## Exercise 37

Let $A, B$ be self adjoint operators in $\mathcal{H}$ and let $\lambda, \mu \in \rho(A) \cap \rho(B)$. Show that

$$(B-\mu)^{-1} - (A-\mu)^{-1} = \big(1 + (\mu-\lambda)(A-\mu)^{-1}\big)\big((B-\lambda)^{-1} - (A-\lambda)^{-1}\big)\big(1 + (\mu-\lambda)(B-\mu)^{-1}\big)$$

holds.
*Hint:* Use the resolvent formula from Exercise 7.

In the following 2 examples we assume that $T$ is a self adjoint operator in the Hilbert space $\mathcal{H}$ and that $V$ is symmetric. The operator $V$ is called *$T$-compact*, if $\operatorname{dom} T \subset \operatorname{dom} V$ and $V : (\operatorname{dom} T, \|\cdot\|_T) \to \mathcal{H}$ is compact (here $\|\cdot\|_T$ denotes the graph norm associated to $T$).

## Exercise 38

Assume that $V$ is $T$-compact. Show that $V$ is $T$-bounded with $T$-bound zero.
*Hint:* each symmetric operator is closable!

## Exercise 39

Assume that $V$ is $T$-compact. Show the following statements:

(i) $T + V$ is self adjoint.

(ii) The graph norms for $T$ and $T + V$ are equivalent to each other and $-V$ is $(T + V)$-compact.

(iii) $\sigma_{ess}(T) = \sigma_{ess}(T + V)$ and $T$ and $T + V$ have the same singular sequences.

## Exercise 40

Let $a, b \in \mathbb{R}$. A function $u \in L^2(a, b)$ is called *weakly differentiable*, if there exists $v \in L^2(a, b)$ such that

$$\int_a^b u(x)\varphi'(x)dx = -\int_a^b v(x)\varphi(x)dx$$

holds for all $\varphi \in C_0^\infty(a, b)$. In this case we write $u' = v$. Show that $u$ is weakly differentiable if and only if $u$ is absolutely continuous on $[a, b]$ and $u' \in L^2(a, b)$. For this, proceed as follows:

(i) Define for $v \in L^2(a, b)$ and $c \in \mathbb{R}$

$$u(x) := c + \int_a^x v(y)\mathrm{d}y, \qquad x \in (a, b).$$

Then $u$ is weakly differentiable and its weak derivative is given by $u' = v$.

(ii) Conversely, let $u$ be weakly differentiable and denote by $u'$ its weak derivative. Then there exists a constant $c \in \mathbb{R}$ such that

$$u(x) = c + \int_a^x u'(y)\mathrm{d}y \tag{1}$$

for almost all $x \in (a, b)$.

HINTS: For $\varphi \in C_0^\infty(a,b)$ and $x \in (a,b)$ the relation $\varphi(x) = -\int_x^b \varphi'(y)\mathrm{d}y$ holds. Moreover, exchanging the order of integration might help.
Finally, it is allowed to use (without proof) the following statement: If

$$\int_a^b u\varphi' \mathrm{d}x = 0 \quad \forall\, \varphi \in C_0^\infty(a,b),$$

then there exists a constant $c$ such that $u(x) = c$ for almost all $x \in (a,b)$.

## Exercise 41

Let $k \in L^2(\mathbb{R}^2)$ and define in $L^2(\mathbb{R})$ the operator

$$K : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad (Kf)(x) := \int_{\mathbb{R}} k(x,y)f(y)dy.$$

prove that $K$ is compact.
*Hint:* You can use (without proof) that functions $\sum_{j=1}^N c_j \mathbb{1}_{Q_j}$ with $Q_j = [a_j, b_j] \times [c_j, d_j]$ and $c_j \in \mathbb{C}$ are dense in $L^2(\mathbb{R}^2)$.

## Exercise 42

Let $T : L^2(\mathbb{R}) \supset H^2(\mathbb{R})$, $Tf = -f''$. Show that

$$(T-\lambda)^{-1} f(x) = \int_{\mathbb{R}} \frac{1}{2\sqrt{-\lambda}} e^{-\sqrt{-\lambda}|x-y|} f(y)dy$$

holds for any $\lambda < 0$.