m(x) := \varphi(|x| - m)$

i.e. $\varphi_m \in C_0^\infty(\mathbb{R})$,
$$\varphi_m(x) = \begin{cases} 1, & |x| \leq m \\ 0, & |x| \geq m+1 \end{cases}$$



Define
$$f_m(x) := \frac{1}{\sqrt{2m}} \varphi_m(x)\, e^{i\sqrt{\lambda}x}, \quad x \in \mathbb{R}$$

Then $f_m \in C_0^\infty(\mathbb{R}) \subset \operatorname{dom} T$.

Goal: $\|f_m\| \xrightarrow[m\to\infty]{} 1$, $\|(T-\lambda) f_m\| \xrightarrow[m\to\infty]{} 0$, i.e. (f_m) is an approximate eigensequence

$$\|f_m\|^2 = \int_{\mathbb{R}} \frac{1}{2m} |\varphi_m(x)|^2 \underbrace{|e^{i\sqrt{\lambda}x}|^2}_{=1} dx$$

$\operatorname{supp}\varphi_m \subset [-m-1, m+1]$

$$\int_{-m-1}^{m+1} \frac{1}{2m} \underbrace{|\varphi_m(x)|^2}_{=1,\ |x|\le m} dx = 1 + \underbrace{\frac{1}{2m}\left(\int_{-m-1}^{-m} |\varphi_m(x)|^2 dx + \int_m^{m+1} |\varphi_m(x)|^2 dx\right)}_{\xrightarrow{m\to\infty} 0}$$

$$\xrightarrow{m\to\infty} 1$$

Since $-\frac{d^2}{dx^2}\left(e^{i\sqrt{\lambda}x}\right) = \lambda e^{i\sqrt{\lambda}x}$, we get

$$-f_m'' - \lambda f_m = \frac{1}{\sqrt{2m}}\left(-\varphi''(|x|-m) - 2i\sqrt{\lambda}\,\operatorname{sgn}(x)\,\varphi'(|x|-m)\right) e^{i\sqrt{\lambda}x}$$

Since φ_m is constant in $(-\infty, -m-1) \cup (-m, m) \cup (m+1, \infty)$, the function $-f_m'' - f_m$ is supported in $[-m-1, -m] \cup [m, m+1] = I_m$

$$\Rightarrow \|-f_m'' - \lambda f_m\|^2 = \int_{I_m} \frac{1}{2m} \underbrace{\left|-\varphi''(|x|-m) - 2i\sqrt{\lambda}\,\operatorname{sgn}(x)\,\varphi'(|x|-m)\right|^2}_{\le C} dx$$

$$\le \frac{2C}{m} \xrightarrow{m\to\infty} 0$$

$\Rightarrow (f_m)$ is an approximate eigensequence

$\Rightarrow [0,\infty) \subset \sigma(T)$, as the above works $\forall \lambda \in [0,\infty)$

$\Rightarrow \boxed{\sigma(T) = [0,\infty)}$.

$\sigma_p(T) = \emptyset$, since eigenfunctions would be of the form $a\sin(\sqrt{\lambda}x) + b\cos(\sqrt{\lambda}x)$, but such functions do not belong to $L^2(\mathbb{R}) \Rightarrow \sigma(T) = \sigma_c(T) = \sigma_{ess}(T)$ $\square$

Theorem 6.10. Let $V \in L^2(\mathbb{R})$ be real valued. Then $\sigma_{ess}(T_V) = [0, \infty)$.

Proof:

We use that for $\lambda < 0$ the resolvent of T is given by

$$((T-\lambda)^{-1} f)(x) = \frac{1}{2\sqrt{-\lambda}} \int_{\mathbb{R}} e^{-\sqrt{-\lambda}|x-y|} f(y)\,dy, \quad \forall f \in L^2(\mathbb{R})$$

(exercise). Hence

$M_V (T-\lambda)^{-1}$ is integral operator, i.e.

$$(M_V (T-\lambda)^{-1} f)(x) = \int_{\mathbb{R}} \underbrace{V(x) \frac{1}{2\sqrt{-\lambda}} e^{-\sqrt{-\lambda}|x-y|}}_{k(x,y)} f(y)\,dy$$

Since $V \in L^2(\mathbb{R})$

$$\|k\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}}\int_{\mathbb{R}} \left| V(x) \frac{1}{2\sqrt{-\lambda}} e^{-\sqrt{-\lambda}|x-y|}\right|^2 dx\,dy$$

$$\overset{z=y-x}{=} \int_{\mathbb{R}} |V(x)|^2 \cdot \frac{1}{4(-\lambda)} \int_{\mathbb{R}} e^{-2\sqrt{-\lambda}|z|}\,dz\,dx < \infty$$

we have $k \in L^2(\mathbb{R}^2)$ and hence $M_V (T-\lambda)^{-1} \in \mathfrak{S}_\infty$ [exercise]. Since T_V is semibounded from below, for $\lambda < 0$ sufficiently small $\lambda \in \rho(T_V)$ and

$$(T_V-\lambda)^{-1} M_V (T-\lambda)^{-1} = (T_V-\lambda)^{-1}[T + M_V - \lambda - (T-\lambda)](T-\lambda)^{-1}$$

$$\overset{\mathfrak{S}_\infty}{=} (T-\lambda)^{-1} - (T_V-\lambda)^{-1} \overset{\text{Theorem 5.12}}{\Longrightarrow} \sigma_{ess}(T_V) = \sigma_{ess}(T) = [0,\infty) \quad \square$$