# Linearizations of a class of elliptic boundary value problems

**Jussi Behrndt***

Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

We construct linearizations for a class of second order elliptic eigenvalue dependent boundary value problems on smooth bounded domains with rational operator-valued Nevanlinna functions in the boundary condition.



## 1 Introduction

Various types of boundary value problems with eigenparameter dependent boundary conditions appear in many physical applications and have extensively been studied in a more or less abstract framework in the last decades. A lot of attention has been drawn to $\lambda$-dependent boundary value problems for ordinary differential operators and for large classes of functions in the boundary condition the theory is well understood. In this note we consider a second order elliptic differential expression $\mathcal{L}$ (which is regarded as an unbounded operator in $L^2(\Omega)$ defined on the so-called Beals space $\mathcal{D}_1(\Omega)$) subject to a $\lambda$-dependent boundary condition involving a rational $\mathfrak{L}(L^2(\partial\Omega))$-valued Nevanlinna function $\tau$ and the traces and conormal derivatives of the functions in $\mathcal{D}_1(\Omega)$, see (1) below and [3] for a more abstract treatment. Here $\mathfrak{L}(L^2(\partial\Omega))$ denotes the space of bounded everywhere defined linear operators in $L^2(\partial\Omega)$. In Theorem 3.1 we construct a self-adjoint operator $\widetilde{A}$ in the product Hilbert space $L^2(\Omega)\oplus L^2(\partial\Omega)\oplus\cdots\oplus L^2(\partial\Omega)$ and we show that its compressed resolvent $P_{L^2(\Omega)}(\widetilde{A}-\lambda)^{-1}|_{L^2(\Omega)}$ onto $L^2(\Omega)$ yields a solution of the $\lambda$-dependent boundary value problem (1). For the special case of $\lambda$-linear boundary condition we retrieve some results from [4, 5] in Corollary 3.2.

## 2 Elliptic differential operators defined on the Beals space

Let $\Omega$ be a bounded domain in $\mathbb{R}^m$ with $C^\infty$ boundary $\partial\Omega$ and closure $\overline{\Omega}$. We consider the differential expression

$$(\mathcal{L}f)(x) := -\sum_{j,k=1}^{m}\big(D_j a_{jk} D_k f\big)(x) + \sum_{j=1}^{m}\big(a_j D_j f - D_j\overline{a_j} f\big)(x) + a(x)f(x), \qquad x\in\Omega,$$

with coefficients $a_{jk}, a_j, a \in C^\infty(\overline{\Omega})$. We assume $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x\in\overline{\Omega}$ and $j,k=1,\ldots,m$, and that $a$ is real valued. Moreover, we assume that $\sum_{j,k=1}^m a_{jk}(x)\xi_j\xi_k \geq C\sum_{k=1}^m \xi_k^2$ holds for some constant $C>0$ and all $x\in\overline{\Omega}$, $(\xi_1,\ldots,\xi_m)^\top\in\mathbb{R}^m$, i.e., $\mathcal{L}$ is a uniformly elliptic differential expression which is symmetric. Denote by $n(x)$ the outward normal vector at $x\in\partial\Omega$. We say that $f\in H^2_{\mathrm{loc}}(\Omega)$ has $L^2$ *boundary value on* $\partial\Omega$ if the limit $f|_{\partial\Omega} := \lim_{\varepsilon\to 0+} f(x-\varepsilon n(x))$ exists in $L^2(\partial\Omega)$. The differential expression $\mathcal{L}$ is then regarded as an operator in $L^2(\Omega)$ which is defined on the so-called *Beals space* $\mathcal{D}_1(\Omega) := \big\{f\in L^2(\Omega) | \mathcal{L}f\in L^2(\Omega), f, \frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_m}$ have $L^2$ boundary values on $\partial\Omega\big\}$, cf. [2] and e.g. [1]. For the general spectral theory of operators associated with $\mathcal{L}$ we refer the reader to [7, 8, 9] and to e.g. [3, 6] for a more abstract extension theory. We recall that the mapping $\frac{\partial f}{\partial\nu}\big|_{\partial\Omega} := \sum_{j,k=1}^m a_{jk} n_j \frac{\partial f}{\partial x_k}\big|_{\partial\Omega} + \sum_{j=1}^m \overline{a_j} n_j f|_{\partial\Omega}$, $f\in\mathcal{D}_1(\Omega)$, is surjective onto $L^2(\partial\Omega)$ and Green's identity $(\mathcal{L}f,g)_\Omega - (f,\mathcal{L}g)_\Omega = (f|_{\partial\Omega}, \frac{\partial g}{\partial\nu}|_{\partial\Omega})_{\partial\Omega} - (\frac{\partial f}{\partial\nu}|_{\partial\Omega}, g|_{\partial\Omega})_{\partial\Omega}$ holds for all $f,g\in\mathcal{D}_1(\Omega)$.

## 3 An eigenvalue dependent elliptic boundary value problem

Let $A_i, B_i\in\mathfrak{L}(L^2(\partial\Omega))$, $i=1,\ldots,n$, be bounded self-adjoint operators in $L^2(\partial\Omega)$ and assume that the $B_i$ are uniformly positive, that is, $\sigma(B_i)\subset(0,\infty)$, $i=1,\ldots,n$. Then the function

$$\mathbb{C}\backslash\mathbb{R}\ni\lambda\mapsto\tau(\lambda) := A_1 + \lambda B_1 + \sum_{j=2}^{n} B_j^{1/2}(A_j-\lambda)^{-1}B_j^{1/2} \in \mathfrak{L}(L^2(\partial\Omega))$$

is an $\mathfrak{L}(L^2(\partial\Omega))$-valued *Nevanlinna function*, i.e., $\tau$ is holomorphic on $\mathbb{C}\backslash\mathbb{R}$, $\tau(\overline{\lambda}) = \tau(\lambda)^*$, $\lambda\in\mathbb{C}\backslash\mathbb{R}$, and Im $\tau(\lambda)$ is a nonnegative operator for all $\lambda\in\mathbb{C}^+$. Note that by the assumption $0\in\rho(B_i)$, $B_i\geq 0$, here the operator Im $\tau(\lambda)$ is even uniformly positive (uniformly negative) for $\lambda\in\mathbb{C}^+$ ($\lambda\in\mathbb{C}^-$, respectively). Moreover $\tau$ can be analytically continued to all real $\lambda$ which belong to $\rho(A_2)\cap\cdots\cap\rho(A_n)$.

* Corresponding author: e-mail: behrndt@math.tu-berlin.de, Phone: +49 30 314 23607, Fax: +49 30 314 21110

We consider the following elliptic boundary value problem: For a given $g \in L^2(\Omega)$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$ find a function $f \in \mathcal{D}_1(\Omega)$ such that

$$(\mathcal{L} - \lambda) f = g \qquad \text{and} \qquad \tau(\lambda)\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} + f|_{\partial\Omega} = 0 \tag{1}$$

holds. In the next theorem we show how this problem can be solved with the help of the compressed resolvent of a self-adjoint operator $\widetilde{A}$ in $L^2(\Omega) \oplus (L^2(\partial\Omega))^n$.

**Theorem 3.1** *The operator* $\widetilde{A}\{f, h_1, \ldots, h_n\} = \{\mathcal{L}f, h_1', \ldots, h_n'\}$ *defined on*

$$\operatorname{dom} \widetilde{A} = \left\{ \{f, h_1, \ldots, h_n\} : \begin{array}{l} f \in \mathcal{D}_1(\Omega),\ h_i, h_i' \in L^2(\partial\Omega),\ i = 1, \ldots, n, \\ \frac{\partial f}{\partial \nu}|_{\partial\Omega} = B_1^{-1/2} h_1 = B_j^{-1/2}(h_j' - A_j h_j),\ j = 2, \ldots, n, \\ f|_{\partial\Omega} = -A_1 B_1^{-1/2} h_1 - B_1^{1/2} h_1' + \sum_{j=2}^n B_j^{1/2} h_j \end{array} \right\}$$

*is self-adjoint in the Hilbert space* $L^2(\Omega) \oplus (L^2(\partial\Omega))^n$*. For all* $\lambda \in \mathbb{C}\backslash\mathbb{R}$ *the unique solution* $f \in \mathcal{D}_1(\Omega)$ *of the boundary value problem* (1) *is given by* $f = P_{L^2(\Omega)}(\widetilde{A} - \lambda)^{-1}|_{L^2(\Omega)} g$.

P r o o f. Let us first verify that $\widetilde{A}$ is well defined as an operator. In fact, if $f = 0$ and $h_1 = \cdots = h_n = 0$ then obviously $\mathcal{L}f = 0$ and the boundary conditions reduce to $0 = \frac{\partial f}{\partial \nu}|_{\partial\Omega} = B_2^{-1/2} h_2' = \cdots = B_n^{-1/2} h_n'$ and $0 = f|_{\partial\Omega} = -B_1^{1/2} h_1'$. Since $0 \in \rho(B_i)$, $i = 1, \ldots, n$ we obtain $h_1' = \cdots = h_n' = 0$, i.e., $\widetilde{A}$ is an operator. Next we check that $\widetilde{A}$ is symmetric in $L^2(\Omega) \oplus (L^2(\partial\Omega))^n$. For this, let $\tilde{f} = \{f, h_1, \ldots, h_n\}$, $\tilde{g} = \{g, k_1, \ldots, k_n\} \in \operatorname{dom} \widetilde{A}$ and $\widetilde{A}\tilde{f} = \{\mathcal{L}f, h_1', \ldots, h_n'\}$, $\widetilde{A}\tilde{g} = \{\mathcal{L}g, k_1', \ldots, k_n'\}$. Making use of Green's identity we obtain

$$\big(\widetilde{A}\tilde{f}, \tilde{g}\big) - \big(\tilde{f}, \widetilde{A}\tilde{g}\big) = \big(f|_{\partial\Omega}, \tfrac{\partial g}{\partial \nu}|_{\partial\Omega}\big)_{\partial\Omega} - \big(\tfrac{\partial f}{\partial \nu}|_{\partial\Omega}, g|_{\partial\Omega}\big)_{\partial\Omega} + \sum_{i=1}^n \big((h_i', k_i)_{\partial\Omega} - (h_i, k_i')_{\partial\Omega}\big) \tag{2}$$

and a straightforward calculation using the boundary conditions satisfied by $\tilde{f}, \tilde{g} \in \operatorname{dom} \widetilde{A}$ shows that (2) is zero and hence $\widetilde{A}$ is symmetric. For the self-adjointness of $\widetilde{A}$ it is now sufficient to prove $\operatorname{ran}(\widetilde{A} - \lambda_+) = \operatorname{ran}(\widetilde{A} - \lambda_-) = L^2(\Omega) \oplus L^2(\partial\Omega)^n$ for some $\lambda_+ \in \mathbb{C}^+$ and $\lambda_- \in \mathbb{C}^-$. This follows from a perturbation argument as in [3, Theorem 5.1].

Let now $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and set $f := P_{L^2(\Omega)}(\widetilde{A} - \lambda)^{-1}|_{L^2(\Omega)} g$. If we denote the element $P_{L^2(\partial\Omega)^n}(\widetilde{A} - \lambda)^{-1}|_{L^2(\Omega)} g$ by $\{h_1, \ldots, h_n\}$, $h_i \in L^2(\partial\Omega)$, then $\{f, h_1, \ldots, h_n\}$ belongs to $\operatorname{dom} \widetilde{A}$ and $\widetilde{A}\{f, h_1, \ldots, h_n\} = \{g + \lambda f, \lambda h_1, \ldots, \lambda h_n\}$ holds. Hence we have $\mathcal{L}f = g + \lambda f$ and it remains to show that the boundary condition in (1) is satisfied. In fact, since $\{f, h_1, \ldots, h_n\} \in \operatorname{dom} \widetilde{A}$ and $h_i' = \lambda h_i$, $i = 1, \ldots, n$, we obtain

$$\begin{aligned} \tau(\lambda)\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} &= \Big(A_1 + \lambda B_1 + \sum_{j=2}^n B_j^{1/2}(A_j - \lambda)^{-1} B_j^{1/2}\Big)\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} \\ &= (A_1 + \lambda B_1) B_1^{-1/2} h_1 + \sum_{j=2}^n B_j^{1/2}(A_j - \lambda)^{-1} B_j^{1/2} B_j^{-1/2}(\lambda h_j - A_j h_j) = -f|_{\partial\Omega} \end{aligned}$$

and hence $f = P_{L^2(\Omega)}(\widetilde{A} - \lambda)^{-1}|_{L^2(\Omega)} g$ solves the boundary value problem (1). The uniqueness follows as in [3]. □

**Corollary 3.2** *Let $A$ and $B$ be bounded self-adjoint operators in $L^2(\partial\Omega)$ and assume that $B$ is uniformly positive. Then*

$$\widetilde{A}\Big\{f, B^{1/2}\tfrac{\partial f}{\partial \nu}\big|_{\partial\Omega}\Big\} = \Big\{\mathcal{L}f, -B^{-1/2}\big(f|_{\partial\Omega} + A\tfrac{\partial f}{\partial \nu}\big|_{\partial\Omega}\big)\Big\}, \qquad f \in \mathcal{D}_1(\Omega),$$

*is a self-adjoint operator in $L^2(\Omega) \oplus L^2(\partial\Omega)$ and the unique solution $f \in \mathcal{D}_1(\Omega)$ of the $\lambda$-linear boundary value problem*

$$(\mathcal{L} - \lambda) f = g, \qquad (A + \lambda B)\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} + f|_{\partial\Omega} = 0,$$

*is given by $f = P_{L^2(\Omega)}(\widetilde{A} - \lambda)^{-1}|_{L^2(\Omega)} g$.*

## References


[1] W. O. Amrein, D. B. Pearson, J. Comput. Appl. Math. **171** (2004), 1–26.
[2] R. Beals, Amer. J. Math. **87** (1965), 315–362.
[3] J. Behrndt, M. Langer, *submitted*.
[4] P. Binding, R. Hryniv, H. Langer, B. Najman, J. Differential Equations **174** (2001), 30–54.
[5] J. Ercolano, M. Schechter, Comm. Pure Appl. Math. **18** (1965), 83–105 and 397–414.
[6] W. N. Everitt, L. Markus, Mem. Amer. Math. Soc. **162**, no. 770, (2003).
[7] G. Grubb, Ann. Scuola Norm. Sup. Pisa **22** (1968), 425–513.
[8] J. Lions, E. Magenes, Springer Verlag, Berlin Heidelberg New York (1972).
[9] M. I. Višik, Trudy Moskov. Mat. Obšč. **1** (1952), 187–246. (Russian)