A REALIZATION THEOREM FOR GENERALIZED NEVANLINNA FAMILIES

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Dedicated to the memory of Branko Najman

Abstract. Boundary relations for a symmetric relation in a Pontryagin space are studied and the corresponding Weyl families are characterized. In particular, it is shown that every generalized Nevanlinna family can be realized as the Weyl family of a boundary relation in a Pontryagin space.

1. Introduction

The notions of boundary triplets and associated Weyl functions play a key role in the extension theory of symmetric operators and relations. By means of a boundary triplet all selfadjoint extensions of a given symmetric operator can be parametrized and their spectral properties can be described efficiently with the help of the Weyl function; see, e.g. [10, 11]. If the underlying space is a Hilbert space or a Pontryagin space, then the Weyl function belongs to the class of Nevanlinna or generalized Nevanlinna functions, respectively, and satisfies an additional strictness condition. Conversely, every Nevanlinna or generalized Nevanlinna function with this additional strictness property can be realized as a Weyl function of a boundary triplet.

The concept of boundary relations and associated Weyl families for symmetric operators and relations in Hilbert spaces was introduced in [8] and studied further in [9]. The notion of boundary relation is a generalization of the notion of boundary triplet which makes it possible to interpret all Nevanlinna functions and even so-called Nevanlinna families as Weyl families. This was shown in [8] with the help of the Naimark dilation theorem; in [2, 3, 7] an alternative realization in reproducing kernel Hilbert spaces was given.

In the present paper the main interest is in the notions of boundary relations and Weyl families in a Pontryagin space setting; for the Kreĭn space case see also [4]. Many of the basic definitions and facts from the Hilbert space case remain the same in the indefinite setting due to one of the key observations in [8]: boundary triplets and relations are unitary relations in a Kreĭn space sense. However, in the Pontryagin (and Kreĭn) space case certain new difficulties arise: it may happen that the so-called main transform of the boundary relation leads to a selfadjoint relation which has an empty resolvent set, see Example 3.7. As becomes clear from the considerations below, the Pontryagin space setting, when compared with the

Key words and phrases. Boundary relation, Weyl family, boundary triplet, Weyl function, Kreĭn space, Pontryagin space, unitary relation.

This research was supported by the grants from the Academy of Finland (project 128085) and the German Academic Exchange Service (DAAD project D/08/08852). The third author thanks the Deutsche Forschungsgemeinschaft (DFG) for the Mercator visiting professorship at the Technische Universität Berlin.

Hilbert space case, yields a somewhat more delicate interplay between the geometric properties of boundary relations and the analytic properties of their Weyl families. The treatments needed here are important in establishing the connection to the class of generalized Nevanlinna families.

In an earlier paper [9] the coupling method was introduced to deal with the selfadjoint extensions in an exit space (being a Hilbert space) of a given symmetric operator or relation in a Hilbert space. The present results make it possible to also consider selfadjoint extensions in an exit space which is allowed to be a Pontryagin space. Hence this paper gives a possibility to extend the scope of the applications of the theory of boundary relations to abstract boundary value problems with eigenparameter dependent boundary conditions.

Here is a short overview over the contents of the paper. In Section 2 some definitions and preparatory facts on linear operators and relations on Kreĭn and Pontryagin spaces are given. In Section 3 boundary relations in Pontryagin and Kreĭn spaces are considered involving a study of their main transforms with empty resolvent sets. In Section 4 the main results of the paper are established: it will be shown that every Weyl family associated to a boundary relation of a symmetric operator or relation in a Pontryagin space is a generalized Nevanlinna family (Theorem 4.8), and that, conversely, every generalized Nevanlinna family can be realized as the Weyl family of a boundary relation in a Pontryagin space (Theorem 4.10). For this converse statement a Pontryagin space variant of the functional model from [2, 3, 7] is established. A partial case of Theorem 4.10 for normalized generalized Nevanlinna pairs was formulated in other terms and proved in [21].

2. Preliminaries

2.1. **Linear relations in Banach spaces.** Let $\mathfrak{H}$ and $\mathfrak{K}$ be linear spaces. A *linear relation* (multivalued operator) S from $\mathfrak{H}$ to $\mathfrak{K}$ is a linear subspace of the product space $\mathfrak{H} \times \mathfrak{K}$. For the usual definitions concerning operations with relations, see for instance [15]. The domain, kernel, range, and multivalued part of a linear relation S from a linear space $\mathfrak{H}$ to a linear space $\mathfrak{K}$ will be denoted by $\operatorname{dom} S$, $\ker S$, $\operatorname{ran} S$, and $\operatorname{mul} S$, respectively. The elements in a linear relation S will usually be written in the form

$$\{f, f'\} \quad \text{or} \quad \begin{pmatrix} f \\ f' \end{pmatrix}, \qquad \text{where } f \in \operatorname{dom} S,\ f' \in \operatorname{ran} S.$$

Linear operators from $\mathfrak{H}$ into $\mathfrak{K}$ are viewed as linear relations via their graphs.

Now consider the case where $\mathfrak{H}$ and $\mathfrak{K}$ are Banach spaces. A linear relation S from $\mathfrak{H}$ to $\mathfrak{K}$ is said to be *closed* if S is closed as a subspace of the product space $\mathfrak{H} \times \mathfrak{K}$. The linear space of bounded linear operators defined on $\mathfrak{H}$ with values in $\mathfrak{K}$ is denoted by $\mathbf{B}(\mathfrak{H}, \mathfrak{K})$ and by $\mathbf{B}(\mathfrak{H})$ when $\mathfrak{K} = \mathfrak{H}$. Let S be a closed linear relation in $\mathfrak{H}$. The set of *points of regular type* $\widehat{\rho}(S)$ of S is the set of all $\lambda \in \mathbb{C}$ such that $(S - \lambda)^{-1}$ is a bounded linear operator (defined on $\operatorname{ran}(S - \lambda)$). The *resolvent set* $\rho(S)$ of S is the set of all $\lambda \in \mathbb{C}$ such that $(S - \lambda)^{-1} \in \mathbf{B}(\mathfrak{H})$; the *spectrum* $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. A point $\lambda \in \mathbb{C}$ is an *eigenvalue* of S if $\ker(S - \lambda) \neq \{0\}$. The set of all eigenvalues of S is denoted by $\sigma_p(S)$. When $\lambda \in \mathbb{C}$ is an eigenvalue of S the corresponding linear space of eigenelements is denoted by $\mathfrak{N}_\lambda(S) := \ker(S - \lambda)$ and, furthermore,

$$\widehat{\mathfrak{N}}_\lambda(S) = \big\{ \{f_\lambda, \lambda f_\lambda\} : \ f_\lambda \in \mathfrak{N}_\lambda(S) \big\}.$$

The following situation occurs frequently.

Lemma 2.1. *Let $\mathfrak{H}$ be a Banach space and let $A, B \in \mathbf{B}(\mathfrak{H})$. Define the linear relation S in $\mathfrak{H}$ by*

$$S = \{\, \{Ah, Bh\} : \, h \in \mathfrak{H} \,\}.$$

If $B - \lambda A$ is boundedly invertible for some $\lambda \in \mathbb{C}$, then the relation S is closed and $\lambda \in \rho(S)$. Moreover, $(S-\lambda)^{-1} = A(B-\lambda A)^{-1}$.

2.2. **Linear relations and operators in Kreĭn spaces.** Let $(\mathfrak{H}, [\cdot,\cdot])$ be a Kreĭn space and let $J_{\mathfrak{H}}$ be a corresponding fundamental symmetry. Then

$$(\cdot,\cdot) := [J_{\mathfrak{H}}\cdot,\cdot]$$

defines a scalar product on $\mathfrak{H}$ such that $[\cdot,\cdot]$ is continuous with respect to the norm induced by $(\cdot,\cdot)$ and $(\mathfrak{H}, (\cdot,\cdot))$ is a Hilbert space; see [1]. In the following all topological notions are understood with respect to the norm $\|\cdot\|$ induced by $(\cdot,\cdot)$.

Let $U \subset \mathfrak{H} \times \mathfrak{K}$ be a linear relation from the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ to the Kreĭn space $(\mathfrak{K}, [\cdot,\cdot]_{\mathfrak{K}})$. The *adjoint* U^+ is defined by

$$U^+ := \left\{ \{\, \tilde{k}, \tilde{h}\} \in \mathfrak{K} \times \mathfrak{H} \,:\, [k, \tilde{k}]_{\mathfrak{K}} = [h, \tilde{h}]_{\mathfrak{H}} \text{ for all } \{h,k\} \in U \right\}.$$

It is a closed linear relation from $\mathfrak{K}$ to $\mathfrak{H}$, i.e., a closed subspace of $\mathfrak{K} \times \mathfrak{H}$ when considered as the direct sum of the corresponding Hilbert spaces. A linear relation $U \subset \mathfrak{H} \times \mathfrak{K}$ is said to be *isometric* if $U^{-1} \subset U^+$ and *unitary* if $U^{-1} = U^+$. Recall that a unitary relation U from $\mathfrak{H}$ to $\mathfrak{K}$ satisfies

$$\ker U = (\operatorname{dom} U)^{[\perp]_{\mathfrak{H}}} \qquad \text{and} \qquad \operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_{\mathfrak{K}}}.$$

Furthermore, recall that $\operatorname{dom} U$ is closed if and only if $\operatorname{ran} U$ is closed (see [23]). Unitary relations may be multivalued; and single-valued unitary relations may be unbounded. A unitary relation from $\mathfrak{H}$ to $\mathfrak{K}$ is said to be a *standard unitary operator* if it is (the graph of) an operator belonging to $\mathbf{B}(\mathfrak{H}, \mathfrak{K})$. The inverse of a standard unitary operator is automatically also a standard unitary operator.

Finally, a linear relation $A \subset \mathfrak{H} \times \mathfrak{H}$ in a Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ is said to be *symmetric* if $A \subset A^+$ and *selfadjoint* if $A = A^+$. A selfadjoint relation in a Kreĭn space (or in a Pontryagin space) may have an empty resolvent set; cf. Example 3.7.

2.3. **Linear relations and operators in Pontryagin spaces.** Symmetric and selfadjoint relations in Pontryagin spaces have some useful properties. If S is a symmetric operator in a Pontryagin space $\mathfrak{H}$, then automatically the set of points of regular type is nonempty: $\widehat{\rho}(S) \neq \emptyset$; in fact, then $\#(\mathbb{C}_\pm \setminus \widehat{\rho}(S)) \leq \kappa$. If A is a selfadjoint operator in a Pontryagin space with κ negative squares, then the upper and lower half planes consist of points of the resolvent set with the exception of at most κ eigenvalues in each of the half planes; see [1], [16]. The situation is different for selfadjoint relations in a Pontryagin space; see [14].

Lemma 2.2. *Let A be a selfadjoint linear relation in a Pontryagin space $\mathfrak{H}$ with κ negative squares. Then $\sigma(A) = \mathbb{C}$ if and only if $\sigma_p(A)$ contains at least $\kappa+1$ points in $\mathbb{C}_+$ or in $\mathbb{C}_-$. In this case*

$$\operatorname{span}\{\, \ker(A-\lambda) : \lambda \in \mathbb{C}_+ \,\} \quad \textit{and} \quad \operatorname{span}\{\, \ker(A-\lambda) : \lambda \in \mathbb{C}_- \,\}$$

are neutral subspaces of $\mathfrak{H}$, which contain at least one nontrivial vector from $\operatorname{mul} A$.

Proof. For the first statement, see [14]. Now let $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{\kappa+1}$ be different eigenvalues of A in $\mathbb{C}_+$ and let

$$\{u_j, \lambda_j u_j\} \in A, \quad u_j \neq 0, \quad j = 1, 2, \ldots, \kappa+1. \tag{2.1}$$

Then $[u_j, u_k] = 0$ for all $j, k = 1, 2, \ldots, \kappa+1$ and thus, the subspace

$$\mathfrak{L} = \operatorname{span}\{u_j : j = 1, 2, \ldots, \kappa+1\}$$

is neutral. Since $\mathfrak{H}$ is a Pontryagin space with negative index κ the dimension d of the neutral subspace $\mathfrak{L}$ is less or equal to κ, and hence the vectors u_j are linearly dependent. Enumerate u_j such that $u_1, \ldots, u_d$ form a basis in $\mathfrak{L}$. Then there are $\alpha_j \in \mathbb{C}$ such that $u_{d+1} = \sum_{j=1}^d \alpha_j u_j$ and

$$\left\{ u_{d+1}, \sum_{j=1}^{d} \alpha_j \lambda_j u_j \right\} \in A. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\left\{ 0\,, \sum_{j=1}^{d} \alpha_j (\lambda_j - \lambda_{d+1}) u_j \right\} \in A,$$

and, hence,

$$u_\infty := \sum_{j=1}^{d} \alpha_j (\lambda_j - \lambda_{d+1}) u_j \in \operatorname{mul} A.$$

If $u_\infty = 0$, then $\alpha_j = 0$ for all $j = 1, \ldots, d$ and hence $u_{d+1} = 0$, a contradiction. Thus u_∞ is not trivial. $\square$

Corollary 2.3. *If A is a selfadjoint linear relation in a Pontryagin space $\mathfrak{H}$ such that $\sigma(A) = \mathbb{C}$, then $\operatorname{mul} A$ contains at least one nontrivial neutral vector. In particular, if $\operatorname{mul} A$ is either a positive subspace or a negative subsoace of $\mathfrak{H}$, then $\rho(A) \neq \emptyset$.*

Lemma 2.4. *Let K be a closed densely defined linear operator acting from a Hilbert space $(\mathcal{L}, (\cdot,\cdot))$ to a Pontryagin space $(\mathfrak{H}, [\cdot,\cdot])$ of negative index κ. Then the operator K^+K is selfadjoint in $\mathcal{L}$ and the form $[K\cdot, K\cdot]$ is semibounded from below on $\operatorname{dom} K$.*

Proof. It follows from $(K^+Kx, y) = [Kx, Ky] = (x, K^+Ky)$, $x, y \in \operatorname{dom}(K^+K)$, that K^+K is a symmetric operator in $\mathcal{L}$. Since K^+K has at most κ negative eigenvalues, one can assume that $-1 \notin \sigma_p(K^+K)$. Then the subspace

$$\mathcal{K} = \{\{Ku, u\} : u \in \operatorname{dom} K\}$$

is a closed nondegenerate subspace of $\mathfrak{H} \oplus \mathcal{L}$ and its orthogonal complement in $\mathfrak{H} \oplus \mathcal{L}$ takes the form

$$\mathcal{K}^{[\perp]} = \left\{\{f, -K^+f\} : f \in \operatorname{dom} K^+\right\}.$$

Since $\mathfrak{H} \oplus \mathcal{L}$ is a Pontryagin space it follows that $\mathfrak{H} \oplus \mathcal{L} = \mathcal{K}[+]\mathcal{K}^{[\perp]}$ holds; cf. [1, Theorem 9.9], [16, Theorem 3.2]. Hence there exist $u \in \operatorname{dom} K$ and $f \in \operatorname{dom} K^+$ such that

$$\{0, h\} = \{Ku, u\} + \{f, -K^+f\}$$

This implies $u \in \operatorname{dom}(K^+K)$, $h = (I+K^+K)u$ and hence $\operatorname{ran}(I+K^+K) = \mathcal{L}$. Thus $-1 \in \rho(K^+K)$ and it follows that K^+K is a selfadjoint operator in $\mathcal{L}$. Furthermore, as $\mathfrak{H}$ is a Pontryagin space with negative index κ it follows that the spectral subspace of K^+K corresponding to the negative spectrum has at most dimension κ and therefore K^+K is semibounded from below.

To see that also the form $[K\cdot, K\cdot]$ is semibounded from below on $\operatorname{dom} K$ it suffices to prove that

$$\mathcal{K}_0 := \{\{Ku, u\} : u \in \operatorname{dom}(K^+K)\}$$

is dense in $\mathcal{K}$. Indeed, if $\{Kv, v\}$ is orthogonal to $\mathcal{K}_0$ for some $v \in \operatorname{dom} K$ then

$$(v, (I + K^+K)u) = (v, u) + [Kv, Ku] = 0$$

for all $u \in \operatorname{dom}(K^+K)$ and hence $v = 0$. □

3. Boundary relations and Weyl families

In this section boundary relations and their Weyl families in Kreĭn spaces will be studied. It will be shown that on an algebraic level there is a close connection between the Kreĭn space and the Hilbert space situations.

3.1. **Boundary relations in Kreĭn spaces.** Let $(\mathfrak{H}, [\cdot,\cdot])$ be a Kreĭn space and let $J_{\mathfrak{H}}$ be a corresponding fundamental symmetry. The product space $\mathfrak{H}^2 = \mathfrak{H} \times \mathfrak{H}$ will be equipped with the indefinite inner product

$$\llbracket \hat{f}, \hat{g} \rrbracket_{\mathfrak{H}^2} = i\big([f, g'] - [f', g]\big), \quad \hat{f} = \{f, f'\},\ \hat{g} = \{g, g'\} \in \mathfrak{H}^2. \tag{3.1}$$

Then $(\mathfrak{H}^2, \llbracket\cdot,\cdot\rrbracket_{\mathfrak{H}^2})$ is also a Kreĭn space and

$$\begin{pmatrix} 0 & -iJ_{\mathfrak{H}} \\ iJ_{\mathfrak{H}} & 0 \end{pmatrix} \in \mathbf{B}(\mathfrak{H}^2)$$

is a corresponding fundamental symmetry. In the following $(\mathcal{H}, (\cdot,\cdot))$ will be a Hilbert space. By replacing the inner product $[\cdot,\cdot]$ on the right hand side of (3.1) with $(\cdot,\cdot)$ the product space $\mathcal{H}^2$ equipped with the corresponding indefinite inner product $\llbracket\cdot,\cdot\rrbracket_{\mathcal{H}^2}$ is also a Krein space with the fundamental symmetry

$$J_{\mathcal{H}^2} = \begin{pmatrix} 0 & -iI_{\mathcal{H}} \\ iI_{\mathcal{H}} & 0 \end{pmatrix} \in \mathbf{B}(\mathcal{H}^2). \tag{3.2}$$

Let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a linear relation from the Kreĭn space $(\mathfrak{H}^2, \llbracket\cdot,\cdot\rrbracket_{\mathfrak{H}^2})$ to the Kreĭn space $(\mathcal{H}^2, \llbracket\cdot,\cdot\rrbracket_{\mathcal{H}^2})$ and denote the adjoint of Γ by $\Gamma^{\llbracket+\rrbracket}$. Then Γ is $\llbracket\cdot,\cdot\rrbracket$-*isometric* or $\llbracket\cdot,\cdot\rrbracket$-*unitary* if Γ is an isometric or unitary relation from $(\mathfrak{H}^2, \llbracket\cdot,\cdot\rrbracket_{\mathfrak{H}^2})$ to $(\mathcal{H}^2, \llbracket\cdot,\cdot\rrbracket_{\mathcal{H}^2})$; i.e., if $\Gamma^{-1} \subset \Gamma^{\llbracket+\rrbracket}$ or $\Gamma^{-1} = \Gamma^{\llbracket+\rrbracket}$, respectively.

Definition 3.1. *Let S be a closed symmetric relation in a Kreĭn space $\mathfrak{H}$. A linear relation $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ is called a boundary relation for S^+ if $\mathcal{H}$ is a Hilbert space, $T := \operatorname{dom}\Gamma$ is dense in S^+ w.r.t. the graph topology on S^+ (induced by the Hilbert space inner product on $\mathfrak{H}^2 = \mathfrak{H} \times \mathfrak{H}$), and Γ is $\llbracket\cdot,\cdot\rrbracket$-unitary.*

One can restate this definition also as follows: a $\llbracket\cdot,\cdot\rrbracket$-unitary relation $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ is a boundary relation for S^+ if and only if $\ker\Gamma = S$; see [8, Proposition 2.3]. The space $\mathcal{H}$ in Definition 3.1 is the parameter space of the boundary relation Γ; it plays the role of the boundary space in applications to ODE's and PDE's.

The product space $\mathfrak{H}^2$ can be provided with another indefinite inner product

$$\begin{aligned} \ll \hat{f}, \hat{g} \gg_{\mathfrak{H}^2} &= i\big((f, g') - (f', g)\big) \\ &= i\big([J_{\mathfrak{H}} f, g'] - [J_{\mathfrak{H}} f', g]\big), \quad \hat{f} = \{f, f'\},\ \hat{g} = \{g, g'\} \in \mathfrak{H}^2, \end{aligned}$$

where $(\cdot,\cdot) = [J_{\mathfrak{H}}\cdot,\cdot]$ is the Hilbert space inner product on $\mathfrak{H}$. The fundamental symmetry $J_{\mathfrak{H}}$ can be used to give a connection between different classes of linear relations in Kreĭn spaces and Hilbert spaces. This is formulated in the next lemma, whose proof is straightforward.

Lemma 3.2. *The linear operator $U_{J_{\mathfrak{H}}}$ from the Kreĭn space $(\mathfrak{H}^2, [\![\cdot,\cdot]\!]_{\mathfrak{H}^2})$ to the Kreĭn space $(\mathfrak{H}^2, \ll\cdot,\cdot\gg_{\mathfrak{H}^2})$ defined by*

$$U_{J_{\mathfrak{H}}}\{f,f'\} = \{f, J_{\mathfrak{H}}f'\}, \quad \{f,f'\} \in \mathfrak{H}^2, \tag{3.3}$$

is a standard unitary operator. Furthermore, for every linear relation A in $\mathfrak{H}^2$,

$$U_{J_{\mathfrak{H}}}(A^+) = (U_{J_{\mathfrak{H}}}(A))^*, \tag{3.4}$$

where * denotes the adjoint in $\mathfrak{H}$ with respect to the Hilbert space inner product $(\cdot,\cdot)$. In particular, $U_{J_{\mathfrak{H}}}$ establishes a bijective correspondence between the symmetric and selfadjoint relations in the Kreĭn space $(\mathfrak{H},[\cdot,\cdot])$ and the symmetric and selfadjoint relations in the Hilbert space $(\mathfrak{H},(\cdot,\cdot))$, respectively.

Using this connection one can conclude the existence and give a description of all boundary relations $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ for an arbitrary closed symmetric relation S in a Kreĭn space setting.

Proposition 3.3. *Let S be a closed symmetric relation in the Kreĭn space $(\mathfrak{H},[\cdot,\cdot])$ with the fundamental symmetry $J_{\mathfrak{H}}$. Then the mapping $U_{J_{\mathfrak{H}}}$ in (3.3) establishes a bijective correspondence between the boundary relations $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ for S^+ and the boundary relations $\widetilde{\Gamma} \subset \mathfrak{H}^2 \times \mathcal{H}^2$ for $\widetilde{S}^*$, where $\widetilde{S} = U_{J_{\mathfrak{H}}}(S)$ is a closed symmetric relation in the Hilbert space $(\mathfrak{H},(\cdot,\cdot))$, via*

$$\Gamma = \widetilde{\Gamma} \circ U_{J_{\mathfrak{H}}}.$$

Proof. Let $\widetilde{S} = U_{J_{\mathfrak{H}}}(S)$ and let $\widetilde{\Gamma} \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for $\widetilde{S}^*$. Then $\widetilde{\Gamma}$ is a unitary relation from the Kreĭn space $(\mathfrak{H}^2, \ll\cdot,\cdot\gg_{\mathfrak{H}^2})$ to the Kreĭn space $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$ with $\ker\widetilde{\Gamma} = \widetilde{S}$. Since $U_{J_{\mathfrak{H}}}$ is a standard unitary operator by Lemma 3.2, it follows from [9, Theorem 2.10 (iv)] that the composition

$$\Gamma = \widetilde{\Gamma} \circ U_{J_{\mathfrak{H}}}$$

is a unitary relation from the Kreĭn space $(\mathfrak{H}^2, [\![\cdot,\cdot]\!]_{\mathfrak{H}^2})$ to the Kreĭn space $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$. Clearly, $\ker\Gamma = S$ and thus Γ defines a boundary relation for S^+.

The inverse $U_{J_{\mathfrak{H}}}^{-1}$ is also a standard unitary operator and, therefore, if $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ is a boundary relation for S^+, then the same argument shows that the composition

$$\widetilde{\Gamma} = \Gamma \circ U_{J_{\mathfrak{H}}}^{-1}$$

is a boundary relation for $\widetilde{S}^*$. This yields the one-to-one correspondence between the boundary relations of S^+ and $\widetilde{S}^*$. □

The existence of boundary relations for closed symmetric operators or relations in the Hilbert space setting was proved in [8, Proposition 3.7], which together with Proposition 3.3 gives the corresponding fact for symmetric relations in a Kreĭn space.

A boundary relation Γ is said to be an *ordinary boundary triplet* for S^+ if $\operatorname{ran}\Gamma = \mathcal{H} \times \mathcal{H}$; see [8, Proposition 5.3]. The existence of ordinary boundary triplets for symmetric relations in a Kreĭn space can now be described as follows.

Proposition 3.4. *Let S be a closed symmetric relation in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ with the fundamental symmetry $J_{\mathfrak{H}}$. Then there exists an ordinary boundary triplet for S^+ if and only if S admits a selfadjoint extension in $(\mathfrak{H}, [\cdot,\cdot])$. In this case the mapping $U_{J_{\mathfrak{H}}}$ in* (3.3) *establishes a bijective correspondence between all ordinary boundary triplets $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^+ and all ordinary boundary triplets $\{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for $\widetilde{S}^*$, where $\widetilde{S} = U_{J_{\mathfrak{H}}}(S)$ is a closed symmetric relation in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$, via*

$$\{\Gamma_0, \Gamma_1\} = \{\widetilde{\Gamma}_0 \circ U_{J_{\mathfrak{H}}}, \widetilde{\Gamma}_1 \circ U_{J_{\mathfrak{H}}}\} = \widetilde{\Gamma} \circ U_{J_{\mathfrak{H}}}.$$

Moreover, $A_0 := \ker \Gamma_0$ and $A_1 := \ker \Gamma_1$ are selfadjoint extensions of S in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ which are transversal, i.e., $A_0 \widehat{+} A_1 = S^+$.

Proof. It follows from (3.4) that S admits a selfadjoint extension in the Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ if and only if $\widetilde{S}$ admits a selfadjoint extension in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$, or equivalently, $\widetilde{S}$ has equal defect numbers in $(\mathfrak{H}, (\cdot,\cdot))$. This is a necessary and sufficient condition for the existence of an ordinary boundary triplet $\{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for $\widetilde{S}^*$; cf. [11], [8, Section 5.1]. On the other hand, it is clear that $\widetilde{\Gamma} = \{\widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is an ordinary boundary triplet for $\widetilde{S}^*$ if and only if $\widetilde{\Gamma} \circ U_{J_{\mathfrak{H}}}$ is an ordinary boundary triplet for S^+, since $U_{J_{\mathfrak{H}}}$ is a standard unitary operator and $U_{J_{\mathfrak{H}}}(S^+) = \widetilde{S}^*$. This yields the bijective correspondence between the ordinary boundary triplets for S^+ and $\widetilde{S}^*$. Clearly, $U_{J_{\mathfrak{H}}}(A_j) = \ker \widetilde{\Gamma}_j$ and hence (3.4) implies that A_0 and A_1 are transversal selfadjoint extensions of S in $(\mathfrak{K}, [\cdot,\cdot])$. □

The existence of an ordinary boundary triplet for a closed symmetric relation S in a Kreĭn space which admits a selfadjoint extension A_0 with a nonempty resolvent set was shown in [5]. Proposition 3.4 implies that all ordinary boundary triplets of S^+ in a fixed parameter space $\mathcal{H}$ can be described in the same way as in the Hilbert space setting; cf., e.g., [11, Proposition 1.7].

Lemma 3.5. *Let Γ be a boundary relation for the adjoint S^+ and let W be a standard unitary operator in the Kreĭn space $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$. Then $W \circ \Gamma$ is also a boundary relation for S^+. Moreover, if Γ is an ordinary boundary triplet for S^+, then the same is true for $W \circ \Gamma$ and all ordinary boundary triplets for S^+ can be obtained in this way.*

The connection between the boundary relations of S^+ of a symmetric relation S in a Kreĭn space $(\mathfrak{H}, [\cdot,\cdot])$ and the boundary relations of $\widetilde{S}^*$ in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ given in Proposition 3.3 makes it possible to extend or easily translate several facts for (different types of) boundary relations as well as various results on their composition and coupling which were proved in the Hilbert space setting in [8, 9] for boundary relations of symmetric relations in Kreĭn spaces. The reason is that many of the results involving the construction of boundary triplets and relations (e.g. for various intermediate extensions of orthogonal sums of symmetric relations as in [9, Sections 3–5]) have been proved by using composition of two unitary operators or relations (cf. [9, Theorem 2.10]), i.e., they are of algebraic nature: when such a result on boundary relations is established for symmetric relations in a Hilbert space it also holds for symmetric relations in a Kreĭn space due to the connection of boundary relations and triplets in Propositions 3.3, 3.4 via the standard unitary mapping $U_{J_{\mathfrak{H}}}$ in (3.3). In this way one gets also various transformation results for Weyl families (see Definition 3.6 below) of boundary relations in the Kreĭn space setting, proved in the Hilbert space setting in [8, 9]; as a simple example see, e.g., Lemma 3.8. It is emphasized that these results give the "algebraic part" involving, e.g., various transformations of Weyl families of symmetric operators in a Kreĭn space; they do not make it possible to derive specific analytic properties of a given Weyl family of some symmetric operator S in a Kreĭn space from the analytic properties of a Weyl family of a symmetric operator $\widetilde{S} = U_{J_{\mathfrak{H}}}(S)$ in a Hilbert space. The reason is that the defect subspaces of S and $\widetilde{S}$ are not connected by the mapping $U_{J_{\mathfrak{H}}}$ and therefore there is no direct connection between the Weyl families of the boundary relations of S^+ and $\widetilde{S}^*$.

3.2. **Weyl families for boundary relations in Kreĭn spaces.** The introduction of Weyl families and γ-fields for the present situation follows the same pattern as in the Hilbert space case; see [8, 9].

Definition 3.6. *Let S be a closed symmetric relation in the Kreĭn space $\mathfrak{H}$ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for S^+ with $T = \operatorname{dom} \Gamma$. The γ-field γ and the Weyl family M of the boundary relation Γ are defined by*

$$\gamma(\lambda) := \{ \{h, f\} : \{\hat{f}, \hat{h}\} \in \Gamma \text{ and } \hat{f} = \{f, f'\} \in \widehat{\mathfrak{N}}_\lambda(T) \}, \quad \lambda \in \mathbb{C},$$

and

$$M(\lambda) := \Gamma\big(\widehat{\mathfrak{N}}_\lambda(T)\big) = \{ \hat{h} : \{\hat{f}, \hat{h}\} \in \Gamma \text{ and } \hat{f} \in \widehat{\mathfrak{N}}_\lambda(T) \}, \quad \lambda \in \mathbb{C}.$$

Let S be a closed symmetric relation in the Kreĭn space $\mathfrak{H}$ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for S^+. Then Γ induces a selfadjoint extension A of S in the Kreĭn space $\mathfrak{H} \times \mathcal{H}$ defined by

$$A := \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ -h' \end{pmatrix} \right\} \in \Gamma \right\}, \tag{3.5}$$

which is called the *main transform* of Γ in [8]. As was shown in [8, Theorem 3.9] the Weyl family M of the boundary relation Γ and the selfadjoint extension A in (3.5) associated to Γ are connected via

$$P_{\mathcal{H}}(A - \lambda)^{-1} \restriction_{\mathcal{H}} = -(M(\lambda) + \lambda)^{-1}, \quad \lambda \in \rho(A), \tag{3.6}$$

and that therefore in this case

$$-(M(\lambda) + \lambda)^{-1} \in \mathbf{B}(\mathcal{H}), \quad \lambda \in \rho(A). \tag{3.7}$$

However, in general, the selfadjoint relation A in (3.5) has an empty resolvent set and (3.7) will not hold in general. In fact, the righthand side of (3.6) suggests that $\rho(A)$ will be empty if the Weyl family of Γ is $M(\lambda) = -\lambda$. In the following example this situation is considered.

Example 3.7. *Let $\mathfrak{H}$ be the one-dimensional Pontryagin space $(\mathbb{C}, [\cdot, \cdot])$, where the inner product is defined by $[f, g] := -f\overline{g}$, $f, g \in \mathbb{C}$. Then*

$$\Gamma := \left\{ \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} f \\ -f' \end{pmatrix} \right\} : f, f' \in \mathbb{C} \right\} \subset \mathfrak{H}^2 \times \mathbb{C}^2$$

is a boundary relation for $T = S^+ = \{ \{f, f'\} : f, f' \in \mathfrak{H} \}$ when the closed symmetric operator S is defined by $S = \{0\}$. In fact,

$$\Gamma^{[\![+]\!]} := \left\{ \left\{ \begin{pmatrix} h \\ h' \end{pmatrix}, \begin{pmatrix} g \\ g' \end{pmatrix} \right\} : -(h, f') - (h', f) = [g, f'] - [g', f] \text{ for all } f, f' \in \mathbb{C} \right\},$$

and simple observations show $\Gamma^{-1} = \Gamma^{[\![+]\!]}$. *The Weyl family associated with* Γ, *defined by*

$$M(\lambda) = \left\{ \begin{pmatrix} f \\ -\lambda f \end{pmatrix} : f \in \mathbb{C} \right\} = -\lambda,$$

is a scalar generalized Nevanlinna function with one negative square. The selfadjoint relation A *in* $\mathfrak{H} \times \mathbb{C}$ *associated with the boundary relation* Γ *is given by*

$$A = \left\{ \left\{ \begin{pmatrix} f \\ f \end{pmatrix}, \begin{pmatrix} f' \\ f' \end{pmatrix} \right\} : f, f' \in \mathbb{C} \right\}.$$

Clearly every $\lambda \in \mathbb{C}$ *is an eigenvalue of* A *and hence* $\rho(A) = \emptyset$. *Note that also* $\mathfrak{H} = \mathfrak{N}_\lambda(T)$, $\lambda \in \mathbb{C}$, *and* $\mathfrak{H} = \operatorname{mul} A$.

Recall that standard unitary operators in $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$ transform boundary relations by interpreting the standard unitary operator W in $\mathcal{H}$ as a transformer (of linear relations) in the sense of Shmuljan [24]; see Lemma 3.5. The corresponding Weyl families transform accordingly, cf. [9, Proposition 3.11].

Lemma 3.8. *Let* S *be a closed symmetric relation in the Kreĭn space* $\mathfrak{H}$ *and let* $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ *be a boundary relation for* S^+ *with corresponding Weyl family* M. *Let* W *be a standard unitary operator in the Kreĭn space* $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$ *decomposed as*

$$W = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix},$$

and let $\Gamma_W = W \circ \Gamma$ *be the transformed boundary relation with corresponding Weyl family* M_W. *Then* $\operatorname{dom} \Gamma_W = \operatorname{dom} \Gamma$ *and*

$$M_W(\lambda) = \left\{ \{W_{00} f + W_{01} f', W_{10} f + W_{11} f'\} : \{f, f'\} \in M(\lambda) \right\}. \tag{3.8}$$

In the following lemma a particularly useful transform of a boundary relation is described with some further details; the first part (in the Hilbert space setting) is contained in [9, Proposition 3.18] and the second part is obtained by applying (3.5) to Γ_W.

Lemma 3.9. *Let* S *be a closed symmetric relation in the Kreĭn space* $\mathfrak{H}$ *and let* $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ *be a boundary relation for* S^+ *with corresponding Weyl family* M. *Let* $X, X^{-1}, Y = Y^* \in \mathbf{B}(\mathcal{H})$ *and define*

$$W = \begin{pmatrix} X^{-1} & 0 \\ Y X^{-1} & X^* \end{pmatrix}. \tag{3.9}$$

Then W *is a standard unitary operator in* $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$ *and the Weyl family* M_W *related to the boundary relation* $\Gamma_W = W \circ \Gamma$ *is given by*

$$M_W(\lambda) = X^* M(\lambda) X + Y. \tag{3.10}$$

If the selfadjoint relations A *and* A_W *correspond to the boundary relations* Γ *and* Γ_W *via* (3.5), *then*

$$A_W = \left\{ \left\{ \begin{pmatrix} f \\ X^{-1} h \end{pmatrix}, \begin{pmatrix} f' \\ X^* h' - Y X^{-1} h \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ h' \end{pmatrix} \right\} \in A \right\},$$

and the subspace $\operatorname{mul} A_W$ *admits the representation*

$$\operatorname{mul} A_W = \left\{ \begin{pmatrix} f' \\ X^* h' \end{pmatrix} : \begin{pmatrix} f' \\ h' \end{pmatrix} \in \operatorname{mul} A \right\}. \tag{3.11}$$

Proof. A direct computation shows

$$W^{[\![+]\!]} = \begin{pmatrix} X & 0 \\ -X^{-*}Y & X^{-*} \end{pmatrix} = W^{-1},$$

so that W is a standard unitary operator in $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$. The identity (3.10) follows from (3.8), which in the present circumstances (3.8) reads as

$$M_W(\lambda) = \left\{ \{X^{-1}f, YX^{-1}f + X^*f'\} : \{f,f'\} \in M(\lambda) \right\}.$$

The remaining statements follow from $\Gamma_W = W \circ \Gamma$ and (3.5). □

Lemma 3.10. *Let S be a closed symmetric relation in the Kreĭn space $\mathfrak{H}$, let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for S^+, and let A be the selfadjoint relation associated with Γ via* (3.5)*. Assume that*

$$\operatorname{mul} A \cap \mathfrak{H} = \{0\}. \tag{3.12}$$

Then there exists a closed linear operator $K : \operatorname{dom} K \subset \mathcal{H} \to \mathfrak{H}$ such that

$$\operatorname{mul} A = \left\{ \begin{pmatrix} Ku \\ u \end{pmatrix} : u \in \operatorname{dom} K \subset \mathcal{H} \right\}. \tag{3.13}$$

The condition (3.12) *is satisfied, if S is an operator.*

Proof. Observe that $\operatorname{mul} A$ is a closed subspace of $\mathfrak{H} \oplus \mathcal{H}$ given by

$$\operatorname{mul} A = \left\{ \begin{pmatrix} f' \\ h' \end{pmatrix} : \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} f' \\ h' \end{pmatrix} \right\} \in A \right\}.$$

Hence, the condition (3.12) is necessary and sufficient for $\operatorname{mul} A$ to be the graph of a linear operator $K : \operatorname{dom} K \subset \mathcal{H} \to \mathfrak{H}$. Clearly, the operator K is closed, since $\operatorname{mul} A$ is a closed subspace of $\mathfrak{H} \oplus \mathcal{H}$.

To prove the last statement assume that $f' \in \operatorname{mul} A \cap \mathfrak{H}$. Then

$$\begin{pmatrix} f' \\ 0 \end{pmatrix} \in \operatorname{mul} A \quad \text{and} \quad \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} f' \\ 0 \end{pmatrix} \right\} \in A,$$

and therefore

$$\left\{ \begin{pmatrix} 0 \\ f' \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \in \Gamma \quad \text{and} \quad \begin{pmatrix} 0 \\ f' \end{pmatrix} \in \ker \Gamma.$$

Hence, $f' \in \operatorname{mul} S$ and if S is an operator this yields $f' = 0$. □

3.3. **Boundary relations in Pontryagin spaces.** In this subsection somewhat more specific results for boundary relations in Pontryagin spaces are given. It is clear from (3.6) that a nonempty resolvent set $\rho(A) \neq \emptyset$ of the selfadjoint relation A in (3.5) simplifies the investigation of the analytic properties of the Weyl families and γ-fields associated with boundary relations for S^+. Another helpful condition in this connection is the notion of minimality of boundary relations; the definition given here is a slight adaption of [8, Definition 3.4] in the case of Hilbert spaces.

Definition 3.11. *Let S be a closed symmetric relation in the Pontryagin space $\mathfrak{H}$ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for S^+ with $\operatorname{dom} \Gamma = T$. The boundary relation Γ is called minimal if $\widehat{\rho}(S) \neq \emptyset$ and*

$$\mathfrak{H} = \overline{\operatorname{span}} \left\{ \mathfrak{N}_\lambda(T) : \lambda \in \widehat{\rho}(S) \right\}. \tag{3.14}$$

Recall that if S is a closed symmetric operator in the Pontryagin space $\mathfrak{H}$, then $\widehat{\rho}(S) \neq \emptyset$.

Proposition 3.12. *Let S be a closed symmetric operator in the Pontryagin space $\mathfrak{H}$ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for S^+ with* $\operatorname{dom}\Gamma = T$*. Then there exists a boundary relation Γ_W for S^+ with the same domain* $\operatorname{dom}\Gamma_W = T$*, such that the selfadjoint relation A_W associated with Γ_W via (3.5) satisfies*

$$\rho(A_W) \neq \emptyset.$$

Proof. Assume that $\rho(A) = \emptyset$ for the main transform A of Γ; otherwise the statement is clear. Since S is an operator it follows from Lemma 3.10 that there exists a closed linear operator $K : \operatorname{dom} K \subset \mathcal{H} \to \mathfrak{H}$ such that (3.13) holds. Due to Lemma 2.4 the form $[K\cdot, K\cdot]$ on $\operatorname{dom} K$ is semibounded from below, that is there is $a > 0$ such that

$$[Ku, Ku] \geq -a^2\|u\|^2, \quad u \in \operatorname{dom} K.$$

Consider a standard unitary operator W in $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$ of the form (3.9). The selfadjoint relation A is transformed into the selfadjoint relation A_W whose multivalued part is given by (3.11). The inner product on $\operatorname{mul} A_W$ is obtained from (3.13) and (3.11):

$$\left[\begin{pmatrix} Ku \\ X^*u \end{pmatrix}, \begin{pmatrix} Ku \\ X^*u \end{pmatrix} \right] = (XX^*u, u) + [Ku, Ku], \quad u \in \operatorname{dom} K. \tag{3.15}$$

By choosing W such that the inner product in (3.15) is positive (for instance $X = (a+ib)I$ with some $b > 0$ and $Y = 0$), it follows from Corollary 2.3 that $\rho(A^W) \neq \emptyset$. By construction, $\operatorname{dom}\Gamma_W = \operatorname{dom}\Gamma$ and this completes the proof. □

Clearly, the above result need not hold if S is not an operator: consider $S \oplus A$, where S is symmetric and A a selfadjoint relation with $\rho(A) = \emptyset$. Proposition 3.12 gives a couple of useful corollaries; it will be also used in establishing the characteristic properties of the Weyl families in the next section.

Corollary 3.13. *Let S be a closed symmetric operator in the Pontryagin space $\mathfrak{H}$ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for S^+ with* $\operatorname{dom}\Gamma = T$*. Then the selfadjoint relation A_W in Proposition 3.12 satisfies*

$$\mathfrak{N}_\lambda(T) = \operatorname{ran} P_{\mathfrak{H}}(A_W - \lambda)^{-1}|_{\mathcal{H}}, \quad \lambda \in \rho(A_W). \tag{3.16}$$

Proof. Due to $\operatorname{dom}\Gamma_W = T$ and $\rho(A_W) \neq \emptyset$ the result is obtained by using the arguments appearing in [8, Lemma 2.14]. □

The next corollary is valid for arbitrary boundary relations in Pontryagin spaces.

Corollary 3.14. *Let S be a closed symmetric operator in the Pontryagin space $\mathfrak{H}$, let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for S^+ with* $\operatorname{dom}\Gamma = T$ *and let A_W be a selfadjoint relation as in Proposition 3.12. Then the following statements hold:*

(i) *$\mathfrak{N}_\lambda(T)$ is dense in $\mathfrak{N}_\lambda(S^+)$, $\lambda \in \rho(A_W)$;*

(ii) *Γ is minimal if and only if S is simple, i.e.,*

$$\mathfrak{H} = \overline{\operatorname{span}}\left\{ \mathfrak{N}_\lambda(S^+) : \lambda \in \widehat{\rho}(S) \right\};$$

(iii) *for $\lambda \in \mathbb{C}_\pm$ except for at most κ points the linear spaces $\mathfrak{N}_\lambda(T)$ in (3.16) satisfy*

$$\|P_{\mathfrak{H}}(A_W - \lambda)^{-1} - P_{\mathfrak{H}}(A_W - \lambda_0)^{-1}\| \to 0, \quad \lambda \to \lambda_0, \quad \lambda, \lambda_0 \in \rho(A_W).$$

Proof. The statements (i) and (ii) are obtained from Corollary 3.13 combined with [8, Lemma 2.14]. To prove (iii) it suffices to apply the resolvent identity for A_W. □

As to part (iii) of Corollary 3.14 note that the linear spaces of eigenelements $\mathfrak{N}_\lambda(T)$ are in general nonclosed and that $S = \ker \Gamma$ need not have equal defect numbers, cf. [8], in which case there are no selfadjoint extensions in $\mathfrak{H}$ connecting e.g. the defect subspaces of S as in the case of ordinary boundary triplets where one may use the resolvent of $A_0 := \ker \Gamma_0$. Note that Corollary 3.14 makes it also possible to replace $\widehat{\rho}(S)$ in the definition of minimality in (3.14) by much smaller subsets in $\mathbb{C}_\pm$.

To this end the following result is given for completeness; it is well-known at least for simple symmetric operators with equal defect numbers.

Lemma 3.15. *Let S be a closed symmetric relation in the Pontryagin space $\mathfrak{H}$ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a minimal boundary relation for S^+. Then $S = \ker \Gamma$ is an operator without eigenvalues.*

Proof. To see that S is an operator, assume that $\{0, g\} \in S$. Then for all $f_\lambda \in \mathfrak{N}_\lambda(T)$, $\lambda \in \mathbb{C}$,

$$0 = [g, f_\lambda] - [0, \lambda f_\lambda] = [g, f_\lambda],$$

and hence it follows from (3.14) that $g = 0$. Next assume that $\{h, \zeta h\} \in S$ for some $\zeta \in \mathbb{C}$. Then

$$0 = [\zeta h, f_\lambda] - [h, \lambda f_\lambda] = (\zeta - \bar{\lambda})[h, f_\lambda],$$

and hence again using (3.14) one easily concludes that $h = 0$. □

4. Generalized Nevanlinna families and Weyl families of boundary relations in Pontryagin spaces

In this section boundary relations and their Weyl families are considered when the space $\mathfrak{H}$ is a Pontryagin space. The corresponding Weyl family turns out to be a generalized Nevanlinna family and it is also shown that, conversely, every generalized Nevanlinna family is the Weyl family corresponding to a boundary relation in a Pontryagin space.

4.1. Generalized Nevanlinna pairs and families.

The next definition generalizes the notion of Nevanlinna pairs which was first introduced apparently in [22], but its prototype can be found in other places, in particular, in the theory of the Bezoutiant, see [20]. The notions of Nevanlinna pairs and families has been used for the description of generalized resolvents in, e.g., [17], and in the theory of boundary value problems with spectral parameter depending boundary conditions, see, e.g., [12], [13].

For a subset $\mathcal{O}$ in $\mathbb{C}$ the notation

$$\mathcal{O}^* := \{\lambda \in \mathbb{C} : \bar{\lambda} \in \mathcal{O}\}$$

will be used in the following.

Definition 4.1. *Let $\mathcal{H}$ be a Hilbert space and let $\kappa \in \mathbb{N}_0$. A pair $\{\Phi, \Psi\}$ of $\mathbf{B}(\mathcal{H})$-valued functions Φ, Ψ holomorphic on a symmetric open set $\mathcal{O} \cup \mathcal{O}^*$, $\mathcal{O} \subset \mathbb{C}_+$, is said to be a generalized Nevanlinna pair with κ negative squares, if*

(P1) *for all $\lambda \in \mathcal{O}$:*

$$\Psi(\bar{\lambda})^* \Phi(\lambda) - \Phi(\bar{\lambda})^* \Psi(\lambda) = 0;$$

(P2) *for some $\mu \in \mathbb{C}_+$ and some $\lambda \in \mathcal{O}$:*

$$0 \in \rho(\Psi(\lambda) + \mu \Phi(\lambda)), \quad 0 \in \rho(\Psi(\bar{\lambda}) + \bar{\mu} \Phi(\bar{\lambda}));$$

(P3) *the kernel*

$$\mathsf{K}_{\Phi,\Psi}(\lambda,\mu) := \frac{\Psi(\bar\lambda)^*\Phi(\bar\mu) - \Phi(\bar\lambda)^*\Psi(\bar\mu)}{\lambda - \bar\mu} \tag{4.1}$$

has κ negative squares on $\mathcal{O}\cup\mathcal{O}^*$.

A generalized Nevanlinna pair with $\kappa = 0$ negative squares is said to be a Nevanlinna pair.

Observe that if the condition (P2) is satisfied, then it also holds for all points in some neighborhood of λ by continuity of the functions Ψ and Φ. Hence, by making $\mathcal{O}$ smaller (if necessary) one can equivalently assume that (P2) actually holds for all $\lambda \in \mathcal{O}$. Recall also that (P3) means that for all $n \in \mathbb{N}$ and every choice of $\lambda_i \in \mathcal{O}\cup\mathcal{O}^*$ and $x_i \in \mathcal{H}$, $i = 1,\dots,n$, the matrix $(\mathsf{K}_{\Phi,\Psi}(\lambda_i,\lambda_j)x_j, x_i)_{i,j=1}^n$ has at most κ negative eigenvalues and that the number κ with this property is minimal.

Two Nevanlinna pairs $\{\Phi,\Psi\}$ and $\{\Phi_1,\Psi_1\}$ are said to be *equivalent*, if for some holomorphic and boundedly invertible operator function $\chi(\cdot) \in \mathbf{B}(\mathcal{H})$ on $\mathcal{O}\cup\mathcal{O}^*$

$$\Phi_1(\lambda) = \Phi(\lambda)\chi(\lambda) \qquad \text{and} \qquad \Psi_1(\lambda) = \Psi(\lambda)\chi(\lambda)$$

holds for all $\lambda \in \mathcal{O}\cup\mathcal{O}^*$. Note that

$$\big\{ \{\Phi(\lambda)h, \Psi(\lambda)h\} : h \in \mathcal{K} \big\} = \big\{ \{\Phi(\lambda)\chi(\lambda)k, \Psi(\lambda)\chi(\lambda)k\} : k \in \mathcal{K} \big\}, \quad \lambda \in \mathcal{O}\cup\mathcal{O}^*,$$

and that the kernel

$$\mathsf{K}_{\Phi\chi,\Psi\chi}(\lambda,\mu) = \chi(\bar\lambda)^*\mathsf{K}_{\Phi,\Psi}(\lambda,\mu)\chi(\bar\mu)$$

also has κ negative squares on $\mathcal{O}\cup\mathcal{O}^*$.

Definition 4.2. *Let $\kappa \in \mathbb{N}_0$. A family of linear relations τ in a Hilbert space $\mathcal{H}$ defined on $\mathcal{O}\cup\mathcal{O}^*$ where $\mathcal{O} \subset \mathbb{C}_+$ is an open set, is called a generalized Nevanlinna family with κ negative squares if*

(Q1) *$\tau(\lambda)^* = \tau(\bar\lambda)$ for all $\lambda \in \mathcal{O}\cup\mathcal{O}^*$;*
(Q2) *for some $\mu \in \mathbb{C}_+$ the operator family $(\tau(\lambda)+\mu)^{-1}$ has values in $\mathbf{B}(\mathcal{H})$ and is holomorphic on $\mathcal{O} \subset \mathbb{C}_+$;*
(Q3) *the kernel $\mathsf{K}_{\widetilde\Phi,\widetilde\Psi}(\lambda,\mu)$ associated with the pair*

$$\widetilde\Phi(\lambda) := \begin{cases} (\tau(\lambda)+\mu)^{-1}, & \lambda \in \mathcal{O}, \\ (\tau(\lambda)+\bar\mu)^{-1}, & \bar\lambda \in \mathcal{O}, \end{cases} \quad \widetilde\Psi(\lambda) := \begin{cases} I - \mu(\tau(\lambda)+\mu)^{-1}, & \lambda \in \mathcal{O}, \\ I - \bar\mu(\tau(\lambda)+\bar\mu)^{-1}, & \bar\lambda \in \mathcal{O}, \end{cases} \tag{4.2}$$

has κ negative squares on $\mathcal{O}\cup\mathcal{O}^*$.

A generalized Nevanlinna family with $\kappa = 0$ negative squares is said to be a Nevanlinna family.

The following proposition shows how generalized Nevanlinna pairs and generalized Nevanlinna families are connected to each other and how they can be extended onto $\mathbb{C}\setminus\mathbb{R}$ with the possible exception of at most κ pairs of points in $\mathbb{C}\setminus\mathbb{R}$.

Proposition 4.3. *Let $\{\Phi,\Psi\}$ be a generalized Nevanlinna pair with κ negative squares on $\mathcal{O}\cup\mathcal{O}^*$ and let τ be defined by*

$$\lambda \mapsto \tau(\lambda) = \big\{ \{\Phi(\lambda)h, \Psi(\lambda)h\} : h \in \mathcal{H} \big\}, \quad \lambda \in \mathcal{O}\cup\mathcal{O}^*. \tag{4.3}$$

Then τ is a generalized Nevanlinna family with κ negative squares and τ can be uniquely extended to a generalized Nevanlinna family with κ negative squares on a domain

$$\mathcal{D}_+ \cup \mathcal{D}_+^*, \quad \textit{where} \quad \mathcal{D}_+ \subset \mathbb{C}_+ \quad \textit{and} \quad \#(\mathbb{C}_+ \setminus \mathcal{D}_+) \leq \kappa. \tag{4.4}$$

Moreover, for every $\zeta \in \mathbb{C}_+$ the condition $0 \in \rho(\tau(\lambda) + \zeta)$ holds for all $\lambda \in \mathcal{D}_+$ except at most κ points.

Conversely, if τ is a generalized Nevanlinna family with κ negative squares, then the pair $\{\widetilde{\Phi}, \widetilde{\Psi}\}$ defined by (4.2) *is a generalized Nevanlinna pair with κ negative squares and admits a unique extension to a generalized Nevanlinna pair with κ negative squares on a domain as in* (4.4).

Proof. Let $\{\Phi, \Psi\}$ be a generalized Nevanlinna pair and let τ be defined by (4.3). Then it follows from (P1) that $\tau(\lambda) \subset \tau(\bar{\lambda})^*$. The definition (4.3) implies that

$$\tau(\lambda) + \mu = \{\{\Phi(\lambda)h, \Psi(\lambda)h + \mu\Phi(\lambda)h\} : h \in \mathcal{H}\}\,, \tag{4.5}$$

and, hence (P2) shows that $0 \in \rho(\tau(\lambda) + \mu)$ for all $\lambda \in \mathcal{O}(\subset \mathbb{C}_+)$. Similarly, one shows that $0 \in \rho(\tau(\bar{\lambda}) + \bar{\mu})$ for all $\lambda \in \mathcal{O}$. This implies that $\tau(\lambda)^* = \tau(\bar{\lambda})$ and also proves (Q1) and (Q2). The property (Q3) follows immediately from (P3) since the pair $\{\widetilde{\Phi}, \widetilde{\Psi}\}$ is equivalent to $\{\Phi, \Psi\}$:

$$\widetilde{\Phi}(\lambda) = \begin{cases} \Phi(\lambda)(\Psi(\lambda) + \mu\Phi(\lambda))^{-1}, & \lambda \in \mathcal{O}, \\ \Phi(\lambda)(\Psi(\lambda) + \bar{\mu}\Phi(\lambda))^{-1} & \lambda \in \mathcal{O}^*, \end{cases}$$

$$\widetilde{\Psi}(\lambda) = \begin{cases} \Psi(\lambda)(\Psi(\lambda) + \mu\Phi(\lambda))^{-1}, & \lambda \in \mathcal{O}, \\ \Psi(\lambda)(\Psi(\lambda) + \bar{\mu}\Phi(\lambda))^{-1} & \lambda \in \mathcal{O}^* \end{cases}.$$

Therefore τ defined in (4.3) is a generalized Nevanlinna family with κ negative squares.

Associate with the Nevanlinna pair $\{\widetilde{\Phi}, \widetilde{\Psi}\}$ the operator valued function

$$\Theta(\lambda) = \begin{cases} \widetilde{\Psi}(\lambda) + \bar{\mu}\widetilde{\Phi}(\lambda), & \lambda \in \mathcal{O}, \\ \widetilde{\Psi}(\lambda) + \mu\widetilde{\Phi}(\lambda), & \lambda \in \mathcal{O}^*. \end{cases} \tag{4.6}$$

Note that the condition (P1) is equivalent to the condition $\Theta(\bar{\lambda})^* = \Theta(\lambda)$. In particular, (P1) implies that

$$\widetilde{\Phi}(\lambda) = \widetilde{\Phi}(\bar{\lambda})^*, \quad \widetilde{\Psi}(\lambda) = \widetilde{\Psi}(\bar{\lambda})^*, \quad \lambda \in \mathcal{O}. \tag{4.7}$$

Associate with Θ the Schur kernel $\mathsf{S}_\Theta(\lambda, \omega)$ in $\mathcal{O}$ by the formula

$$\mathsf{S}_\Theta(\lambda, \omega) := \frac{I - \Theta(\lambda)\Theta(\omega)^*}{-2i(\lambda - \bar{\omega})}, \quad \lambda, \omega \in \mathcal{O}.$$

It follows from (4.6), (4.7) and the equalities

$$\widetilde{\Psi}(\lambda) + \mu\widetilde{\Phi}(\lambda) = I \quad \widetilde{\Psi}(\bar{\lambda}) + \bar{\mu}\widetilde{\Phi}(\bar{\lambda}) = I, \quad \lambda \in \mathcal{O},$$

that the Schur kernel $\mathsf{S}_\Theta(\lambda, \omega)$ is connected to the Nevanlinna kernel $\mathsf{K}_{\widetilde{\Phi},\widetilde{\Psi}}(\lambda, \omega)$ on $\mathcal{O}$ via

$$\mathsf{S}_\Theta(\lambda, \omega) = \operatorname{Im} \mu\, \mathsf{K}_{\widetilde{\Phi},\widetilde{\Psi}}(\lambda, \omega). \tag{4.8}$$

Hence, in the terminology of [18] Θ belongs to the generalized Schur class $\mathbf{S}_\kappa(\mathcal{H})$ and according to [18, Satz 3.2, Folgerung 3.3] Θ admits a unique holomorphic continuation $\widetilde{\Theta}(\lambda)$ to a set $\mathcal{D}_+$ in the open upper half plane $\mathbb{C}_+$ with $\#(\mathbb{C}_+ \setminus \mathcal{D}_+) \leq$

κ. Let $\widetilde{\Theta}$ be extended to the lower half plane $\mathbb{C}_-$ by $\widetilde{\Theta}(\lambda) := \Theta(\bar{\lambda})^*$, $\lambda \in \mathcal{D}_+^*$ and let the family of linear relations $\widetilde{\tau}$ be defined by

$$\widetilde{\tau}(\lambda) = \begin{cases} \{\, \{(I - \widetilde{\Theta}(\lambda))f, (\mu\widetilde{\Theta}(\lambda) - \bar{\mu})f\} : \, f \in \mathcal{H}\,\}, & \lambda \in \mathcal{D}_+, \\ \{\, \{(I - \widetilde{\Theta}(\lambda))f, (\bar{\mu}\widetilde{\Theta}(\lambda) - \mu)f\} : \, f \in \mathcal{H}\,\}, & \lambda \in \mathcal{D}_+^*. \end{cases} \tag{4.9}$$

Then the family of linear relations $\widetilde{\tau}$ is a continuation of τ to the domain $\mathcal{D}_\tau := \mathcal{D}_+ \cup \mathcal{D}_+^*$ and the property (Q1) is easily verified on $\mathcal{D}_\tau$.

Introduce the kernel $\mathsf{D}_{\widetilde{\Theta}}(\lambda, \omega)$ on $(\mathcal{D}_+ \cup \mathcal{D}_+^*) \times (\mathcal{D}_+ \cup \mathcal{D}_+^*)$ by

$$\mathsf{D}_{\widetilde{\Theta}}(\lambda, \omega) := \begin{cases} \frac{I - \widetilde{\Theta}(\lambda)\widetilde{\Theta}(\omega)^*}{-2i(\lambda - \bar{\omega})}, & \omega \in \mathcal{D}_+, \, \lambda \in \mathcal{D}_+, \\ \frac{\widetilde{\Theta}(\lambda) - \widetilde{\Theta}(\omega)^*}{-2i(\lambda - \bar{\omega})}, & \omega \in \mathcal{D}_+, \, \lambda \in \mathcal{D}_+^*, \\ \frac{\widetilde{\Theta}(\lambda) - \widetilde{\Theta}(\omega)^*}{2i(\lambda - \bar{\omega})}, & \omega \in \mathcal{D}_+^*, \, \lambda \in \mathcal{D}_+, \\ \frac{I - \widetilde{\Theta}(\lambda)\widetilde{\Theta}(\omega)^*}{2i(\lambda - \bar{\omega})}, & \omega \in \mathcal{D}_+^*, \, \lambda \in \mathcal{D}_+^*. \end{cases}$$

Due to [12] (see also [6] in the present notation) the kernel $\mathsf{D}_{\widetilde{\Theta}}(\lambda, \omega)$ has κ negative squares on $(\mathcal{D}_+ \cup \mathcal{D}_+^*) \times (\mathcal{D}_+ \cup \mathcal{D}_+^*)$. Similar calculations as in (4.8) show that

$$\mathsf{D}_{\widetilde{\Theta}}(\lambda, \omega) = \operatorname{Im} \mu \, \mathsf{K}_{\widetilde{\Phi}, \widetilde{\Psi}}(\lambda, \omega),$$

where $\widetilde{\Phi}$, $\widetilde{\Psi}$ are determined by (4.2) with $\widetilde{\tau}$ instead of τ. Therefore, the property $(Q3)$ for $\widetilde{\tau}$ is satisfied.

It follows from (4.9) that for $\zeta \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_+ \cap \mathcal{D}_\tau$,

$$\widetilde{\tau}(\lambda) + \zeta = \{\, \{(I - \widetilde{\Theta}(\lambda))f, ((\mu - \zeta)\widetilde{\Theta}(\lambda) - (\bar{\mu} - \zeta))f\} : \, f \in \mathcal{H}\,\}.$$

Let $\nu = (\mu - \zeta)/(\bar{\mu} - \zeta)$. Then by [18, Satz 3.2, Lemma 3.5] $0 \in \rho(I - \nu\widetilde{\Theta}(\lambda))$ or, equivalently, $0 \in \rho(\widetilde{\tau}(\lambda) + \zeta)$ for all $\lambda \in \mathcal{D}_+$ except at most κ points. In particular, also (Q2) is satisfied.

Conversely, assume that $\tau(\cdot)$ is a generalized Nevanlinna family with κ negative squares and let the pair $\{\widetilde{\Phi}, \widetilde{\Psi}\}$ be defined by (4.2) on $\mathcal{O} \cup \mathcal{O}^*$. The first part of the proof implies that τ can be extended to a generalized Nevanlinna family with κ negative squares on the domain $\mathcal{D}_\tau = \mathcal{D}_\tau^*$, $\#(\mathbb{C}_+ \setminus \mathcal{D}_\tau) \leq \kappa$; compare [19, Satz 3.4]. Now clearly (P1) is implied by (Q1). It follows from (4.5) and (Q2) that $0 \in \rho(\widetilde{\Psi}(\lambda) + \mu\widetilde{\Phi}(\lambda))$ for all $\lambda \in \mathcal{D}_\tau \cap \mathbb{C}_+$. Similarly, (Q1) and (Q2) imply that $0 \in \rho(\tau(\bar{\lambda}) + \bar{\mu})$ and, hence, $0 \in \rho(\widetilde{\Psi}(\bar{\lambda}) + \bar{\mu}\widetilde{\Phi}(\lambda))$ for all $\lambda \in \mathcal{D}_\tau \cap \mathbb{C}_-$. This proves (P2). Finally, (P3) is clear from (Q3). $\square$

Remark 4.4. *Proposition 4.3 contains the continuation property for generalized Nevanlinna families and generalized Nevanlinna pairs with κ negative squares on $\mathcal{O} \cup \mathcal{O}^*$. A similar fact for generalized Schur functions with κ negative squares on some open set in the unit disk was proved by M.G. Kreĭn and H. Langer in* [18], *and the present result is obtained from their result by applying Caley transforms. The continuation property in the case of bounded operator functions τ whose Nevanlinna kernel has κ negative squares on a set $\mathcal{D}_0 \subset \mathbb{C}_+$, which contains at least one accumulation point was proved earlier in* [12, Section 2].

By replacing the generalized Nevanlinna pair $\{\Phi, \Psi\}$ by the extended generalized Nevanlinna pair $\{\widetilde{\Phi}, \widetilde{\Psi}\}$ in Proposition 4.3 one can consider generalized Nevanlinna pairs being defined in the maximal domain of holomorphy $\mathcal{D}_+ \cup \mathcal{D}_+^*$ of a given generalized Nevanlinna family τ. In what follows such a pair is often still denoted by $\{\Phi,\Psi\}$ with the indication that its domain of holomorphy is denoted by $\mathcal{D}_{\Phi,\Psi} := \mathcal{D}_+ \cup \mathcal{D}_+^*$, instead of being a pair defined on some open subset $\mathcal{O}\cup\mathcal{O}^* \subset \mathbb{C}\setminus\mathbb{R}$.

4.2. **Standard unitary transforms of generalized Nevanlinna pairs and families.** Let $\mathcal{H}$ be a Hilbert space and let the generalized Nevanlinna pair $\{\Phi,\Psi\}$ and the generalized Nevanlinna family $\tau(\lambda)$ be connected by

$$\tau(\lambda) = \big\{ \{\Phi(\lambda)h, \Psi(\lambda)h\} : h \in \mathcal{K} \big\}, \tag{4.10}$$

where $\lambda \in \mathcal{D}_{\Phi,\Psi}$; see Proposition 4.3. Let $W = (W_{ij})_{i,j=0}^1$ be a standard unitary operator in the Kreĭn space $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$. Then W transforms the generalized Nevanlinna pair $\{\Phi,\Psi\}$ as follows:

$$\begin{pmatrix}\Phi_W(\lambda)\\ \Psi_W(\lambda)\end{pmatrix} = \begin{pmatrix}W_{00} & W_{01}\\ W_{10} & W_{11}\end{pmatrix}\begin{pmatrix}\Phi(\lambda)\\ \Psi(\lambda)\end{pmatrix}. \tag{4.11}$$

Moreover, by considering W as a transformer (in the sense of [24]) of the generalized Nevanlinna family τ one obtains the family

$$\tau_W(\lambda) = \{ \{W_{00}f + W_{01}f', W_{10}f + W_{11}f'\} : \{f,f'\} \in \tau(\lambda) \}. \tag{4.12}$$

Clearly, the pair $\{\Phi_W,\Psi_W\}$ and the family $\tau_W(\lambda)$ are connected by

$$\tau_W(\lambda) = \big\{ \{\Phi_W(\lambda)h, \Psi_W(\lambda)h\} : h \in \mathcal{K} \big\}. \tag{4.13}$$

The next purpose is to show that for suitable standard unitary operators W the pair $\{\Phi_W,\Psi_W\}$ in (4.11) is a generalized Nevanlinna pair with κ negative squares and that τ_W in (4.12) is a generalized Nevanlinna family with κ negative squares. Due to the correspondence in (4.13) it suffices to restrict attention to standard unitary transforms of generalized Nevanlinna pairs. Corresponding results for standard unitary transforms of generalized Nevanlinna families follow accordingly.

Proposition 4.5. *Let $\mathcal{H}$ be a Hilbert space, let $\{\Phi,\Psi\}$ be a generalized Nevanlinna pair with κ negative squares, and let W be a standard unitary operator in $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$. Then the pair $\{\Phi_W,\Psi_W\}$ defined in* (4.11) *satisfies the properties (P1) and (P3) of Definition 4.1. Moreover, for every $\zeta \in \mathbb{C}_+$ the operator functions*

$$\begin{aligned}&(W_{00}^* - \zeta W_{01}^*)\Psi_W(\lambda) - (W_{10}^* - \zeta W_{11}^*)\Phi_W(\lambda),\\ &(W_{00}^* - \bar\zeta W_{01}^*)\Psi_W(\bar\lambda) - (W_{10}^* - \bar\zeta W_{11}^*)\Phi_W(\bar\lambda),\end{aligned} \tag{4.14}$$

are boundedly invertible for all $\lambda \in \mathcal{D}_{\Phi,\Psi}\cap\mathbb{C}_+$ except at most κ points.

Proof. Let $\{\Phi,\Psi\}$ be a generalized Nevanlinna pair with κ negative squares on $\mathcal{D}_{\Phi,\Psi}$. Note that the condition (P1) can be rewritten as

$$\begin{pmatrix}\Phi(\bar\lambda)\\ \Psi(\bar\lambda)\end{pmatrix}^*\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\begin{pmatrix}\Phi(\lambda)\\ \Psi(\lambda)\end{pmatrix} = 0 \tag{4.15}$$

and that the kernel in (P3) can be written as

$$\mathsf{K}_{\Phi,\Psi}(\lambda,\mu) = \frac{1}{\lambda-\bar\mu}\begin{pmatrix}\Phi(\bar\lambda)\\ \Psi(\bar\lambda)\end{pmatrix}^*\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\begin{pmatrix}\Phi(\bar\mu)\\ \Psi(\bar\mu)\end{pmatrix}. \tag{4.16}$$

Since W is a standard unitary operator

$$W^* J_{\mathcal{H}^2} W = W J_{\mathcal{H}^2} W^* = J_{\mathcal{H}^2} \tag{4.17}$$

holds, where $J_{\mathcal{H}^2}$ is as in (3.2). Hence, it follows from (4.15) and (4.16) that the pair $\{\Phi_W, \Psi_W\}$ also satisfies the conditions (P1) and (P3). Furthermore, by means of (4.17) the following identities are easily checked

$$(W_{00}^* - \zeta W_{01}^*)\Psi_W(\lambda) - (W_{10}^* - \zeta W_{11}^*)\Phi_W(\lambda) = \Psi(\lambda) + \zeta\Phi(\lambda),$$
$$(W_{00}^* - \bar\zeta W_{01}^*)\Psi_W(\bar\lambda) - (W_{10}^* - \bar\zeta W_{11}^*)\Phi_W(\bar\lambda) = \Psi(\bar\lambda) + \bar\zeta\Phi(\bar\lambda).$$

Hence, these identities and the condition (P2) of Definition 4.1 yield (4.14). □

Remark 4.6. *Assume that the standard unitary operator W is of the form* (3.9)*, so that the pair $\{\Phi_W, \Psi_W\}$ in* (4.11) *is given by*

$$\Phi_W(\lambda) = X^{-1}\Phi(\lambda), \quad \Psi_W(\lambda) = YX^{-1}\Phi(\lambda) + X^*\Psi(\lambda),$$

and the family $\tau_W(\lambda)$ in (4.12) *is given by*

$$\tau_W(\lambda) = X^*\tau(\lambda)X + Y. \tag{4.18}$$

Then the condition (4.14) *is equivalent to*

$$\Psi_W(\lambda) + (\zeta X^*X - Y)\Phi_W(\lambda) \quad \text{and} \quad \Psi_W(\bar\lambda)^* + \Phi_W(\bar\lambda)^*(\zeta X^*X - Y) \tag{4.19}$$

to be boundedly invertible for all $\lambda \in \mathcal{D}_{\Phi,\Psi} \cap \mathbb{C}_+$ except at most κ points.

Moreover, if the operator $\zeta X^*X - Y$ in (4.19) *is of the form μI with $\mu \in \mathbb{C}^+$, then* (4.19) *is equivalent to (P2) of Definition 4.1. Hence, in this case $\{\Phi_W, \Psi_W\}$ in* (4.11) *is a generalized Nevanlinna pair with κ negative squares and $\tau_W(\lambda)$ in* (4.12) *is a generalized Nevanlinna family with κ negative squares.*

The previous situation occurs in the particular case that W is of the form (3.9) *with*

$$X = xI_{\mathcal{H}}, \quad Y = yI_{\mathcal{H}}, \quad x \in \mathbb{C}, \quad y \in \mathbb{R}. \tag{4.20}$$

Then

$$\zeta X^*X - Y = (\zeta|x|^2 - y)I \quad \text{and} \quad \zeta|x|^2 - y \in \mathbb{C}_+. \tag{4.21}$$

Therefore, under these circumstances, $\{\Phi_W, \Psi_W\}$ in (4.11) *is a generalized Nevanlinna pair with κ negative squares.*

Proposition 4.5 and its special cases as discussed in Remark 4.6 now lead to the following corollary (see (4.20) and (4.21)).

Corollary 4.7. *Assume that the generalized Nevanlinna pair $\{\Phi, \Psi\}$ satisfies*

$$0 \in \rho(\Psi(\lambda_0) + \zeta\Phi(\lambda_0)) \quad \text{and} \quad 0 \in \rho(\Psi(\bar\lambda_0) + \bar\zeta\Phi(\bar\lambda_0)),$$

with $\lambda_0 = a + ib$, $a \in \mathbb{R}$, $b > 0$, and $\zeta = c + id$, $c \in \mathbb{R}$, $d > 0$. Let W be the standard unitary operator of the form (3.9)*, where X and Y are given by* (4.20) *with*

$$x = \sqrt{\frac{b}{d}}, \quad y = \frac{cb - ad}{d}.$$

Then the pair $\{\Phi_W, \Psi_W\}$ in (4.11) *is a generalized Nevanlinna pair with κ negative squares which satisfies the additional properties*

$$0 \in \rho(\Psi_W(\lambda_0) + \lambda_0\Phi_W(\lambda_0)) \quad \text{and} \quad 0 \in \rho(\Psi_W(\bar\lambda_0) + \bar\lambda_0\Phi_W(\bar\lambda_0)).$$

It will be a consequence of the main realization result, Theorem 4.10 below, that the result stated in Corollary 4.7 holds actually for every standard unitary operator W in $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$, that is, $\{\Phi_W, \Psi_W\}$ in Proposition 4.5 is a generalized Nevanlinna pair, i.e. it satisfies also the invertibility conditions (P2) in Definition 4.1; see Corollary 4.12.

4.3. **Weyl families as generalized Nevanlinna families.** In this subsection it will be shown that every Weyl family corresponding to a boundary relation for the adjoint of a symmetric relation in a Pontryagin space is a generalized Nevanlinna family.

Theorem 4.8. *Let S be a closed symmetric relation in a Pontryagin space $\mathfrak{H}$ with negative index κ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a minimal boundary relation for S^+ with the corresponding Weyl family M. Then M is a generalized Nevanlinna family with κ negative squares.*

Proof. Let A be the selfadjoint relation in $\mathfrak{H} \times \mathcal{H}$ defined by the boundary relation Γ as in (3.5). Since $\mathfrak{H} \times \mathcal{H}$ is a Pontryagin space there is an alternative for A: either $\rho(A) \neq \emptyset$, in which case the nonreal spectrum of A consists of at most finitely many eigenvalues, or $\rho(A) = \emptyset$; cf. Lemma 2.2. The proof will be given in two steps corresponding to these cases.

Step 1. Assume that $\rho(A) \neq \emptyset$. Denote the compressed resolvent $P_{\mathcal{H}}(A-\lambda)^{-1}|_{\mathcal{H}}$ by $\Phi(\lambda)$, $\lambda \in \rho(A)$. Then by (3.6) $\Phi(\lambda) = -(M(\lambda)+\lambda)^{-1}$ for all $\lambda \in \rho(A)$ and therefore

$$M(\lambda) = \left\{ \left\{ \Phi(\lambda)h, -(I+\lambda\Phi(\lambda))h \right\} : \, h \in \mathcal{H} \right\}. \tag{4.22}$$

Setting $\Psi(\lambda) = -(I+\lambda\Phi(\lambda))$ it is clear that Φ and Ψ are meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and it follows from $\Phi(\lambda) = \Phi(\bar{\lambda})^*$ and $\Psi(\lambda) = \Psi(\bar{\lambda})^*$ that the symmetry condition (P1) in Definition 4.1 holds. Observe that $\Psi(\lambda) + \lambda\Phi(\lambda) = -I$, which shows that condition (P2) of Definition 4.1 is satisfied. Furthermore, a simple calculation shows that the kernel $\mathsf{K}_{\Phi,\Psi}$ takes the form

$$\begin{aligned} \mathsf{K}_{\Phi,\Psi}(\lambda,\mu) &= \frac{\Phi(\lambda) - \Phi(\bar{\mu})}{\lambda - \bar{\mu}} - \Phi(\lambda)\Phi(\bar{\mu}) \\ &= P_{\mathcal{H}}(A-\lambda)^{-1}(I-P_{\mathcal{H}})(A-\bar{\mu})^{-1}|_{\mathcal{H}}. \end{aligned}$$

Thus for $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}_+$, and $x_i \in \mathcal{H}$, $i = 1, \dots, n$, it follows that

$$\left(\mathsf{K}_{\Phi,\Psi}(\lambda_i,\lambda_j)x_i, x_j\right)_{i,j=1}^n = \left[P_{\mathfrak{H}}\left(A - \bar{\lambda}_j\right)^{-1} \begin{pmatrix} 0 \\ x_i \end{pmatrix}, P_{\mathfrak{H}}\left(A - \bar{\lambda}_i\right)^{-1} \begin{pmatrix} 0 \\ x_j \end{pmatrix} \right]_{i,j=1}^n,$$

where $[\cdot,\cdot]$ denotes the inner product in $\mathfrak{H}$. Hence the number of negative squares of $\mathsf{K}_{\Phi,\Psi}(\cdot,\cdot)$ is less or equal to κ, the negative index $\kappa_-(\mathfrak{H} \times \mathcal{H}) = \kappa_-(\mathfrak{H})$.

If the boundary relation Γ is assumed to be minimal, then the condition

$$\mathfrak{H} = \overline{\operatorname{span}} \left\{ \operatorname{ran} P_{\mathfrak{H}}(A-\lambda)^{-1}|_{\mathcal{H}} : \lambda \in \rho(A) \cap (\mathbb{C} \setminus \mathbb{R}) \right\}$$

is satisfied; see Corollaries 3.13, 3.14. Therefore the numbers of negative squares of $\mathsf{K}_{\Phi,\Psi}(\cdot,\cdot)$ is equal to $\kappa_-(\mathfrak{H})$, and the condition (P3) of Definition 4.2 is satisfied. Thus $\{\Phi,\Psi\}$ is a generalized Nevanlinna pair with κ negative squares. It follows from (4.22) that M is a generalized Nevanlinna family with κ negative squares; cf. Proposition 4.3.

Step 2. In the general case the resolvent set $\rho(A)$ may be empty. The minimality of Γ implies that S is an operator; cf. Lemma 3.13. By Proposition 3.12 there exists a standard unitary operator W such that the main transform A_W corresponding to the boundary relation $\Gamma_W = W \circ \Gamma$ satisfies the condition $\rho(A_W) \neq \emptyset$. Furthermore, the boundary relation Γ_W is automatically minimal, since $\operatorname{dom}\Gamma_W = \operatorname{dom}\Gamma$ (see also Corollary 3.14).

According to Step 1 the Weyl family M_W corresponding to Γ_W is a generalized Nevanlinna family with κ negative squares with the property

$$0 \in \rho(M_W(\lambda) + \lambda), \tag{4.23}$$

cf. (3.6). Let $\{\widetilde{\Phi}, \widetilde{\Psi}\}$ be a generalized Nevanlinna pair with κ negative squares, induced by M_W. Then (4.23) implies that

$$0 \in \rho(\widetilde{\Psi}(\lambda) + \lambda\widetilde{\Phi}(\lambda)). \tag{4.24}$$

Define the pair $\{\Phi, \Psi\}$ by

$$\begin{pmatrix} \Phi(\lambda) \\ \Psi(\lambda) \end{pmatrix} = W^{-1} \begin{pmatrix} \widetilde{\Phi}(\lambda) \\ \widetilde{\Psi}(\lambda) \end{pmatrix}.$$

According to Proposition 4.5 (applied with W replaced by W^{-1}) it follows that $\{\Phi, \Psi\}$ has the properties (P1) and (P3). Moreover, the standard unitary operator W can be chosen of the form (3.9), where

$$X = (a + ib)I, \quad Y = 0,$$

with $a > 0$ sufficiently large and $b > 0$ (see the proof of Proposition 3.12). Then Remark 4.6 yields

$$\widetilde{\Psi}(\lambda) + \lambda\widetilde{\Phi}(\lambda) = (a - ib)\left(\Psi(\lambda) + \frac{\lambda}{a^2 + b^2}\,\Phi(\lambda)\right). \tag{4.25}$$

It follows from (4.24) and (4.25) that $\{\Phi, \Psi\}$ also has the property (P2). Hence $\{\Phi, \Psi\}$ is a generalized Nevanlinna pair with κ negative squares. According to Proposition 4.3 the corresponding Weyl family M is a generalized Nevanlinna family with κ negative squares. $\square$

Observe that if S is a closed symmetric operator in a Pontryagin space $\mathfrak{H}$ with negative index κ and if $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ is a boundary relation for S^+ which is not necessarily minimal, then the corresponding Weyl family is a generalized Nevanlinna family with $\kappa' \leq \kappa$ negative squares. The next statement for the Hilbert space case is known from [8].

Corollary 4.9. *Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ be a boundary relation for S^*. Then the corresponding Weyl family is a Nevanlinna family.*

4.4. **Generalized Nevanlinna families as Weyl families.** Let τ be a generalized Nevanlinna family with κ negative squares in a Hilbert space $\mathcal{H}$. Let τ have the representation

$$\tau(\lambda) = \big\{\{\Phi(\lambda)h, \Psi(\lambda)h : h \in \mathcal{H}\}\big\} = \big\{\{k, k'\} : \Psi(\bar{\lambda})^*k = \Phi(\bar{\lambda})^*k'\big\} \tag{4.26}$$

with a generalized Nevanlinna pair $\{\Phi, \Psi\}$ with κ negative squares on $\mathcal{D}_{\Phi,\Psi}$, see Proposition 4.3; here the second equality follows from the property (Q1) in Definition 4.2. The reproducing kernel space $\mathfrak{H}(\Phi, \Psi)$ induced by the pair $\{\Phi, \Psi\}$ is characterized by the properties

(i) the mappings $\lambda \mapsto \mathsf{K}_{\Phi,\Psi}(\lambda, \mu)h \in \mathfrak{H}(\Phi, \Psi)$ for all $h \in \mathcal{H}$ and all μ in the domain of holomorphy $\mathcal{D}_{\Phi,\Psi}$ of Φ and Ψ form a dense set in $\mathfrak{H}(\Phi, \Psi)$;

(ii) for every $f \in \mathfrak{H}(\Phi, \Psi)$ the following identity holds:

$$\langle f(\cdot), \mathsf{K}_{\Phi,\Psi}(\cdot, \mu)h\rangle = (f(\mu), h), \quad h \in \mathcal{H}. \tag{4.27}$$

This function space equipped with (the extension) of the inner product

$$\langle \mathsf{K}_{\Phi,\Psi}(\cdot,\nu)k, \mathsf{K}_{\Phi,\Psi}(\cdot,\mu)h\rangle := (\mathsf{K}_{\Phi,\Psi}(\mu,\nu)k,h), \quad \nu,\mu \in \mathcal{D}_{\Phi,\Psi}, \ \ h,k\in\mathcal{H},$$

is a Pontryagin space with κ negative squares.

Multiplication by the independent variable is a closed symmetric operator in the reproducing kernel Pontryagin space $\mathfrak{H}(\Phi,\Psi)$. In the following theorem it will be shown that every generalized Nevanlinna family can be realized as the Weyl family of a boundary relation corresponding to the multiplication operator in $\mathfrak{H}(\Phi,\Psi)$. The proof given here is based on the approach which in the Hilbert space setting was used in [7, Theorem 2.5 and Remark 2.6]; for an other approach which uses Cayley transforms, see [3, Theorem 6.1].

Theorem 4.10. *Let τ be a generalized Nevanlinna family with κ negative squares in a Hilbert space $\mathcal{H}$ and let τ be represented in the form* (4.26) *by a generalized Nevanlinna pair $\{\Phi,\Psi\}$. Then*

$$S = \bigl\{\{f,f'\}\in\mathfrak{H}(\Phi,\Psi)^2 : \ f'(\lambda)=\lambda f(\lambda)\bigr\} \tag{4.28}$$

is a closed symmetric operator in the reproducing kernel Pontryagin space $\mathfrak{H}(\Phi,\Psi)$ and

$$\Gamma = \left\{\left\{\begin{pmatrix} f\\ f'\end{pmatrix},\begin{pmatrix} h\\ h'\end{pmatrix}\right\} : \begin{array}{c} f,f'\in\mathfrak{H}(\Phi,\Psi),\ h,h'\in\mathcal{H},\\ f'(\lambda)-\lambda f(\lambda)=\Psi(\bar\lambda)^*h-\Phi(\bar\lambda)^*h'\end{array}\right\} \tag{4.29}$$

is a minimal boundary relation for S^+ such that the corresponding Weyl family coincides with the generalized Nevanlinna family τ.

Proof. The generalized Nevanlinna pair $\{\Phi,\Psi\}$ with κ negative squares is defined in Definition 4.1. The maximality condition (P2) is weaker than the condition that

$$0\in\rho(\Psi(\lambda_0)+\lambda_0\Phi(\lambda_0)),\quad 0\in\rho(\Psi(\bar\lambda_0)+\bar\lambda_0\Phi(\bar\lambda_0)) \tag{4.30}$$

hold for some $\lambda_0\in\mathbb{C}\setminus\mathbb{R}$ in the domain $\mathcal{D}_{\Phi,\Psi}$ of holomorphy of Φ and Ψ. The proof of the theorem will be given in two steps, depending on whether (4.30) is satisfied or not.

Step 1. Assume that $\{\Phi,\Psi\}$ is a generalized Nevanlinna pair with κ negative squares for which (4.30) is satisfied. In this case one can proceed in a similar way as in the Hilbert space setting; cf. the proof of [7, Theorem 2.5]. For the convenience of the reader a short sketch will be given; for some further details see also [21], where the result is formulated in terms of the main transform. Using (4.30) it will be shown that the linear relation

$$A = \left\{\left\{\begin{pmatrix} f\\ h\end{pmatrix},\begin{pmatrix} f'\\ -h'\end{pmatrix}\right\} : \begin{array}{c} f,f'\in\mathfrak{H}(\Phi,\Psi),\ h,h'\in\mathcal{H},\\ f'(\lambda)-\lambda f(\lambda)=\Psi(\bar\lambda)^*h-\Phi(\bar\lambda)^*h'\end{array}\right\}$$

is selfadjoint in $\mathfrak{H}(\Phi,\Psi)\oplus\mathcal{H}$. For this purpose define

$$B = \operatorname{span}\left\{\left\{\begin{pmatrix} \mathsf{K}_{\Phi,\Psi}(\cdot,\bar\mu)k\\ -\Phi(\mu)k\end{pmatrix},\begin{pmatrix} \mu\mathsf{K}_{\Phi,\Psi}(\cdot,\bar\mu)k\\ \Psi(\mu)k\end{pmatrix}\right\} : \ k\in\mathcal{H},\ \mu\in\mathcal{D}_{\Phi,\Psi}\right\}. \tag{4.31}$$

By means of (4.1) one can immediately check that $B\subset A$ and, moreover, by using (4.27) it is seen that B is symmetric in $\mathfrak{H}(\Phi,\Psi)\oplus\mathcal{H}$. The elements in $\operatorname{ran}(B-\lambda_0)$, have the form

$$\begin{pmatrix} (\mu-\lambda_0)\mathsf{K}_{\Phi,\Psi}(\cdot,\bar\mu)k\\ (\Psi(\mu)+\lambda_0\Phi(\mu))k\end{pmatrix}.$$

Therefore, choosing $\mu = \lambda_0$ and taking into account that

$$\operatorname{ran}(\Psi(\lambda_0) + \lambda_0 \Phi(\lambda_0)) = \mathcal{H},$$

it follows that $\{0\} \oplus \mathcal{H} \subset \operatorname{ran}(B - \lambda_0)$; hence also the elements of the form

$$\begin{pmatrix} \mathsf{K}_{\Phi,\Psi}(\cdot, \bar{\mu})k \\ 0 \end{pmatrix}, \quad k \in \mathcal{H}, \quad \mu \in \mathcal{D}_{\Phi,\Psi}, \quad \mu \neq \bar{\lambda}_0,$$

belong to $\operatorname{ran}(B - \lambda_0)$. Therefore, $\operatorname{ran}(B - \lambda_0)$ is dense in $\mathfrak{H}(\Phi, \Psi) \oplus \mathcal{H}$. Similarly, one shows that $\operatorname{ran}(B - \bar{\lambda}_0)$ is dense in $\mathfrak{H}(\Phi, \Psi) \oplus \mathcal{H}$. Furthermore, by means of (4.27) it is straightforward to check that the adjoint of B coincides with A. Hence, A is a closed symmetric relation with $\operatorname{ran}(A - \lambda)$ dense in $\mathfrak{H}(\Phi, \Psi) \oplus \mathcal{H}$ for $\lambda = \lambda_0$ and $\lambda = \bar{\lambda}_0$. Consequently, A is selfadjoint.

The Weyl family M associated with the boundary relation Γ is defined by

$$M(\lambda) = \left\{ \{h, h'\} : \left\{ \begin{pmatrix} f \\ \lambda f \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\}.$$

Therefore,

$$M(\lambda) = \left\{ \{h, h'\} : \Psi(\bar{\lambda})^* h = \Phi(\bar{\lambda})^* h' \right\},$$

and it follows from (4.26) that $M(\lambda) = \tau(\lambda)$. Moreover, it is clear from the formula (4.31) and the inclusion $B \subset A$ that the boundary relation Γ is minimal; see (3.14).

Step 2. Now assume that $\{\Phi, \Psi\}$ is a generalized Nevanlinna pair with κ negative squares for which (4.30) is not necessarily satisfied. By Definition 4.1

$$0 \in \rho(\Psi(\lambda_0) + \zeta \Phi(\lambda_0)), \quad 0 \in \rho(\Psi(\bar{\lambda}_0) + \bar{\zeta} \Phi(\bar{\lambda}_0))$$

for some $\lambda_0 = a + ib \in \mathcal{D}_{\Phi,\Psi}$ and some $\zeta = c + id$ with $b, d > 0$. Then the standard unitary operator W in Corollary 4.7 provides a generalized Nevanlinna pair of the form

$$\begin{pmatrix} \Phi_W(\lambda) \\ \Psi_W(\lambda) \end{pmatrix} = W \begin{pmatrix} \Phi(\lambda) \\ \Psi(\lambda) \end{pmatrix}$$

with κ negative squares on $\mathcal{D}_{\Phi,\Psi}$ which additionally satisfies the condition (4.30) for $\lambda_0 \in \mathcal{D}_{\Phi,\Psi} \cap \mathbb{C}_+$ (equivalently, for all λ in a small neighborhood $\mathcal{O}$ of λ_0). Hence it follows from the first part of the proof that

$$\Gamma^{(W)} := \left\{ \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} : \begin{matrix} f, f' \in \mathfrak{H}(\Phi_W, \Psi_W), \; h, h' \in \mathcal{H}, \\ f'(\lambda) - \lambda f(\lambda) = \Psi_W(\bar{\lambda})^* h - \Phi_W(\bar{\lambda})^* h' \end{matrix} \right\} \tag{4.32}$$

is a minimal boundary relation for S^+ such that the corresponding Weyl family coincides with the generalized Nevanlinna family

$$\tau_W(\lambda) = \left\{ \{\Phi_W(\lambda)h, \Phi_W(\lambda)h\} : \; h \in \mathcal{H} \right\}.$$

Due to Lemma 3.9 $\Gamma = W^{-1} \circ \Gamma^{(W)}$ is a boundary relation for S^+ and the corresponding Weyl family is precisely τ. It remains to note that due to the above choice of W (cf. (4.18), Corollary 4.7 the equality $\mathfrak{H}(\Phi_W, \Psi_W) = \mathfrak{H}(\Phi, \Psi)$ holds. Moreover, using the formulas for Φ_W and Ψ_W one can rewrite the formula for $\Gamma^{(W)}$ in (4.32) equivalently in the form (4.29) with $\binom{h}{h'} = W^{-1}\binom{\tilde{h}}{\tilde{h}'}$. $\square$

For the special case $\kappa = 0$, i.e. τ is a Nevanlinna family, Theorem 4.10 implies the following result; cf. [2, 3] and [8, Theorem 3.9]

Corollary 4.11. *Let τ be a Nevanlinna family represented in the form* (4.26) *by a Nevanlinna pair $\{\Phi, \Psi\}$. Then the multiplication operator S in* (4.28) *is a closed symmetric operator in the reproducing kernel Hilbert space $\mathfrak{H}(\Phi, \Psi)$ and the relation Γ in* (4.29) *is a minimal boundary relation for S^* such that the corresponding Weyl family coincides with the Nevanlinna family τ.*

The next corollary is an improvement of the statement in Proposition 4.5: the maximality condition (4.14) can actually be replaced by the maximality condition as in Definition 4.1; cf. Remark 4.6.

Corollary 4.12. *Let $\{\Phi, \Psi\}$ is a generalized Nevanlinna pair with κ negative squares on $\mathcal{D}_{\Phi,\Psi}$ and let W be a standard unitary operator in $(\mathcal{H}^2, [\![\cdot,\cdot]\!]_{\mathcal{H}^2})$. Then the transformed pair $\{\Phi_W, \Psi_W\}$ in* (4.11) *is a generalized Nevanlinna pair with κ negative squares on $\mathcal{D}_{\Phi,\Psi}$.*

Proof. If $\{\Phi, \Psi\}$ be a generalized Nevanlinna pair with κ negative squares on $\mathcal{D}_{\Phi,\Psi}$, then by Theorem 4.10 it corresponds to a Weyl family of a (minimal) boundary relation Γ via (4.10); cf. (4.29). By Lemma 3.8 $\Gamma_W = W \circ \Gamma$ is also a boundary relation whose Weyl family M_W is given by (3.8). According to Theorem 4.8 M_W is also a generalized Nevanlinna family with κ negative squares. This shows that the corresponding transformed pair $\{\Phi_W, \Psi_W\}$ in Proposition 4.5 is actually a generalized Nevanlinna pair with κ negative squares on $\mathcal{D}_{\Phi,\Psi}$; cf. Proposition 4.3. $\square$

In the following corollary condition (4.30) in the proof of Theorem 4.10 is connected to the resolvent set of the selfadjoint relation A in $\mathfrak{H}(\Phi, \Psi) \times \mathcal{H}$ associated with the minimal boundary relation $\Gamma \subset \mathfrak{H}(\Phi, \Psi)^2 \times \mathcal{H}^2$ in (4.29).

Corollary 4.13. *Let τ be a generalized Nevanlinna family with κ negative squares in a Hilbert space $\mathcal{H}$ and let τ be represented in the form* (4.26) *by a generalized Nevanlinna pair $\{\Phi, \Psi\}$. Let A be the main transform of the boundary relation Γ in* (4.29),

$$A = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \begin{array}{c} f, f' \in \mathfrak{H}(\Phi, \Psi),\, h, h' \in \mathcal{H}, \\ f'(\lambda) - \lambda f(\lambda) = \Psi(\bar{\lambda})^* h - \Phi(\bar{\lambda})^* h' \end{array} \right\},$$

and let $\lambda_0 \in \mathbb{C}_+$. Then $\lambda_0 \in \rho(A)$ if and only if

$$(\tau(\lambda_0) + \lambda_0)^{-1} \in \mathbf{B}(\mathcal{H}) \quad \text{and} \quad (\tau(\bar{\lambda}_0) + \bar{\lambda}_0)^{-1} \in \mathbf{B}(\mathcal{H}). \tag{4.33}$$

Proof. It is clear form (3.6) that $\lambda_0 \in \rho(A)$ implies the first condition in (4.33). Since $\rho(A)$ is symmetric with respect to the real line also $\bar{\lambda}_0 \in \rho(A)$ and hence the second condition in (4.33) is satisfied.

Conversely, if (4.33) holds, then the Nevanlinna pair $\{\Phi, \Psi\}$ in (4.26) satisfies the conditions (4.30) in the proof of Theorem 4.10. It follows as in the proof of Theorem 4.10 that λ_0 and $\bar{\lambda}_0$ belong to $\rho(A)$. $\square$

Observe, that by taking adjoints and using the property (Q1) in Definition 4.2 it follows that the conditions in (4.33) are actually equivalent to each other:

$$(\tau(\lambda_0) + \lambda_0)^{-1} \in \mathbf{B}(\mathcal{H}) \quad \Leftrightarrow \quad (\tau(\bar{\lambda}_0) + \bar{\lambda}_0)^{-1} \in \mathbf{B}(\mathcal{H}).$$

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