Advanced functional analysis
3. Exercise sheet
(Functional calculus for bounded, self adjoint operators)

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Exercise 19

Let $A = A^*$ be a bounded operator in a Hilbert space $\mathcal{H}$ and $f, g \in C(\sigma(A))$. Prove the following statements.

1. $\|f(A)\| = \|f\|_\infty$.
2. $f \geq 0$ implies $f(A) \geq 0$.[1]
3. $Ax = \lambda x$ implies $f(A)x = f(\lambda)x$ for each $\lambda \in \sigma(A)$.
4. $f(A)g(A) = g(A)f(A)$, $f(A)$ is a normal operator,[2] and $f(A)$ is self adjoint if and only if f is real-valued.

Exercise 20

Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a bounded sesquilinear form, i.e. a map satisfying

$$a[\alpha x+\beta y, z] = \alpha a[x,z]+\beta a[y,z], \quad a[x, \alpha y+\beta z] = \overline{\alpha} a[x,y]+\overline{\beta} a[x,z], \quad \text{and} \quad |a[x,y]| \leq M\|x\|\cdot\|y\|$$

for all $x, y, z \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$ and a constant $M \geq 0$. Prove that there exists a unique operator $A \in \mathcal{L}(\mathcal{H})$ such that

$$(Ax, y) = a[x, y]$$

holds for all $x, y \in \mathcal{H}$.

Exercise 21

Let $A = A^*$ be a bounded operator in a Hilbert space $\mathcal{H}$ and let $B \in \mathcal{L}(\mathcal{H})$ satisfy $AB = BA$. Prove that

$$f(A)B = Bf(A)$$

remains valid for each $f \in B(\sigma(A))$.

Exercise 22

Show that the operator

$$A : L^2(0,1) \to L^2(0,1), \quad (Af)(x) = e^x f(x), \quad x \in (0,1), f \in L^2(0,1),$$

is self adjoint and compute $g(A)$ for each $g \in B(\sigma(A))$. (Hint: Make use of the uniqueness of the measurable functional calculus!)

Moreover, prove or disprove that the spectral mapping theorem

$$g(\sigma(A)) = \sigma(g(A)), \quad g \in B(\sigma(A)), \tag{1}$$

is true for each self adjoint operator A.[3]

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[1] That is, $(f(A)x, x) \geq 0$ holds for all $x \in \mathcal{H}$.
[2] $B \in \mathcal{L}(\mathcal{H})$ is called *normal*, if $B^*B = BB^*$ holds.
[3] Recall that (1) is true for all $g \in C(\sigma(A))$, cf. the lecture.

Exercise 23

Let A be a self adjoint operator in the Hilbert space $\mathbb{C}^n$ having mutually distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, $m \leq n$, and let P_j denote the orthogonal projection onto the eigenspace corresponding to the eigenvalue λ_j, $j = 1, \ldots, m$. For each Borel set $B \subseteq \mathbb{R}$ we define

$$E_B := \sum_{\lambda_j \in B} P_j.$$

Show that the mapping $B \mapsto E_B$ provides a spectral measure and that A can be represented as

$$A = \int_{\mathbb{R}} \lambda dE = \sum_{j=1}^{m} \lambda_j P_j.$$

Moreover, find polynomials p and q such that $p(A) = 0$ and $q(A) = \arctan(e^A)$ hold.

Exercise 24

Let $\mathcal{H}$ be a Hilbert space and $A = A^* \in \mathcal{L}(\mathcal{H})$. Denote by $\Sigma \ni B \mapsto E_B$ the spectral measure of A. The set

$$\sigma_{\text{ess}}(A) = \left\{\lambda \in \sigma(A) : \lambda \text{ is an accumulation point of } \sigma(A) \text{ or } \dim\ker(A - \lambda) = \infty\right\}$$

is called the *essential spectrum* of A. Show that λ belongs to $\sigma_{\text{ess}}(A)$ if and only if $\dim \operatorname{ran} E_B = \infty$ holds for each open neighborhood B of λ.