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Bounds on Non-Real Eigenvalues of Indefinite Sturm-Liouville Problems

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It is known since the early 20th century that regular indefinite Sturm-Liouville problems may possess non-real eigenvalues. However, finding bounds for this set in terms of the coefficients of the differential expression has remained an open problem until recently. In this note we prove a variant of a recent result in [1] on the bounds for the non-real eigenvalues of an indefinite Sturm-Liouville problem with Dirichlet boundary conditions.

1 Introduction

We consider regular weighted Sturm-Liouville eigenvalue problems of the form

$$-\phi''(x) + q(x)\phi(x) = \lambda w(x)\phi(x), \qquad \phi(a) = \phi(b) = 0, \qquad x \in [a,b],\ \ \lambda \in \mathbb{C}; \tag{1}$$

where the weight function w and the potential q are assumed to be real-valued integrable functions. Moreover it is assumed that w takes on positive and negative values on subsets of $[a,b]$ with positive Lebesgue measure, so that the problem becomes *indefinite*.

The theory of indefinite Sturm-Liouville problems goes back to works by Haupt [3] and Richardson [6] from the early 20th century. In both papers it is pointed out that (1) may possess a finite number of non-real eigenvalues. Mingarelli noted in an interesting survey article [5] that no a priori bounds on these eigenvalues in terms of the coefficients w and q and the boundary conditions have been found so far and that it would be desirable to obtain such (see also [4, Remark 4.4] for a similar remark concerning the singular case).

The situation has developed rapidly in the very recent past. The first results of the desired form were obtained by the first and third author jointly with Trunk in [2] for a singular problem. Independently, the second and fourth author found a priori bounds in [7] for the regular case. Both contributions [2, 7] consider special cases such as, e.g., only one turning point of the weight w. The regular problem was then solved almost completely in the even more recent paper [1]. In fact, a priori bounds on the non-real eigenvalues were obtained for all selfadjoint boundary conditions, all potentials q, and all weight functions w for which there exists an absolutely continuous function $g \in H^1(a,b)$ such that $\operatorname{sgn}(g) = \operatorname{sgn}(w)$ a.e. on (a,b). In the present note we illustrate the core of our methods by proving a variant of the main theorem in [1] for the special case of Dirichlet boundary conditions.

2 Main Result

We state and prove a variant of Theorem 3.2 in [1] which provides a priori bounds on the non-real eigenvalues of the indefinite Sturm-Liouville problem (1). For a real-valued function f on $[a,b]$ we set $f_\pm(x) := \max\{0, \pm f(x)\}$, so that $f = f_+ - f_-$.

Theorem 2.1 *Assume that there exists a function $g \in H^1(a,b)$ such that $gw > 0$ a.e. on (a,b) and let $\varepsilon > 0$ be such that*

$$\big|\{x \in (a,b) : g(x)w(x) < \varepsilon\}\big| \;\le\; \frac{1}{8(b-a)\|q_-\|_1^2}\,.$$

Then for any non-real eigenvalue $\lambda \in \mathbb{C}\setminus\mathbb{R}$ of problem (1) *we have*

$$|\operatorname{Im}\lambda| \;\le\; \frac{8}{\varepsilon}\sqrt{b-a}\,\|q_-\|_1^2\|g'\|_2 \qquad \textit{and} \qquad |\operatorname{Re}\lambda| \;\le\; \frac{8}{\varepsilon}\|q_-\|_1^2\left(\sqrt{b-a}\|g'\|_2 + 2(b-a)\|q_-\|_1\|g\|_\infty\right).$$

P r o o f. Let ϕ be an eigenfunction corresponding to λ, that is, ϕ satisfies (1). WLOG we may assume $\|\phi\|_2 = 1$. Multiplication of the ODE in (1) by $\overline{\phi}$, followed by integration over $[x,b]$, yields

$$\lambda\int_x^b w|\phi|^2 = \phi'(x)\overline{\phi(x)} + \int_x^b \left(|\phi'|^2 + q|\phi|^2\right). \tag{2}$$

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Taking the real and the imaginary part of (2) gives

$$(\operatorname{Re}\lambda)\int_x^b w|\phi|^2 = \operatorname{Re}\left(\phi'(x)\overline{\phi(x)}\right) + \int_x^b \left(|\phi'|^2 + q|\phi|^2\right) \tag{3}$$

and

$$(\operatorname{Im}\lambda)\int_x^b w|\phi|^2 = \operatorname{Im}\left(\phi'(x)\overline{\phi(x)}\right). \tag{4}$$

Hence, by setting $x = a$ in (4) and (3) we obtain

$$\int_a^b w|\phi|^2 = \int_a^b \left(|\phi'|^2 + q|\phi|^2\right) = 0. \tag{5}$$

Now, for each $x \in [a, b]$ we have

$$|\phi(x)| = \left|\int_a^x \phi'\right| \le \int_a^x |\phi'| \le \sqrt{b-a}\,\|\phi'\|_2. \tag{6}$$

Moreover, putting $Q(x) := \int_a^x q_-(t)\,dt$ for $x \in [a, b]$, the relation (5) yields

$$\|\phi'\|_2^2 = -\int_a^b q|\phi|^2 \le \int_a^b q_-|\phi|^2 = \int_a^b Q'|\phi|^2 = -2\int_a^b Q\operatorname{Re}(\phi\overline{\phi'}) \le 2\|q_-\|_1\|\phi'\|_2,$$

and thus $|\|\phi'\|_2 - \|q_-\|_1| \le \|q_-\|_1$. This together with (6) leads to

$$\|\phi\|_\infty \le 2\sqrt{b-a}\,\|q_-\|_1 \qquad \text{and} \qquad \|\phi'\|_2 \le 2\|q_-\|_1. \tag{7}$$

Now, let $\Omega := \{x \in (a, b) : g(x)w(x) < \varepsilon\}$. Then we obtain from (5) and (7) that

$$\int_a^b g'(x)\int_x^b w(t)|\phi(t)|^2\,dt\,dx = \int_a^b gw|\phi|^2 \ge \varepsilon\int_{\Omega^c}|\phi|^2 = \varepsilon\left(1 - \int_\Omega |\phi|^2\right) \ge \varepsilon\big(1 - \|\phi\|_\infty^2|\Omega|\big) \ge \frac{\varepsilon}{2}.$$

Hence, the relations (4) and (7) imply

$$\frac{\varepsilon}{2}|\operatorname{Im}\lambda| \le \left|\int_a^b g'\operatorname{Im}(\phi'\overline{\phi})\right| \le \int_a^b |g'\phi\phi'| \le \|\phi\|_\infty\|g'\|_2\|\phi'\|_2 \le 4\sqrt{b-a}\|q_-\|_1^2\|g'\|_2,$$

so that the estimate on $|\operatorname{Im}\lambda|$ is established. For the real part we exploit (3) and (5) to derive

$$\frac{\varepsilon}{2}|\operatorname{Re}\lambda| \le \left|\int_a^b g'(x)\left(\operatorname{Re}\left(\phi'(x)\overline{\phi(x)}\right) + \int_x^b \left(|\phi'|^2 + q|\phi|^2\right)\right)dx\right| \le \|\phi\|_\infty\|g'\|_2\|\phi'\|_2 + \left|\int_a^b g\big(|\phi'|^2 + q|\phi|^2\big)\right|.$$

Now, setting $D_+ := |\phi'|^2 + q_+|\phi|^2$, $D_- := q_-|\phi|^2$, and $D := D_+ - D_- = |\phi'|^2 + q|\phi|^2$, we obtain

$$\left|\int_a^b gD\right| \le \int_a^b (g_\pm D_+ + g_\mp D_-) \le \|g\|_\infty\int_a^b (D + 2D_-) = 2\|g\|_\infty\int_a^b q_-|\phi|^2 \le 2\|g\|_\infty\|\phi\|_\infty^2\|q_-\|_1.$$

The estimate on $|\operatorname{Re}\lambda|$ in the theorem now follows from (7). □

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