Advanced functional analysis
2. Exercise sheet
(Spectrum of closed operators)

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Exercise 7

Let X be a Banach space and $T : X \supseteq \operatorname{dom} T \to X$ a closed, linear operator in X. Show that for all $\lambda, \mu \in \rho(T)$ the *resolvent formula*

$$(T-\lambda)^{-1} - (T-\mu)^{-1} = (\lambda-\mu)(T-\lambda)^{-1}(T-\mu)^{-1}$$

holds.

Exercise 8

Show that the operator

$$T : C([0,1]) \to C([0,1]), \quad (Tu)(x) = \int_0^x u(t)dt, \quad x \in [0,1],$$

is bounded[1] and that its spectral radius is smaller than its norm, that is, $\inf_{n\in\mathbb{N}} \|T^n\|^{1/n} < \|T\|$. Finally, compute the spectrum of T.

Exercise 9

Compute the point, continuous, and residual spectrum of the following operators:

1. T as given in Exercise 1 on the first exercise sheet;
2. $U : \ell^2 \to \ell^2, (x_n)_{n\in\mathbb{N}} \mapsto (m_n x_n)_{n\in\mathbb{N}}$, where the sequence $(m_n)_{n\in\mathbb{N}} \in \ell^2$ is given by
$$m_n = \begin{cases} \frac{1}{n}, & n \leq 17, \\ 0, & n > 17; \end{cases}$$
3. $W : \ell^2 \to \ell^2, (x_n)_{n\in\mathbb{N}} \mapsto (\frac{1}{n} x_n)_{n\in\mathbb{N}}$.
4. $Z : \ell^2 \supseteq \operatorname{dom} Z \to \ell^2, (x_n)_{n\in\mathbb{N}} \mapsto (n x_n)_{n\in\mathbb{N}}$, $\operatorname{dom} Z = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : (n x_n)_{n\in\mathbb{N}} \in \ell^2\}$.

Are all of these operators bounded?

A short repetition: The adjoint of a bounded, everywhere defined operator.
Let $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}}), (\mathcal{K}, (\cdot,\cdot)_{\mathcal{K}})$ be Hilbert spaces and T a bounded, everywhere defined linear operator from $\mathcal{H}$ to $\mathcal{K}$, $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. It can be shown with the help of the Fréchet-Riesz theorem that there exists a unique bounded operator $S \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that

$$(Th, k)_{\mathcal{K}} = (h, Sk)_{\mathcal{H}} \quad \forall h \in \mathcal{H}, k \in \mathcal{K}$$

is satisfied. The operator S is said to be the *adjoint operator* of T and one writes T^* instead of S.

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[1] As usual we consider $C([0,1])$ to be equipped with the norm $\|\cdot\|_\infty$.

Exercise 10

Let $\mathcal{H}$ be a Hilbert space. Proof the following statements.

1. If $A \in \mathcal{L}(\mathcal{H})$ is selfadjoint, that is, $A = A^*$ holds, then $\sigma_p(A) \subseteq \mathbb{R}$ holds.
2. If $A : \mathcal{H} \to \mathcal{H}$ is an operator (which, a priori, may be unbounded) satisfying

$$(Ax, y)_{\mathcal{H}} = (x, Ay)_{\mathcal{H}} \quad \forall x, y \in \mathcal{H},$$

   then A is bounded and, hence, selfadjoint.

HINT: Which important criterion for boundedness of an operator do you know?

Exercise 11

Consider the operator

$$A : L^2([0,1]) \to L^2([0,1]), \quad (Af)(t) = tf(t), \quad t \in [0,1].$$

Show that A is bounded and selfadjoint, and compute the operator norm of A. Moreover, compute $\sigma_p(A)$.

Exercise 12

Let S be a densely defined linear operator in a Hilbert space $\mathcal{H}$. Show that the identity

$$\operatorname{dom} S^* = \{g \in \mathcal{H} : \operatorname{dom} S \ni f \mapsto (Sf, g) \text{ is continuous}\}$$

holds true.