Advanced functional analysis
3. Exercise sheet
(Symmetric and self adjoint operators)

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Markus Holzmann
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Exercise 13
In the Hilbert space ℓ^2 consider the multiplication operator

$$T(x_n)_{n\in\mathbb{N}} = (nx_n)_{n\in\mathbb{N}}, \quad \operatorname{dom} T = \big\{(x_n)_{n\in\mathbb{N}} : \exists\, N \in \mathbb{N}\,\text{s.t.}\, x_n = 0\,\forall\, n \geq N\big\}.$$

Investigate whether T is symmetric, essentially selfadjoint, or selfadjoint, respectively. If it is not selfadjoint, compute its adjoint T^*.

Exercise 14
Let $\mathcal{H}$ be a Hilbert space and R, S densely defined linear operators in $\mathcal{H}$. Show the following statements.

1. If $R+S$ is densely defined, then $R^* + S^* \subseteq (R+S)^*$,[1] but in general the converse inclusion does not hold.
2. If $S \in \mathcal{L}(\mathcal{H})$, then equality holds in (i).
3. If RS is densely defined, then $S^*R^* \subseteq (RS)^*$, but in general the converse inclusion does not hold.
4. If $R \in \mathcal{L}(\mathcal{H})$, then equality holds in (iii).

Hint: Remember carefully the definitions of $R+S$ and RS, especially of its domains! One sets $\operatorname{dom}(R+S) = \operatorname{dom} R \cap \operatorname{dom} S$ and $\operatorname{dom}(RS) = \{f \in \operatorname{dom} S : Sf \in \operatorname{dom} R\}$.

Exercise 15
Prove that the domain of the adjoint operator of

$$Tu = u', \quad \operatorname{dom} T = \big\{u \in L^2(0,1) : u' \text{ exists almost everywhere}, u' \in L^2(0,1)\big\}$$

in $L^2(0,1)$ is trivial, i.e., $\operatorname{dom} T^* = \{0\}$.
Idea: Prove first that each $u \in \operatorname{dom} T^*$ is absolutely continuous and satisfies $T^*u = -u'$. For this use that $C_0^\infty(0,1)$ is dense in $L^2(0,1)$.

Exercise 16
For a continuous function $f : \mathbb{R} \to \mathbb{R}$ consider the *multiplication operator*

$$T : L^2(\mathbb{R}) \supseteq \operatorname{dom} T \to L^2(\mathbb{R}), \quad (Tg)(x) = (fg)(x), \quad x \in \mathbb{R},$$

on $\operatorname{dom} T = \{g \in L^2(\mathbb{R}) : fg \in L^2(\mathbb{R})\}$; cf. an example from the lecture.[2] Prove the following statements.

1. λ is an eigenvalue of T, if and only if the set $\{x \in \mathbb{R} : f(x) = \lambda\}$ has positive Lebesgue measure.
2. λ is contained in the continuous spectrum of T, if and only if $\lambda \in \overline{\{f(x) : x \in \mathbb{R}\}}$ and the Lebesgue measure of $\{x \in \mathbb{R} : f(x) = \lambda\}$ is zero.
3. If f ist bounded and strictly monotonous, then $\sup_{x\in\mathbb{R}} f(x)$ belongs to $\sigma_c(T)$.

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[1] The inclusion $A \subseteq B$ of two operators A and B means that $\operatorname{dom} A \subseteq \operatorname{dom} B$ holds and both operators coincide on the smaller set.

[2] There it was shown that T is selfadjoint and satisfies $\sigma(T) = \overline{\{f(x) : x \in \mathbb{R}\}}$.

Exercise 17

Let $\mathcal{H}$ be a Hilbert space and S a symmetric operator in $\mathcal{H}$. Prove the following criteria for (essential) selfadjointness.

1. S is essentially selfadjoint, if and only if there exists $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $\operatorname{ran}(S - \bar{\lambda})$ and $\operatorname{ran}(S - \lambda)$ are dense in $\mathcal{H}$.
2. If there exists a **real** λ such that $\operatorname{ran}(S - \lambda) = \mathcal{H}$ holds, then S is selfadjoint.

Exercise 18

Let $\mathcal{H}$ be a Hilbert space and S a symmetric operator in $\mathcal{H}$. Further, let A be an extension of S with $S \subseteq A \subseteq S^*$. Show that the direct sum decomposition

$$\operatorname{dom} S^* = \operatorname{dom} A \dotplus \ker(S^* - \lambda)$$

holds for each $\lambda \in \rho(A)$.[3] Conclude from this the following statement: If S is a symmetric operator and there exists $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $\operatorname{ran}(\overline{S} - \bar{\lambda}) \neq \mathcal{H}$ and $\operatorname{ran}(\overline{S} - \lambda) = \mathcal{H}$, then S does not posses any selfadjoint extension.

[3] Of course the formula only makes sense, if such a λ exists – which in general might be wrong.