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Direct Schur Complement Method by Domain Decomposition based on the $\mathcal{H}$-Matrix Techniques

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Joint work with

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- *Motivation* and brief excurse to the domain decomposition and the $\mathcal{H}$-matrix technique
- FEM- and BEM-Galerkin approximations to the Schur complement $\mathbf{T}_i$ on substructures
- $\mathcal{H}$-matrix representation to the interface Schur complement $\mathbf{B}_\Gamma$ and its inverse $\mathbf{B}_\Gamma^{-1}$ (key point)
- Numerics for the FEM-Galerkin method
- Remarks on possible application in the FETI/BETI *iterative* methods and *conclusions*

Interface Formulation by Domain Decomposition (Main Concept)

- Natural parallelization
- Reduction of spacial dimension $d \to d-1$, providing the complexity $O(N_\Gamma)$
- FEM and BEM reciprocally complement each other nicely

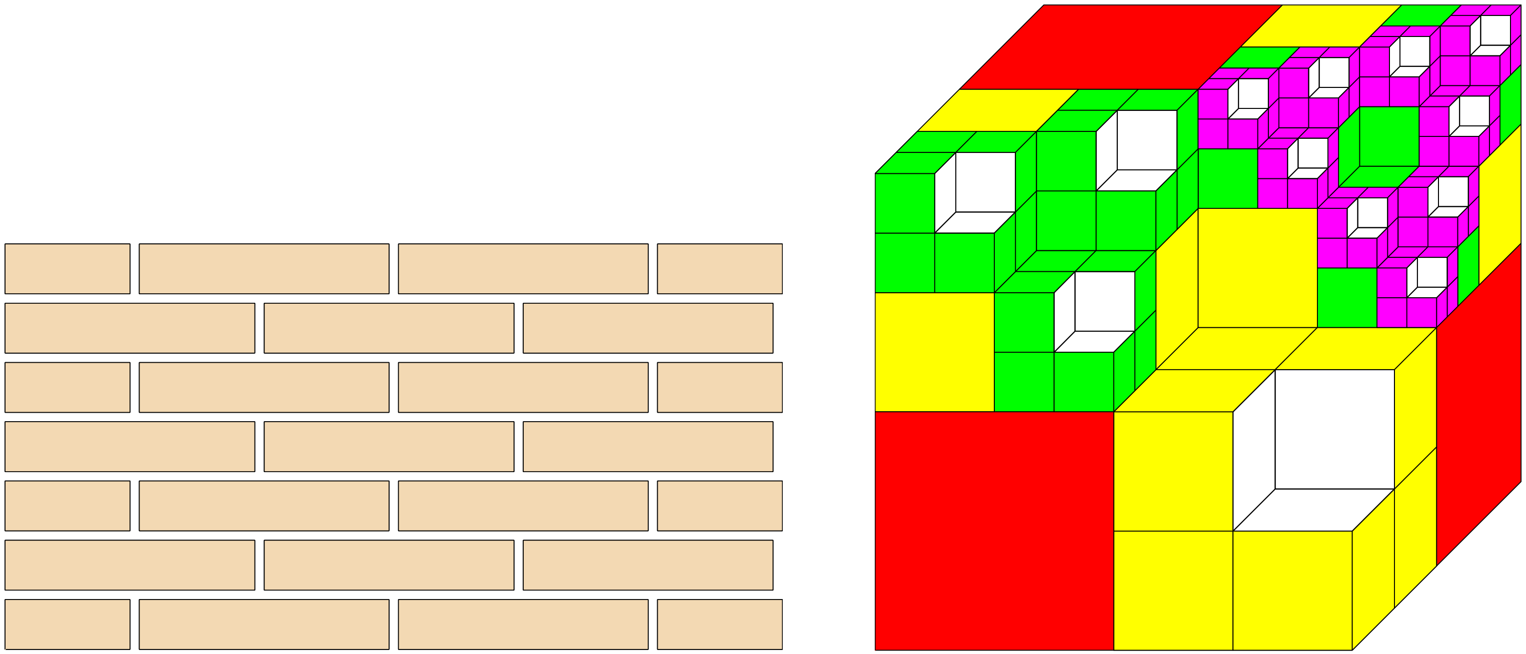

Figure 1: Skin problem (left), multiple Fichera cube (right).

Features and Applications of the $\mathcal{H}$-Matrices

$\mathcal{M}_{\mathcal{H},k}(T_{I\times I},\mathcal{P})$, *the class of data-sparse* $\mathcal{H}$*-matrices,* introduced by **Hackbusch '99**.
Further developments and applications: **Hackbusch, BNK, Sauter, Grasedyck, Bebendorf '99 - '03**.
A direct descendant of *panel clustering, fast multipole and mosaic-skeleton approximation*,
the $\mathcal{H}$-matrix technique allows, in addition, data-sparse matrix-matrix operations.

Main features:

- matrix arithmetic of $O(N \log^q N)$ - complexity, $N := |I|$ - cardinality
- accurate approximation to general class of nonlocal (integral) operators and operator-valued functions including the elliptic operator inverse $\mathcal{L}^{-1}$ and the Poincaré-Steklov operators
- rigorous theoretical analysis

Thm. 1 *(***complexity of the $\mathcal{H}$-matrix arithmetic***)*
Let $k \in \mathbb{N}$ denote the block-wise rank and $T_{I\times I}$ be an $\mathcal{H}$-tree with depth $L > 1$.
Then the arithmetic of matrices belonging to $\mathcal{M}_{\mathcal{H},k}(T_{I\times I},\mathcal{P})$ has the complexity

$$N_{\mathcal{H},store} \le 2C_{\rm sp}kLN, \qquad N_{\mathcal{H}\cdot v} \le 4C_{\rm sp}kLN, \qquad N_{\mathcal{H}\oplus\mathcal{H}} \le C_{\rm sp}k^2N(C_1L + C_2k),$$

$$N_{\mathcal{H}\odot\mathcal{H}} \le C_0C_{\rm sp}^2k^2LN\max\{k,L\}, \qquad N_{\widetilde{Inv(\mathcal{H})}} \le N_{\mathcal{H}\odot\mathcal{H}} \qquad (\,C_{\rm sp}-\textit{sparsity constant}).$$

Hierarchical Partitionings: $\mathcal{P}_{1/2}(I \times I)$ - standard; $\mathcal{P}_W(I \times I)$ - Weak admissible

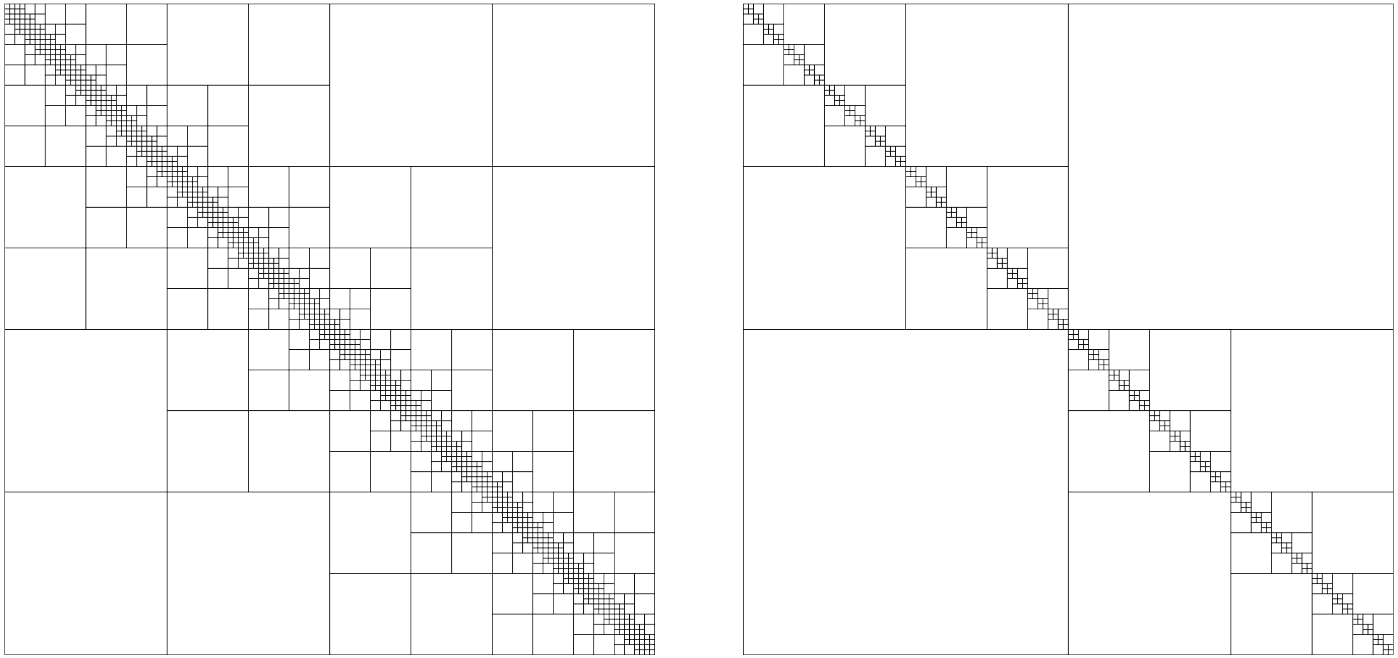

Figure 2: Standard- (left) and Weak-admissible hierarchical partitionings for $d = 1$.

Numerics I: $\mathcal{H}$-Matrix Approximation in 2D BEM

	$Adm_{1/2}$				Adm_W		
n	k	$\frac{\|\|A-A_{\mathcal{H}}\|\|_F}{\|\|A\|\|_F}$	storage	CPU/sec	k	$\frac{\|\|A-A_{\mathcal{H}}\|\|_F}{\|\|A\|\|_F}$	storage
256	2	2.0_{10}-5	0.1	-	5	9.1_{10}-6	0.1
512	2	1.5_{10}-5	0.3	0.04	5	1.1_{10}-5	0.3
1024	2	1.0_{10}-5	0.7	0.11	5	1.1_{10}-5	0.7
2048	2	7.4_{10}-6	1.7	0.25	5	8.8_{10}-6	1.5
4096	2	5.3_{10}-6	3.8	0.57	5	6.7_{10}-6	3.3
8192	2	3.7_{10}-6	8.3	1.27	5	5.0_{10}-6	7.4
8192	2	3.8_{10}-6	8.3	-	5	5.0_{10}-6	7.4
16384	2	2.8_{10}-6	18.2	-	5	3.7_{10}-6	16.1
32768	2	2.0_{10}-6	39.5	-	5	2.7_{10}-6	34.8

Table 1: Accuracy and storage size in the strongly and weakly admissible case

	$Adm_{1/2}$			Adm_W		
n	k	$\left\|I - AA_{\mathcal{H}}^{-1}\right\|_F$	CPU	k	$\left\|I - AA_{\mathcal{H}}^{-1}\right\|_F$	CPU
256	2	8.0_{10}-5	0.2	5	1.8_{10}-5	0.04
512	2	8.1_{10}-5	0.4	5	3.4_{10}-5	0.1
1024	2	8.1_{10}-5	1.1	5	4.6_{10}-5	0.3
2048	2	8.1_{10}-5	2.8	5	1.4_{10}-4	0.7
4096	2	8.1_{10}-5	6.7	5	1.5_{10}-4	1.8
8192	2	8.0_{10}-5	15.9	5	1.5_{10}-4	4.4
16384	2	8.0_{10}-5	37.3	5	1.5_{10}-4	10.5
32768	2	8.1_{10}-5	86.0	5	1.5_{10}-4	25.2

	Adm$_{1/2}$		Adm$_W$		
n	k	CPU	k	CPU	$\\|A_{\mathcal{H},s} - A_{\mathcal{H},w}\\|_2 \,/\, \\|A_{\mathcal{H},s}\\|_2$
131072	4	1031	12	475	$1_{10} - 6$

Table 2: Error of the inverse and CPU time needed

Formulation of the problem

The elliptic operator $\mathcal{L} : V \to V'$ with $V = H_0^1(\Omega)$ and $V' = H^{-1}(\Omega)$, and associated with

$$a_\Omega(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}\partial_j u \partial_i v + \sum_{j=1}^{d} b_j \partial_j u v + a_0 u v\Big) dx, \qquad (0.1)$$

is supposed to be V-elliptic, implying the unique solvability of

$$u \in V : \qquad a_\Omega(u,v) = f(v) \quad \forall v \in V.$$

$\Omega \in \mathbb{R}^d$ is composed of $M \geq 1$ matching, non-overlapping subdomains, $\overline{\Omega} = \cup_{i=1}^{M} \overline{\Omega}_i$ s.t. $a_\Omega(u,v) = \sum a_{\Omega_i}(u|_{\Omega_i}, v|_{\Omega_i})$.
The interface (skeleton) of the decomposition of Ω: $\Gamma = \cup \Gamma_i$ with $\Gamma_i := \partial\Omega_i$.

Distinguish three versions of a *direct method*:

(**A**) Rather general variable coefficients of $\mathcal{L}$ (**Hackbuch ’02; Hackbuch, BNK, Kriemann ’03**).

(**B**) Smooth coefficients in subdomains (*boundary concentrated FEM*: **BNK, Melenk ’02-’03**).

(**C**) Piecewise constant coefficients (*FFT based compression*: **BNK, Wittum ’96-’98, ’03**; *BEM representation of PSO*: **Langer, Steinbach, Wendland, ...**).

Approximate Direct Solver in the General Case (variable coefficients)

Let $\mathbf{A}_h \in \mathbb{R}^{I_\Omega \times I_\Omega}$ be the FEM-Galerkin stiffness matrix, solve

$$\begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{I\Gamma} \\ \mathbf{A}_{\Gamma I} & \mathbf{A}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} U_I \\ U_\Gamma \end{pmatrix} = \begin{pmatrix} F_I \\ F_\Gamma \end{pmatrix},$$

where Γ is the *interface index set* and $I = I_\Omega \setminus \Gamma$ is the *complementary* ('interior') one.

1. $\mathcal{H}$-matrix approximation to the local inv. $\mathbf{A}_i^{-1}$, $i = 1, ..., M$, and comp. particular solut. $\mathbf{A}_{II}^{-1} F_I$, where $\mathbf{A}_{II}^{-1} = blockdiag\{\mathbf{A}_1^{-1}, ..., \mathbf{A}_M^{-1}\}$, $\mathbf{A}_{II}$ is the stiffness matr. of $\mathcal{L}$ subject to $u|_\Gamma = 0$.
2. Solve the interface equation for U_Γ,

$$U_\Gamma = \mathbf{B}_\Gamma^{-1}(F_\Gamma - \mathbf{A}_{\Gamma I}\mathbf{A}_{II}^{-1} F_I), \qquad U_\Gamma, F_\Gamma \in \mathbb{R}^{I_\Gamma} \tag{0.2}$$

with the FEM Schur complement matrix

$$\mathbf{B}_\Gamma := \mathbf{A}_{\Gamma\Gamma} - \mathbf{A}_{\Gamma I}\mathbf{A}_{II}^{-1}\mathbf{A}_{I\Gamma} \in \mathbb{R}^{I_\Gamma \times I_\Gamma}. \tag{0.3}$$

3. Find U_I from $U_I = \mathbf{A}_{II}^{-1} F_I - \mathbf{A}_{II}^{-1}\mathbf{A}_{I\Gamma} U_\Gamma$, by fast extension $\mathcal{E}^{harm}_{V_h \leftarrow Y_h} \Rightarrow -\mathbf{A}_{II}^{-1}\mathbf{A}_{I\Gamma} U_\Gamma$.

Note 1: The "substructure" matrices $\mathbf{A}_i^{-1}$, $i = 1, ..., M$, can be represented in the $\mathcal{H}$-matrix format with cost $O(N_{\Omega_i})$. Moreover, the implementation of $\mathbf{A}_{II}^{-1}$ can be done in parallel.

Compose $\mathbf{B}_\Gamma$ and Compute Its Inverse

In **Step** $(\mathbf{2})$, let $\mathbf{A}_{i,FEM}$, $i = 1, ..., M$, be the local FEM stiffness matrix

$$\mathbf{A}_{i,FEM} = \begin{pmatrix} \mathbf{A}_i & \mathbf{A}_{i,\Gamma_i} \\ \mathbf{A}_{\Gamma_i,i} & \mathbf{A}_{\Gamma_i\Gamma_i} \end{pmatrix}.$$

By the $\mathcal{H}$-matrix arithmetics, compute the local FEM Schur complement matrices

$$\mathbf{T}_{i,h} := \mathbf{A}_{\Gamma_i\Gamma_i} - \mathbf{A}_{\Gamma_i,i}\mathbf{A}_i^{-1}\mathbf{A}_{i,\Gamma_i}, \tag{0.4}$$

where $\mathbf{A}_i$ is the stiffness matrix for $a_{\Omega_i}(\cdot,\cdot)$ with $\{Dirichlet = 0\}$ on Γ_i.
Note 2: $\mathbf{A}_i^{-1}$ can be represented in the $\mathcal{H}$-matrix form and then multiplied with simple matrices $\mathbf{A}_{\Gamma_i,i}\mathbf{A}_i^{-1}\mathbf{A}_{i,\Gamma_i}$.

Compute the interface Schur complement in the $\mathcal{H}$-matrix format

$$\langle \mathbf{B}_\Gamma U, Z\rangle_{I_\Gamma} = \sum_{i=1}^{M} \langle \mathbf{T}_{i,h} U_i, Z_i\rangle_{I_{\Gamma_i}}, \qquad \mathbf{B}_\Gamma \in \mathbb{R}^{I_\Gamma \times I_\Gamma}. \tag{0.5}$$

Here $U_i, Z_i \in \mathbb{R}^{I_{\Gamma_i}}$, $i = 1, ..., M$, are the local vector components $U_i = \mathbf{R}_{\Gamma_i} U$, where the connectivity matrix $\mathbf{R}_{\Gamma_i} \in \mathbb{R}^{I_{\Gamma_i} \times I_\Gamma}$ restricts onto I_{Γ_i}.

Approximate Inverse $\mathbf{B}_\Gamma^{-1}$ in the $\mathcal{H}$-matrix format

Algorithm AII (approximate interface inverse)

- *Evaluate the local Schur complements* $\mathbf{T}_{i,h} \in \mathbb{R}^{N_{\Gamma_i} \times N_{\Gamma_i}}$, $i = 1, ..., M$, *in the* $\mathcal{H}$*-matrix format.*
- *Construct an admissible block partitioning* $\mathcal{P}_\Gamma$ *of the product index set* $I_\Gamma \times I_\Gamma$ *and fill in the corresponding blocks of* $\mathbf{B}_\Gamma$ *by low-rank matrices, using* $\mathbf{T}_{i,h}$ *as the input data.*
- *Compute the inverse matrix* $\mathbf{B}_\Gamma^{-1}$ *by using the* $\mathcal{H}$*-matrix arithmetics.*

Cost estimates to compute $\mathbf{T}_{i,h}$ (in all cases storage is $O(N_{\Gamma_i} \log^q N_{\Gamma_i})$):

(i) Rather general variable coefficients of $\mathcal{L}$; FEM $\Rightarrow O(N_{\Omega_i} \log^q N_{\Omega_i})$.

(ii) Smooth coefficients in subdomains; boundary concentrated hp-FEM $\Rightarrow O(N_{\Gamma_i} \log^q N_{\Gamma_i})$.

(iii) Piecewise constant coefficients; BEM $\Rightarrow O(N_{\Gamma_i} \log^q N_{\Gamma_i})$.

Cost estimates for $\mathbf{B}_\Gamma \Rightarrow O(N_\Gamma \log^q N_\Gamma)$.

Final complexity for $\mathbf{B}_\Gamma^{-1} \Rightarrow O(N_\Gamma \log^q N_\Gamma)$.

On $\mathcal{H}$-Matrix Approximation to Poincaré-Steklov operators in case (B) (p.w. smooth coefficients)

Boundary concentrated hp-FEM (set a_{ij}, a_0 – analytic) **Kh., Melenk '02-'03** :

– Galerkin approximation by p.w. polynomials of degree $\leq \mathbf{p}$, $V_h = S^{\mathbf{p}}(\Omega, \mathcal{T})$

– Compute the local $\mathcal{H}$-matrix inverse $\mathbf{A}_{II}^{-1}$, $dim V_h = O(h^{-1})$

– Calculate the local Schur complement defined on the "interface" index set I_{Γ_i}

– *Advantage*: $O(N_\Gamma)$ complexity; handling locally complicated interface ($p = 1$ - preconditioning)

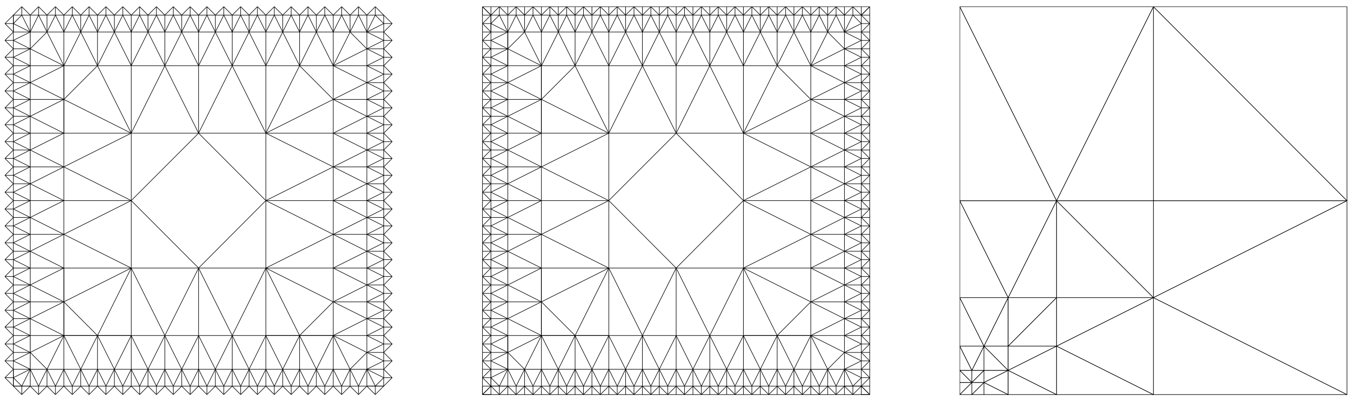

Figure 3: Refinement to the boundary (left, center) and to the corner (right).

BEM-Galerkin Approximation in case (C) (piecewise constant coefficients)

$$a_{\Omega_i}(u,v) := \mu_i \int_{\Omega_i} \nabla u \nabla v dx, \quad \mu_i > 0.$$

$$(\mathcal{V}_i u)(x) = \int_{\Gamma_i} g(x,y)u(y)dy, \qquad (\mathcal{K}_i u)(x) = \int_{\Gamma_i} \frac{\partial}{\partial n_y} g(x,y)u(y)dy,$$

$$(\mathcal{K}'_i u)(x) = \int_{\Gamma_i} \frac{\partial}{\partial n_x} g(x,y)u(y)dy, \qquad (\mathcal{D}_i u)(x) = -\frac{\partial}{\partial n_x} \int_{\Gamma_i} \frac{\partial}{\partial n_y} g(x,y)u(y)dy.$$

Introduce the *modified Calderon projection* $\mathcal{C}_{\Gamma_i}$

$$\mathcal{C}_{\Gamma_i} \begin{pmatrix} u_i \\ \delta_i \end{pmatrix} := \begin{pmatrix} \mu_i \mathcal{D} & \frac{1}{2} I + \mathcal{K}'_i \\ -\frac{1}{2} I - \mathcal{K}_i & \mu_i^{-1} \mathcal{V}_i \end{pmatrix} \begin{pmatrix} u_i \\ \delta_i \end{pmatrix} = \begin{pmatrix} \delta_i \\ 0 \end{pmatrix}, \qquad (0.6)$$

applied to the $\mathcal{L}_i$-harmonic function with $\delta_i = \mu_i \frac{\partial \overline{u}}{\partial n}$, and $-\Delta \overline{u} = 0$ in Ω_i.

The key point is that the Schur complement equation corresponding to (0.6) reads as

$$\mathcal{T}_i u_i := \mu_i \left(\mathcal{D}_i + (\frac{1}{2} I + \mathcal{K}'_i) \mathcal{V}_i^{-1} (\frac{1}{2} I + \mathcal{K}_i) \right) u_i = \delta_i,$$

$$\Rightarrow \quad \mathbf{T}_{ih} := \mu_i (\mathbf{D}_{ih} + (\frac{1}{2} \mathbf{I}_{ih} + \mathbf{K}'_{ih}) \mathbf{V}_{ih}^{-1} (\frac{1}{2} \mathbf{I}_{ih} + \mathbf{K}_{ih})). \qquad (0.7)$$

Implication of (0.7) and Error Analysis

Use the trace space $\Sigma_\Gamma := Y_\Gamma \times \Lambda_\Gamma$ with $\Lambda_\Gamma := \prod_{i=1}^{M} H^{-1/2}(\Gamma_i)$, equipped with the weighted norm

$$\|P\|^2_{\Sigma_\Gamma} = \|u\|^2_{Y_\Gamma} + \sum_{j=1}^{M} \mu_j^{-1} \|\lambda_j\|^2_{H^{-1/2}(\Gamma_j)}, \qquad P = (u,\lambda) \in \Sigma_\Gamma,\ \lambda = (\lambda_1, \dots, \lambda_M).$$

Define the interface bilinear form $c_\Gamma : \Sigma_\Gamma \times \Sigma_\Gamma \to \mathbb{R}$ by

$$c_\Gamma(P,Q) := \sum_{i=1}^{M} c_{\Gamma_i}(P_i; Q_i) \qquad \forall P = (u,\lambda), Q = (v,\eta) \in \Sigma_\Gamma,$$

with

$$\begin{aligned} c_{\Gamma_i}(u,\lambda; v,\eta) := \quad & \mu_i(\mathcal{D}_i u, v) + ((\tfrac{1}{2} I + \mathcal{K}_i')\lambda, v) \\ & -((\tfrac{1}{2} I + \mathcal{K}_i) u, \eta) + \mu_i^{-1}(\mathcal{V}_i \lambda, \eta) = (\delta_i, v) \quad \forall (v,\eta) \in Y_i \times \Lambda_i. \end{aligned}$$

The equation for u will be reduced to the following skew-symmetric variational interface problem (cf. **Hsiao, BNK, Wendland '01**):

Given $\Psi_\Gamma \in Y_\Gamma'$, *find* $P = (u,\lambda) \in \Sigma_\Gamma$ *such that*

$$c_\Gamma(P,Q) \ = \ \langle \Psi_\Gamma, v \rangle_\Gamma \qquad \text{for all } Q = (v,\eta) \in \Sigma_\Gamma. \tag{0.8}$$

BEM-Galerkin Saddle Point System

Using the FE Galerkin ansatz space of piecewise linear functions $\Sigma_h := Y_h \times \Lambda_h$ with $\Lambda_h := \prod_{i=1}^{M} \Lambda_{ih}$, we arrive at the BEM-Galerkin saddle-point system of equations:

Given $\Psi_\Gamma \in Y'_\Gamma$, find $P_h = (u_h, \lambda_h) \in Y_h \times \Lambda_h$ such that

$$c_\Gamma(P_h, Q) \; = \; \langle \Psi_\Gamma, v \rangle_\Gamma \qquad \text{for all } Q = (v, \eta) \in Y_h \times \Lambda_h. \tag{0.9}$$

Thm. 2 *(i) The bilinear form $c_\Gamma : \Sigma_\Gamma \times \Sigma_\Gamma \to \mathbb{R}$ is continuous and Σ_Γ-elliptic.*

(ii) Let P_h solve (0.9), then

$$\|P_h - P\|^2_{\Sigma_\Gamma} \leq c \inf_{(w,\mu)\in\Sigma_h} \left[\sum_{i=1}^{M} \mu_i \|u_i - w_i\|^2_{H^{1/2}(\Gamma_i)} + \sum_{i=1}^{M} \mu_i^{-1} \|\lambda_i - \mu_i\|^2_{H^{-1/2}(\Gamma_i)} \right].$$

(iii) Let $\mathbf{T}_{i,h}$ be the local BEM Schur complement given by (0.7). Then the BEM Schur complement matrix $\mathbf{B}_\Gamma \in \mathbb{R}^{I_\Gamma \times I_\Gamma}$, takes the form

$$\langle \mathbf{B}_\Gamma Z, V \rangle_{I_\Gamma} = \sum_{i=1}^{M} \langle \mathbf{T}_{i,h} Z_i, V_i \rangle_{I_{\Gamma_i}} = \sum_{i=1}^{M} \langle \mathbf{R}^T_{\Gamma_i} \mathbf{T}_{i,h} \mathbf{R}_{\Gamma_i} Z, V \rangle_{I_\Gamma}$$

which implies the explicit representation $\mathbf{B}_\Gamma = \sum_{i=1}^{M} \mathbf{R}^T_{\Gamma_i} \mathbf{T}_{i,h} \mathbf{R}_{\Gamma_i}$.

$\mathcal{H}$-Matrix Representation to $\mathbf{B}_\Gamma$

Given $\mathbf{T}_{i,h}$, how to calculate a low-rank approximation of blocks in $\mathcal{P}(I_\Gamma \times I_\Gamma)$?

1. SVD recompression of $b \in \mathcal{P}(I_\Gamma \times I_\Gamma)$ obtained as a sum of a fixed number of blocks extracted as rank-k sub-matrices in $\mathbf{T}_{i,h}$.

2. Compute only few entries of $b \in \mathbb{R}^{N_b \times N_b}$ and use the adaptive cross approximation (ACA) (cf. **Bebendorf, Rjasanow**) or WACA (cf. **Hackbusch, BNK, Kriemann '02**) on the target block.

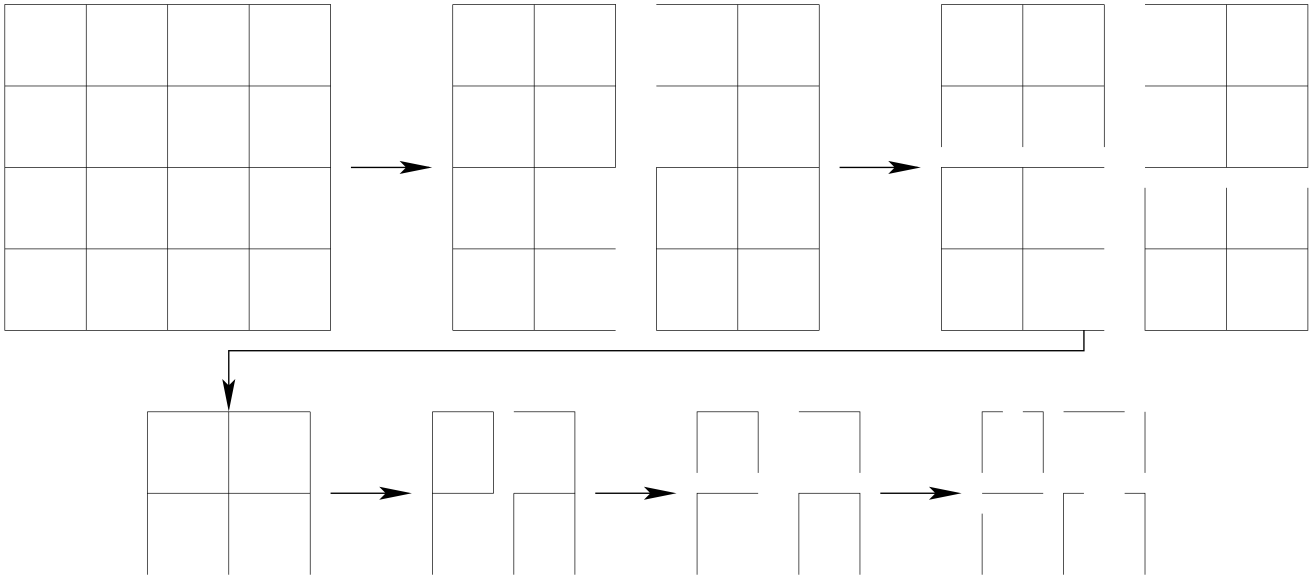

Figure 4: Construction of the cluster tree $T(I_\Gamma)$.

Numerics II: FEM-Galerkin Schur complement $\mathbf{B}_\Gamma$

$-\mu_i\Delta$, randomly chosen $\mu_i > 0$ (almost linear cost). $t(\mathbf{T}_{i,BEM}) \to$ OSTBEM-2D (O. Steinbach)

6×6 domains ($k = 9$)

N_Ω	N_Γ	$t(\mathbf{T}_{i,N})$	$t(\mathbf{T}_{i,BEM})$	$t(\mathbf{B}_{\Gamma,N}^{-1})$	$t(MV)$	$\|I - AA_{\mathcal{H}}^{-1}\|_2$
16 641	1 245	0.6 s	0.06 s	10.7 s	1.36_{10}-2 s	7.7_{10}-6
66 049	2 525	12.2 s	0.26 s	30.3 s	3.98_{10}-2 s	8.0_{10}-6
263 169	5 085	105.1 s	1.4 s	94.2 s	9.43_{10}-2 s	4.6_{10}-5
1 050 625	10 205	696.2 s	9.8 s	218.1 s	1.85_{10}-1 s	7.1_{10}-5

8×8 domains ($k = 9$)

N_Ω	N_Γ	$t(\mathbf{T}_{i,N})$	$t(\mathbf{B}_{\Gamma,N}^{-1})$	$t(MV)$	$\|I - AA_{\mathcal{H}}^{-1}\|_2$
16 641	1 729	0.1 s	13.9 s	2.26_{10}-2 s	6.9_{10}-6
66 049	3 521	3.8 s	41.2 s	5.38_{10}-2 s	2.3_{10}-5
263 169	7 105	43.3 s	126.8 s	1.27_{10}-1 s	3.9_{10}-5
1 050 625	14 273	180.7 s	326.7 s	2.66_{10}-1 s	4.4_{10}-5

Adaptive Choice of the Local Rank

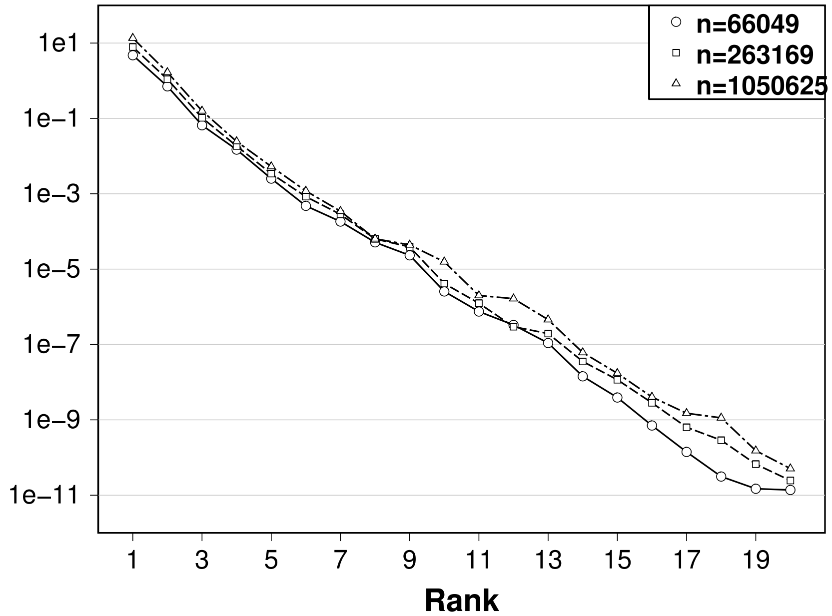

Figure 5: Preconditioning with low local ranks.

For *preconditioning* needs the local rank k can be chosen adaptively to achieve the required tolerance ε.

Multilevel (recursive) Computation of Local Schur Complements

$$\mathbf{B}_{\Gamma,i} = \begin{pmatrix} \mathbf{B}_{II} & \mathbf{B}_{I\Gamma_i} \\ \mathbf{B}_{\Gamma_i I} & \mathbf{B}_{\Gamma_i \Gamma_i} \end{pmatrix} \quad \text{then} \quad \mathbf{T}_{i,h} := \mathbf{B}_{\Gamma_i\Gamma_i} - \mathbf{B}_{\Gamma_i I}\mathbf{B}_{II}^{-1}\mathbf{B}_{I\Gamma_i}$$

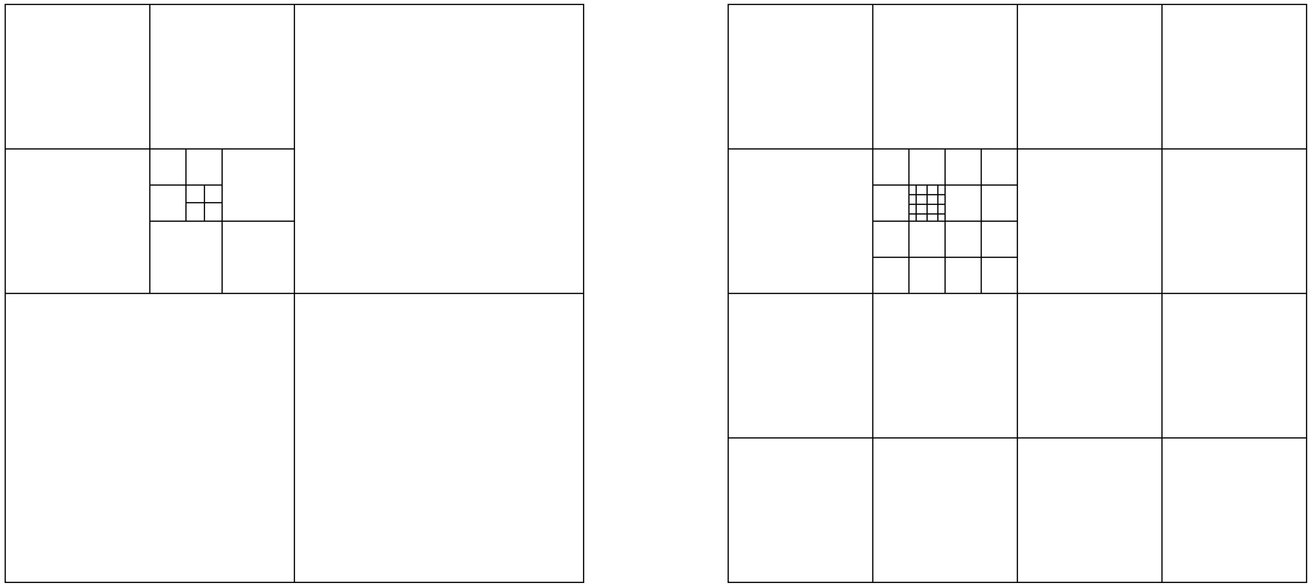

Figure 6: Multilevel 2×2 (left) and 4×4 decompositions.

The complexity bound satisfies a recursion

$$W(A_{i,\ell}^{-1}) = 16W(A_{i,\ell-2}^{-1}) + W(B_{\Gamma,\ell-2}^{-1}).$$

4 $\times$ 4 domains ($k = 9$)

n_Ω	n_Γ	$t(\Omega_i)$	$t(Inv)$	$t(MV)$	$\|I - AA_{\mathcal{H}}^{-1}\|_2$
16 641	753	3.8 s	3.7 s	3.20_{10}-3 s	4.2_{10}-6
66 049	1 521	43.2 s	16.9 s	9.10_{10}-3 s	7.7_{10}-6
263 169	3 057	317.4 s	48.3 s	4.18_{10}-2 s	1.3_{10}-5
1 050 625	6 129	2 020.1 s	118.8 s	8.92_{10}-1 s	2.1_{10}-5

$$W^{ML}(A_{4,\ell}^{-1}) = 16(16 \times 0.1 + 0.8) + 16.9 \approx 1\, min$$

shows that we gain a factor about 33 compared with $2020\; sec$ depicted in the last line.

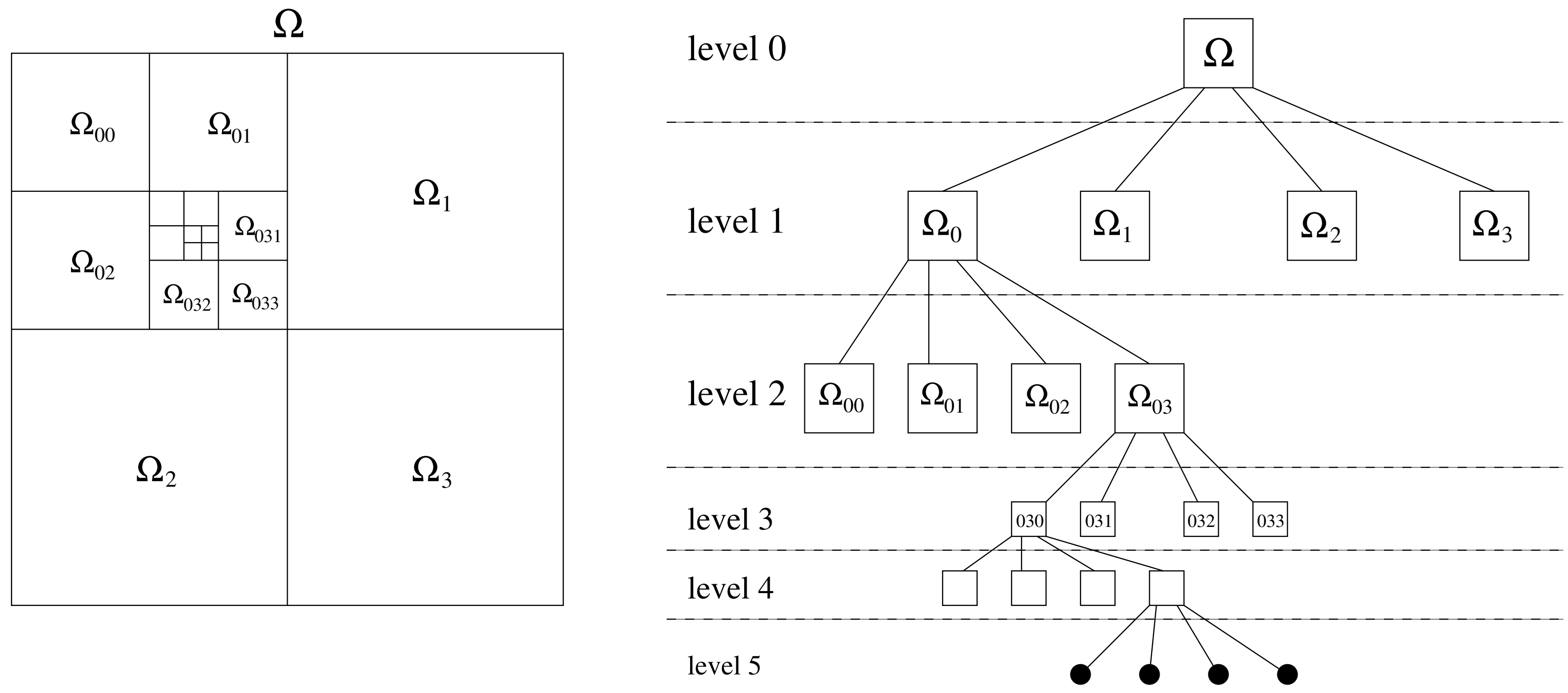

Figure 7: Multilevel parallel algorithm based on 2×2 decomposition.

Application to FETI/BETI Iterative Methods

FETI: **Farhat, ..., Widlund, Brenner**; BETI: **Langer, Steinbach '03**

FETI/BETI applies to a system that is algebraically equivalent to the Schur complement eq.

$$\mathbf{B}_\Gamma U = F := \sum_{i=1}^{M} \mathbf{R}_{\Gamma_i}^T F_i, \quad U, F \in \mathbb{R}^{I_\Gamma}, \tag{0.10}$$

where $F_i = \{\langle \Psi_i, \phi_j \rangle\}_{j \in I_{\Gamma_i}}$, and the matrix $\mathbf{B}_\Gamma = \sum_{i=1}^{M} \mathbf{R}_{\Gamma_i}^T \mathbf{T}_{i,h} \mathbf{R}_{\Gamma_i}$ can be derived by *any of the above described approaches*.

Now (0.10) is equvalent to the solution of a contraint minimization problem

$$\Phi(U) = \min_{V_1,\ldots,V_M:\ \sum_{i=1}^{M} \mathbf{B}_i V_i = 0} \Phi(V), \tag{0.11}$$

$$\Phi(V) := \sum_{i=1}^{M} \left[\frac{1}{2} \langle \mathbf{T}_{i,h} V_i, V_i \rangle_{I_{\Gamma_i}} - \langle F_i, V_i \rangle_{I_{\Gamma_i}} \right],$$

where each row of matching matrices $\mathbf{B}_i \in \mathbb{R}^{I_\Gamma \times I_{\Gamma_i}}$ is related with a pair of matching nodes in I_Γ. Each row has the entries $1, -1$ for the indices corresponding to the matching nodes and 0 otherwise.

Preconditioned Iteration for Dual Problem

Introducing the Lagrange multiplier Λ, (0.11) is reduced to a saddle point system

$$\begin{pmatrix} \mathbf{T}_{1,h} & ... & 0 & \mathbf{B}_1^T \\ \vdots & \ddots & \vdots & \vdots \\ 0 & ... & \mathbf{T}_{M,h} & \mathbf{B}_M^T \\ \mathbf{B}_1 & ... & \mathbf{B}_M & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ \vdots \\ U_M \\ \Lambda \end{pmatrix} = \begin{pmatrix} F_1 \\ \vdots \\ F_M \\ 0 \end{pmatrix}. \tag{0.12}$$

With $\mathbf{T}_D := blockdiag\{\mathbf{T}_{1,h}, ..., \mathbf{T}_{M,h}\}, \mathbf{B} := \{\mathbf{B}_1, ..., \mathbf{B}_M\}$ and $F := \{F_1, ..., F_M\}$, we obtain the dual formulation

$$\mathbf{B}\mathbf{T}_D^{-1}\mathbf{B}^T\Lambda = \mathbf{B}\mathbf{T}_D^{-1}F$$

which can be solved by the iterative PCG method using spectrally close preconditioner $\mathbf{C}$ of the form $\mathbf{C}^{-1} = \mathbf{G}^T\mathbf{T}_D\mathbf{G}$. Different proposals for $\mathbf{G}$ can be found in the *literature on the FETI methods*.

The key point: both $\mathbf{T}_D$ and $\mathbf{T}_D^{-1}$ can be computed and stored in the $\mathcal{H}$-matrix format with almost linear cost. Hence, the same is true for the corresponding matrix-by-vector multiplication with $\mathbf{B}\mathbf{T}_D^{-1}\mathbf{B}^T$ and $\mathbf{G}^T\mathbf{T}_D\mathbf{G}$ provided that $\mathbf{G}$ can be implemented with the linear expense.

Conclusions

1. Our *geometric* direct solver ($\mathcal{H}$-matrix based Schur complement/DD) is preferable vs. its *algebraic* version (global $\mathcal{H}$-matrix inverse):

(a) Sequential computation: $Mem_G = O(N_\Omega/M) + N_\Gamma < Mem_A = O(N_\Omega); T_G \leq 0.5 T_A$.

(b) Parallel computation: $T_G = O(N_\Omega/p + N_\Gamma) < T_A = O(N_\Omega + C N_\Omega \log^q N_\Omega/p)$; $Mem_G = Mem_A$.

2. Depending on the input data (variable, p.w. smooth or p.w. constant coefficients), one can apply three versions of the direct Schur complement/DD method.

3. In the FEM-version, the computation of $\mathbf{T}_{i,h}$ in subdomains dominates vs. the interface solver resulting in $O(N_\Omega \log^q N_\Omega)$-complexity.
In the BEM- and hp-FEM versions, we achieve $O(N_\Gamma)$ complexity (with $f = 0$ in $\Omega \setminus \Gamma$)

4. Further developments:
– implement the BEM-version
(promising 2D numerics based on the $\mathcal{H}$-matrix arithmetics $+$ OST code) ;
– couple with boundary concentrated hp-FEM;
– realise the recursive FEM-version (*multilevel, well parallelisable direct method*).