Theorem 6.10. Let $V \in L^2(\mathbb{R})$ be real valued. Then $\sigma_{ess}(T_V) = [0, \infty)$.

So far: self-adjointness of $-\frac{d^2}{dx^2} + V$

essential spectrum $-\frac{d^2}{dx^2} + V$

open question: Can we say something about the discrete spectrum of T_V?

Observations

1) T_V is semibounded from below, i.e. $(T_V f, f) \geq \gamma \|f\|^2$

since $\sigma(T_V) \subset [\gamma, \infty)$, the discrete eigenvalues do not converge to $-\infty$.

2) If $V \geq 0$ a.e. Then

$$(T_V f, f) = \underbrace{(T f, f)}_{\geq 0} + (V f, f) = \int_{\mathbb{R}} \underbrace{V(x) |f(x)|^2}_{\geq 0} dx \geq 0$$

$\Rightarrow \sigma(T_V) \subset [0, \infty)$

$\Rightarrow \sigma(T_V) = \sigma_{ess}(T_V) = [0, \infty)$

6.1 Estimates for the discrte spectrum of $-\frac{d^2}{dx^2} + V$

In the following T is a self adjoint operator which is bounded from below. The discrete eigenvalues of T which are below its essential spectrum are denoted by

$$\mu_1 \leq \mu_2 \leq \mu_3 \leq \dots .$$

Moreover, for vectors $x_1, \dots, x_n \in \mathcal{H}$ we define

$$U(x_1, \dots, x_n) := \{x \in \operatorname{dom} T : x \in \operatorname{span}\{x_1, \dots, x_n\}^\perp, \|x\| = 1\}.$$

Theorem 6.11 (Min-max-principle)**.** Define the numbers

$$\boxed{\lambda_n := \sup_{x_1, \dots, x_{n-1} \in \mathcal{H}} \inf_{x \in U(x_1, \dots, x_{n-1})} \{(Tx, x)\}.}$$

Then, the following is true:

(i) (λ_n) is a non-decreasing sequence and $\lambda_n \to \lambda_\infty \leq \infty$, as $n \to \infty$.

(ii) $\lambda_n = \mu_n$, if $\lambda_n < \inf \sigma_{ess}(T)$.

(iii) $\lambda_\infty = \inf \sigma_{ess}(T)$ or $\lambda_\infty = \infty$, if $\sigma_{ess}(T) = \emptyset$.

+ good approximations of discrete evs below $\inf \sigma_{ess}(T)$
+ good tool to show existence of one discrete ev ($n=1$ no supremum)
- does not work in gaps of $\sigma_{ess}(T)$

Proof:

(i) clear by def of λ_n

(ii) Let e_k be the orthonormal eigenvectors for μ_k below $\inf \sigma_{ess}(T)$.

- Let $x_1, \dots, x_{n-1}$ be arbitrary, but fixed vectors and choose $x = \sum_{j=1}^{n} \alpha_j e_j$ such that $\|x\| = 1$ and $x \in U(x_1, \dots, x_{n-1})$

$$\Rightarrow \underline{(Tx,x)} = \left(T \sum_{j=1}^{n} \alpha_j e_j , \sum_{k=1}^{n} \alpha_k e_k\right)$$

$$\overset{Te_j=\mu_j e_j}{=} \left(\sum_{j=1}^{n} \alpha_j \mu_j e_j , \sum_{k=1}^{n} \alpha_k e_k\right)$$

$$\overset{e_j \perp e_k,\ j\neq k}{=} \sum_{j=1}^{n} |\alpha_j|^2 \underbrace{\mu_j}_{\leq \mu_n} \leq \mu_n \underbrace{\sum_{j=1}^{n} |\alpha_j|^2}_{=\|x\|^2 = 1} = \underline{\mu_n}$$

$$\Rightarrow \inf_{y \in U(x_1,\dots,x_{n-1})} (Ty,y) \leq \mu_n$$

Since $x_1,\dots,x_{n-1}$ were arbitrary

$$\Rightarrow \underline{\mu_n \geq} \sup_{x_1,\dots,x_{n-1} \in \mathcal{H}} \inf_{y \in U(x_1,\dots,x_{n-1})} (Ty,y) = \underline{\lambda_n}$$

- Conversely, choose $x_1 = e_1, \dots, x_{n-1} = e_{n-1}$.

Then one has for any $x \in U(e_1,\dots,e_{n-1}) \subset \underline{\operatorname{span}\{e_1,\dots,e_{n-1}\}^{\perp}}$

$$= \operatorname{ran} \underline{E_{\mathbb{R}\setminus(-\infty,\mu_n)}}$$

$$\Rightarrow \underline{(Tx,x)} = \int_{\mathbb{R}} t \, d(E(t)x,x) = \int_{\mathbb{R}} t \, d(E(t)E_{\mathbb{R}\setminus(-\infty,\mu_n)}x,x)$$

$$= \int_{\mathbb{R}} t \, d(E_{\mathbb{R}\setminus(-\infty,\mu_n)}(t)x,x) = \int_{\mu_n}^{\infty} t \, d(E(t)x,x)$$

$$\geq \mu_n \cdot \|x\|^2 = \underline{\mu_n}$$

$$\Rightarrow \inf_{x \in U(e_1,\dots,e_{n-1})} (Tx,x) \geq \mu_n \quad (\text{as } x \in U(e_1,\dots,e_{n-1}) \text{ was arbitrary})$$

$\Rightarrow \lambda_n \geq \mu_n \Rightarrow \boxed{\lambda_n = \mu_n}$

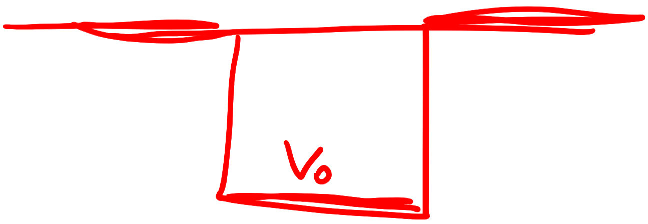
(iii) is similar to (ii), replace $e_1, \dots, e_{n-1}$ by orthonormal singular sequence

Example 6.12 (Existence of eigenvalues of T_V)**.** Let T_V be defined as in Definition 6.7 such that $V \leq 0$, $V \neq 0$. Then, if V is sufficiently negative, then $\sigma_d(T_V) \neq \emptyset$.

Let $f \in H^2(\mathbb{R})$ such that $\|f\| = 1$, $Vf \neq 0$

$$\rightsquigarrow (T_V f, f) = \underbrace{(Tf, f)}_{\text{fixed number independent of } V} + \underbrace{(Vf, f)}_{\substack{\text{negative number becomes smaller than} \\ (Tf,f) \text{, if } V \text{ is sufficiently negative}}} < 0 = \inf \sigma_{ess}(T_V)$$

Min-Max $\Rightarrow$ $\sigma_d(T_V) \neq \emptyset$



real-valued, $V \leq 0$

Theorem 6.13 (Birman-Schwinger principle). Let $\boxed{V \in C_0^\infty(\mathbb{R})}$, let T and T_V be defined as in Definitions 6.3 and 6.7, respectively. Then

$$\boxed{\lambda \in \sigma_d(T_V)} \quad \Leftrightarrow \quad \boxed{1 \in \sigma_d\big(\sqrt{|V|}(T-\lambda)^{-1}\sqrt{|V|}\big).}$$

The Birman Schwinger principle allows to reduce the ev-problem for T_V to the ev-problem for $\sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}$

+ ev-problem for unbounded op T_V is transformed to a family of ev-problems for bounded op $\sqrt{|V|}(T-\lambda)^{-1}\sqrt{|V|}$, which are sometimes well accessible

Proof:

"$\Rightarrow$" Let $\lambda \in \sigma_d(T_V)$, i.e. $\exists f \neq 0$, $f \in \operatorname{dom} T_V = H^2(\mathbb{R})$

i.e. $0 = (T_V - \lambda) f = (T + V - \lambda) f$

$\Rightarrow -Vf = (T-\lambda) f \Rightarrow \cancel{0}\, f = -(T-\lambda)^{-1} \overbrace{V f}^{\neq 0} \quad | \cdot \sqrt{|V|}$

$\underbrace{V}_{V \leq 0 \Rightarrow V = -\sqrt{|V|}^2}$

$\Rightarrow \underbrace{\sqrt{|V|}\, f}_{g \neq 0} = \sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}\,\underbrace{\sqrt{|V|}\, f}_{g \neq 0}$

$\Rightarrow 1 \in \sigma_p\big(\sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}\big)$

"$\Leftarrow$" $1 \in \sigma_d\big(\underbrace{\sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}}_{\text{compact}}\big)$ $\quad \sigma(\cdot) = \{0\} \cup \sigma_d(\cdot)$

$\Rightarrow \exists f \neq 0: \; f = \sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}\, f \quad | \cdot \sqrt{|V|} \quad \Rightarrow \sqrt{|V|}\, f \neq 0$

$\Rightarrow \underbrace{\sqrt{|V|}\, f}_{=:g} = \underbrace{(\sqrt{|V|})^2}_{=-V}\,(T-\lambda)^{-1}\underbrace{\sqrt{|V|}\, f}_{=:g}$

Choose $h := (T-\lambda)^{-1} g \Leftrightarrow g = (T-\lambda) h$

$\Rightarrow (T-\lambda)h = -Vh \Rightarrow (T+V-\lambda)h = 0$

$\Rightarrow \lambda \in \sigma_d(T_V)$ $\square$

Theorem 6.14. Let $V \in C_0^\infty(\mathbb{R})$ such that $\boxed{V \leq 0,\ V \neq 0}$. Then $\sigma_d(T_V)$ is non-empty and finite.

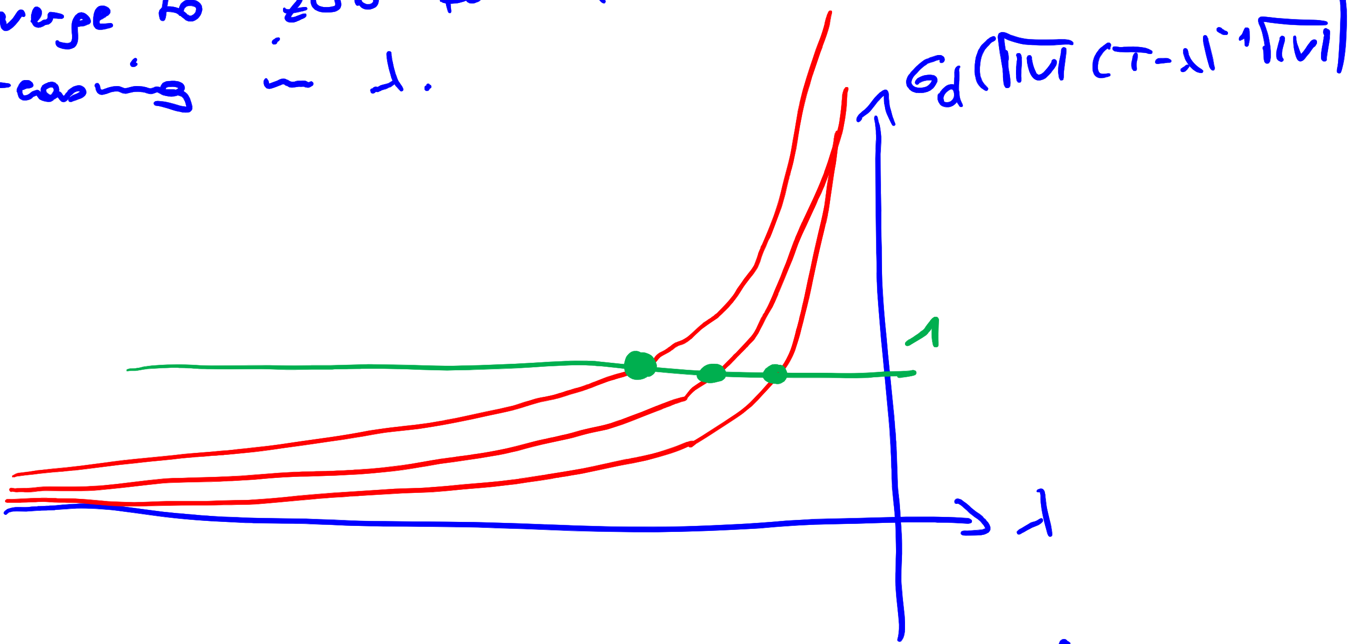
Proof: Use the Birman-Schwinger principle. Note:

$$\left(\sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}\, f\right)(x) = \int_{\mathbb{R}} \underbrace{\sqrt{|V(x)|}\,\frac{1}{2\sqrt{-\lambda}}\, e^{-\sqrt{-\lambda}|x-y|}\sqrt{|V(y)|}}_{k(x,y)}\, f(y)\, dy$$

$k(x,y) \in L^2(\mathbb{R}^2)$

$\Rightarrow \sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}$ is compact, self-adjoint.

One can show that the eigenvalues of $\sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}$ depend continuously on λ, they converge to zero for $\lambda \to -\infty$ and they are non-decreasing in λ.



Since $k(x,y)$ is monotonously increasing in λ, we get for $f \geq 0$: $\|\sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}\, f\|^2 \nearrow \infty$, as $\lambda \to 0$ (monotone convergence) $\Rightarrow \|\sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}\| \xrightarrow{\lambda\to 0} \infty$

$\Rightarrow$ the largest eigenvalue of $\sqrt{|V|}\,(T-\lambda)^{-1}\sqrt{|V|}$ converges to infinity.

Since each intersection point of the eigenvalue lines with the line $y = 1$ corresponds to a discrete ev of T_V, we see that there is always one discrete eigenvalue and at most finitely many ev. $\square$

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