Advanced functional analysis
6. Exercise sheet
(Relatively bounded operators)

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Exercise 31

Let $\mathcal{H}$ be a Hilbert space and let $S : \mathcal{H} \supseteq \operatorname{dom} S \to \mathcal{H}$, $T : \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$ be linear operators. Assume that $\operatorname{dom} T \subseteq \operatorname{dom} S$, that T is closed and that S is closable. Show that S is T-bounded.

Exercise 32

Let T be a self adjoint operator in $\mathcal{H}$ and let $p, q \in \mathbb{R}$ with $0 \leq q < p$. Show that T^q is relatively bounded with respect to T^p with T^p-bound zero.[1]

Exercise 33

Define in $L^2(0, 2\pi)$ the operator

$$
\begin{aligned}
Tu &:= -u'', \\
\operatorname{dom} T &:= \left\{ u \in L^2(0, 2\pi) : u, u' \text{ are absolutely continuous, } u(0) = u(2\pi) = 0, \text{ and } u'' \in L^2(0, 2\pi) \right\}.
\end{aligned} \tag{1}
$$

Prove that T is essentially self adjoint. For this, proceed as follows:

(i) Show that T is symmetric.

(ii) Compute $\sigma_p(T)$ and compute all associated eigenfunctions.

(iii) Use Exercise 17 to conclude that T is essentially self adjoint.[2]

Hints: You are allowed to perform integration by parts as for smooth functions. For (iii) show for $f \in L^2(0, 2\pi)$ that the function $g(x) := -\frac{1}{2\pi}\big((2\pi - x)\int_0^x s f(s) ds + x \int_x^{2\pi} (2\pi - s) f(s) ds\big)$ belongs to $\operatorname{dom} T$ and solves $Tg = f$.

Exercise 34

Let T be given by (1) and define V by $\operatorname{dom} V = \operatorname{dom} T$ and $Vu = u'' + iu'$.

(i) Show that V is symmetric and T-bounded with T-bound one.

(ii) Prove that $e^{\pm x} \in \ker(T + V \mp i)^*$.

(iii) Conclude that $T + V$ is not essentially self-adjoint.

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[1] The powers of T are defined via the functional calculus for T.
[2] One can show that T is even self adjoint.

Exercise 35

Let T be given by (1) and let $V \in L^2(0, 2\pi)$ be real valued. Define the operator T_V by $\operatorname{dom} T_V = \operatorname{dom} T$ and $T_V u = -u'' + Vu$.

(i) Show that T is non-negative, i.e. $(Tx, x) \geq 0$ for all $x \in \operatorname{dom} T$.

(ii) Show that V is symmetric and T-bounded with T-bound less than one.

(iii) Show that T_V is semibounded from below and give an estimate for a lower bounded.

Hints: You are allowed to perform integration by parts as for smooth functions. Moreover, the main theorem of calculus for absolutely continuous functions may help for (ii).

Exercise 36

Let T be a non-negative self adjoint operator (i.e. $T \geq 0$). Assume that S is symmetric with $\operatorname{dom} T \subset \operatorname{dom} S$ and $\|Sx\| \leq \|Tx\|$ for all $x \in \operatorname{dom} T$. Show that

$$|(Sx, x)| \leq (Tx, x)$$

holds for all $x \in \operatorname{dom} T$.
Hint: Apply Theorem 5.8 to $T + \kappa S$ for $\kappa \in (-1, 1)$.