Advanced functional analysis
5. Exercise sheet
(Spectral theorem for unbounded s.a. operators, relatively bounded operators)

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Markus Holzmann
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Exercise 25
Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be the spectral measure associated to A. Moreover, let $B \in \Sigma$ and set $\mathcal{H}_B := \operatorname{ran} E_B$. Show that the following statements hold:

(i) $A\mathcal{H}_B \subset \mathcal{H}_B$, $\mathcal{H}_B^\perp = \operatorname{ran} E_{\mathbb{R}\setminus B}$ and $A\mathcal{H}_B^\perp \subset \mathcal{H}_B^\perp$.

(ii) $A_B := A \restriction \mathcal{H}_B$ is bounded and self-adjoint in $\mathcal{H}_B$.[1]

(iii) $(\sigma(A) \cap B^\circ) \subset \sigma(A_B) \subset (\sigma(A) \cap \overline{B})$, where B° is the interior part of B.

Hint for (iii): How does the spectral measure associated to A_B look like?

Exercise 26
For an operator $A = A^* \in \mathcal{L}(\mathcal{H})$ in a Hilbert space $\mathcal{H}$ with spectral measure $B \mapsto E_B$ its *compression*[2] A_B to $\operatorname{ran} E_B$ is well-defined for each Borel set $B \subseteq \mathbb{R}$, cf. exercise 25. Find examples such that

(i) $\partial B \subseteq \sigma(A_B)$ and

(ii) $\partial B \cap \sigma(A_B) = \emptyset$.

Hint: Multiplication operators in $L^2(\mathbb{R})$ can do the job.

Exercise 27
Let E be a spectral measure in the Hilbert space $\mathcal{H}$.

(i) Let $x \in \mathcal{H}$ be fixed. Show that the map $\Sigma \ni B \to (E_B x, x) \in \mathbb{R}$ is a measure.

(ii) Assume that E has compact support and that $A = A^* \in \mathcal{L}(\mathcal{H})$ is the self adjoint operator associated to E. Show that for any bounded and measurable function f the following formula holds:
$$\int_{\mathbb{R}} |f(\lambda)|^2 d(E(\lambda)x, x) = \|f(A)x\|^2.$$

Exercise 28
Let A and V be linear operators in the Hilbert space $\mathcal{H}$. Show that V is A-bounded if and only if $\operatorname{dom} A \subset \operatorname{dom} V$ and there exist $\alpha, \beta \geq 0$ such that
$$\|Vx\|^2 \leq \alpha\|x\|^2 + \beta\|Ax\|^2$$
holds for all $x \in \operatorname{dom} A$. Moreover, prove that the infimum over all $\sqrt{\beta}$, such that there exists an α so that the above inequality holds, coincides with the A-bound of V.

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[1] The inner product in $\mathcal{H}_B := \operatorname{ran} E_B$ is simply the restriction of the inner product in $\mathcal{H}$ to $\mathcal{H}_B$.

[2] This means, the operator A is restricted and its restriction is understood as an operator in the smaller space $\operatorname{ran} E_B$.

Exercise 29

Consider in ℓ^2 the operator A given by

$$A(x_n)_{n\in\mathbb{N}} = (nx_n)_{n\in\mathbb{N}}, \quad \operatorname{dom} A = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : (nx_n)_{n\in\mathbb{N}} \in \ell^2\}.$$

Then $A = A^*$, cf. exercise 13. Consider in ℓ^2 the operator

$$V(x_n)_{n\in\mathbb{N}} = (\sqrt{n}x_n)_{n\in\mathbb{N}}, \quad \operatorname{dom} V = \{(x_n)_{n\in\mathbb{N}} : (\sqrt{n}x_n)_{n\in\mathbb{N}} \in \ell^2\}.$$

Use the Kato-Rellich theorem to show that $A + V$ is self adjoint.

Exercise 30

Prove or disprove the following statements.

(i) If A is self adjoint in the Hilbert space $\mathcal{H}$ and V is symmetric and A-bounded with A-bound one, then $A + V$ is self adjoint.

(ii) If A is self adjoint in $\mathcal{H}$, V symmetric und $\overline{A + V} = (A + V)^*$ holds, then V is A-bounded.[3]

[3] Here the reverse statement of the Kato-Rellich theorem is investigated.