ON KREIN-TYPE EXTENSIONS OF SYMMETRIC SEMIBOUNDED OPERATORS AND THEIR SPECTRAL INSTABILITY WITH APPLICATIONS TO ODE'S AND PDE'S ON BOUNDED DOMAINS

JUSSI BEHRNDT, FRITZ GESZTESY, AND HENK DE SNOO

Dedicated to our friend and collaborator Marius Mitrea on the occasion of his 60th birthday.

Abstract. For a densely defined, closed, semibounded (hence, symmetric) operator in a Hilbert space, a family of self-adjoint extensions is considered, which can be viewed as natural generalizations of the classical Krein–von Neumann extension of a nonnegative symmetric operator. We review various properties of these so-called Krein-type extensions and we discuss their weak coupling spectral instability. The abstract results are illustrated for regular Sturm–Liouville operators and the multidimensional Laplacian on a bounded Lipschitz domain.

1. Introduction

In this paper we consider a family of self-adjoint extensions of symmetric operators that can be viewed as natural generalizations of the classical Krein–von Neumann extension of a nonnegative symmetric operator. To set the stage, let S be a densely defined, closed, symmetric operator in a complex, separable Hilbert space $\mathfrak{H}$ and recall that under the nonnegativity assumption

$$(Sf, f) \geq 0, \quad f \in \operatorname{dom}(S), \tag{1.1}$$

the Krein–von Neumann extension S_K is defined as the smallest nonnegative self-adjoint extension of S; see [62], [63], and Remark 2.9 for more references. Therefore, S_K can be viewed as the counterpart of the Friedrichs extension, which is defined as the largest nonnegative self-adjoint extension of S. In other words, we have

$$S_K \leq H \leq S_F \tag{1.2}$$

in the sense of the corresponding quadratic forms for any other nonnegative self-adjoint extension H of S or, equivalently, the inequalities

$$(S_F - zI)^{-1} \leq (H - zI)^{-1} \leq (S_K - zI)^{-1}, \quad z < 0, \tag{1.3}$$

hold for the resolvents. In the context of linear relations S_K can also be expressed as $S_K = ((S^{-1})_F)^{-1}$. Furthermore, in the special case that S is uniformly positive,

---

Date: January 27, 2026.

2020 *Mathematics Subject Classification.* Primary: 35J10, 47A10, 47A75, 47F05, 81Q10; Secondary: 34L05, 35P05, 47A55, 47E05.

Key words and phrases. Weak coupling, spectral instability, Krein-type extension, boundary triplet, Weyl–Titchmarsh function.

that is, for some $\kappa > 0$, $(Sf, f) \geq \kappa\|f\|^2$, $f \in \operatorname{dom}(S)$, the Krein–von Neumann extension admits the following simple and more explicit form

$$S_K f = S^* f, \quad \operatorname{dom}(S_K) = \operatorname{dom}(S) \dotplus \ker(S^*), \tag{1.4}$$

where S^* is the adjoint of the densely defined, closed, symmetric operator S in $\mathfrak{H}$.

Inspired by (1.4) one can define a family of extensions $S_{K,x}$ of the densely defined, closed, symmetric operator S with lower bound $\kappa \in \mathbb{R}$ by

$$S_{K,x} f = S^* f, \quad \operatorname{dom}(S_{K,x}) = \operatorname{dom}(S) \dotplus \ker(S^* - xI), \quad x < \kappa. \tag{1.5}$$

The extensions $S_{K,x}$ are self-adjoint in $\mathfrak{H}$ and they will be referred to as Krein-type extensions of S; clearly, in the case $\kappa > 0$ and $x = 0$ this definition reduces to (1.4), so that $S_{K,0} = S_K$. The situation becomes more challenging when one tries to define a Krein-type extension at the lower bound κ: Although the definition (1.5) is still meaningful it does not lead to a self-adjoint operator in general (since $\ker(S^* - \kappa I)$ can be trivial) and thus $S_{K,\kappa}$ needs to be defined differently. In fact, as for the Krein–von Neumann extension in the general nonnegative case, one can define $S_{K,\kappa}$ as the smallest self-adjoint extension of S with lower bound κ. Equivalenty, one has that $S_{K,\kappa}$ is the strong resolvent limit of the monotone family of Krein-type extensions $S_{K,x}$ for $x \uparrow \kappa$, that is,

$$S_{K,\kappa} = \text{sr-lim}_{x\uparrow\kappa} S_{K,x}. \tag{1.6}$$

For a more detailed investigation and spectral analysis of the Krein-type extensions it is natural to employ the concept of boundary triplets and their Weyl (resp., Weyl–Titchmarsh) functions. In the present situation, where a symmetric operator S with lower bound $\kappa \in \mathbb{R}$ is given, one can construct a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ such that the Friedrichs extension S_F of S is induced by the boundary mapping Γ_0, that is,

$$S_F f = S^* f, \quad \operatorname{dom}(S_F) = \big\{ f \in \operatorname{dom}(S^*) \,|\, \Gamma_0 f = 0 \big\}. \tag{1.7}$$

If $M(z)$, $z \in \rho(S_F)$, denotes the corresponding Weyl–Titchmarsh function, an analytic (operator-valued) function in $\mathcal{G}$, then it follows that the Krein-type extensions $S_{K,x}$ for $x < \kappa$ can also be characterized by

$$S_{K,x} f = S^* f, \quad \operatorname{dom}(S_{K,x}) = \big\{ f \in \operatorname{dom}(S^*) \,|\, M(x)\Gamma_0 f = \Gamma_1 f \big\}, \quad x < \kappa, \tag{1.8}$$

and, if $M(\kappa)$ denotes the strong resolvent limit of the monotone family $M(x)$ as $x \uparrow \kappa$, one also has

$$S_{K,\kappa} f = S^* f, \quad \operatorname{dom}(S_{K,x}) = \big\{ f \in \operatorname{dom}(S^*) \,|\, \{\Gamma_0 f, \Gamma_1 f\} \in M(\kappa) \big\}, \tag{1.9}$$

where, in general, $M(\kappa)$ is a self-adjoint relation in $\mathcal{G}$.

After discussing the general theory of Krein-type extensions in Section 2, we then turn to a more specific topic afterwards. Following the theme in [15, 16] we are interested in spectral instability and the weak coupling phenomenon of Krein-type extensions $S_{K,x}$ for $x \leq \kappa$ under relatively compact and relatively form compact perturbations V. If $V \geq 0$ it follows in the case $x < \kappa$ under mild additional conditions on the perturbation that $S_{K,x}$ exhibits a spectral instability in the sense that

$$\sigma(S_{K,x} + \alpha V) \cap (-\infty, x) \neq \emptyset \text{ for } \textbf{any } \alpha < 0. \tag{1.10}$$

The situation is much more delicate for $x = \kappa$ and has been discussed in great detail in [16] in the case that the deficiency indices of S are $(1, 1)$. However, if one imposes the additional assumption that $\kappa \in \sigma_p(S_{K,\kappa})$, then it is easy to see that a spectral instability in the sense of (1.10) appears also at κ (see [15]). We also mention in this context that the classical weak coupling phenomenon for Schrödinger operators in $L^2(\mathbb{R}^n)$, $n = 1, 2$, goes back to Simon [77, 78] and we refer the reader to [16] for more details and the explicit connection between (1.10) and Simon's results in the context of Sturm–Liouville operators with an interface condition. For additional references regarding the weak coupling phenomenon of Schrödinger operators in dimensions $n = 1, 2$, we also refer to [20], [21], [22], [24], [27], [29], [30], [34], [47], [48], [54], [55], [56], [57], [58], [59], [64], [66], [67], [69], [70], [72], [73, p. 336–338].

The abstract considerations in Section 2 and 3 are then illustrated for the Krein-type extensions of regular Sturm–Liouville operators in Section 4 and for the Krein-type extensions of the Laplacian on a bounded Lipschitz domain in Section 5. More precisely, in Section 4 we consider the Sturm–Liouville differential expression

$$\ell = \frac{1}{r}\left[-\frac{d}{dt}p\frac{d}{dt} + q\right] \tag{1.11}$$

on a bounded open interval (a, b), where it is assumed that the coefficients are real functions on (a, b) such that $1/p, q, r \in L^1((a, b))$ and $p(t) > 0$, $r(t) > 0$ for almost all $t \in (a, b)$. In this case both endpoints a and b are regular and the minimal symmetric operator associated to ℓ in the weighted L^2-space $L^2((a, b); rdt)$ is bounded from below and has deficiency indices $(2, 2)$. We then characterize the boundary conditions of the Krein-type extensions $S_{K,x}$ of S and show that they are spectrally unstable. Similarly, in Section 5 we are interested in the Krein-type extensions of the Laplacian on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. Here we make use of results on the Dirichlet and Neumann trace operators on the maximal domain from [14, 19] and provide explicit boundary conditions of the Krein-type extensions of the minimal operator $S = -\Delta \restriction_{H_0^2(\Omega)}$; see [13, 14]. In order to show their spectral instability we proceed slightly differently than for ordinary differential operators. We find it convenient to first verify spectral instability of Robin Laplacians in a similar way as in [15] and to then conclude spectral instability of the Krein-type extension $S_{K,x}$ by using the inequality $S_{K,x} \leq A_R^{(\beta_x)}$, where the parameter β_x in the boundary condition of the Robin Laplacian $A_R^{(\beta_x)}$ is chosen such that x coincides with the smallest eigenvalue of $A_R^{(\beta_x)}$.

Notation. The inner product in a separable (complex) Hilbert space $\mathfrak{H}$ is denoted by $(\,\cdot\,, \cdot\,)$ and is assumed to be linear with respect to the first argument; the symbol I denotes the identity operator in $\mathfrak{H}$. If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ and $\operatorname{ran}(T)$ denote the domain and range of T, respectively. The resolvent set and spectrum of a closed linear operator T in $\mathfrak{H}$ are abbreviated by $\rho(T)$ and $\sigma(T)$, respectively. The set of eigenvalues is denoted by $\sigma_p(T)$ and for a self-adjoint operator T, $\sigma_d(T)$ and $\sigma_{ess}(T)$ denotes the essential and discrete spectrum, respectively. The Banach space of bounded (resp., compact) linear operators on $\mathfrak{H}$ is denoted by $\mathcal{B}(\mathfrak{H})$ (resp., $\mathcal{B}_\infty(\mathfrak{H})$). For $p \in [1, \infty)$, the corresponding ℓ^p-based trace ideals will be denoted by $\mathcal{B}_p(\mathfrak{H})$ with norms abbreviated by $\|\cdot\|_{\mathcal{B}_p(\mathfrak{H})}$. Finally, $L^p_r((a, b)) := L^p((a, b); rdt)$, $p \in [1, \infty)$, represents weighted L^p-spaces with weight $0 \leq r \in L^1_{loc}((a, b))$.

2. Abstract Krein-Type Extensions, Boundary Triplets,

AND WEYL–TITCHMARSH FUNCTIONS

In this section we first introduce the notion of abstract Krein-type extensions of a semibounded symmetric operator in $\mathfrak{H}$ and provide some properties of these self-adjoint operators. These extensions can be viewed as natural generalizations of the classical Krein–von Neumann extension in the nonnegative case. Using the concept of boundary triplets and Weyl–Titchmarsh functions, we show how the Krein-type extensions can be related to the values of the corresponding Weyl–Titchmarsh function in the boundary space $\mathcal{G}$. This observation is particularly useful in applications to differential operators as it reduces the problem to determine the Krein-type extensions to the computation of the Weyl–Titchmarsh function. For most of the statements in this section we provide elementary direct proofs and refer the reader to [18, Chapter 5] for a slightly more abstract treatment and further references.

2.1. **Abstract Krein-type extensions.** Throughout this section suppose that S is a densely defined, closed, symmetric operator in a Hilbert space $\mathfrak{H}$, assume that S is semibounded from below, and that $\kappa \in \mathbb{R}$ is the maximal lower bound[1]

$$(Sf, f) \geq \kappa(f, f), \quad f \in \operatorname{dom}(S); \tag{2.1}$$

such an inequality will be denoted by $S \geq \kappa I$ throughout this paper. For obvious reasons, the case $S = S^*$ will be excluded in the following.

Definition 2.1. The family of *Krein-type extensions* $S_{K,x}$, $x < \kappa$, of S is defined by

$$S_{K,x} = S^* \upharpoonright \operatorname{dom}(S_{K,x}), \quad \operatorname{dom}(S_{K,x}) = \operatorname{dom}(S) \dot{+} \ker(S^* - xI), \quad x < \kappa. \tag{2.2}$$

One notes that the sum in $\operatorname{dom}(S_{K,x})$ is indeed direct as otherwise $x < \kappa$ would be an eigenvalue of S; this is not possible as $S \geq \kappa I$. If we agree to denote the elements $f \in \operatorname{dom}(S_{K,x}) = \operatorname{dom}(S) \dot{+} \ker(S^* - xI)$ in the form $f = f_S + f_x$, where $f_S \in \operatorname{dom}(S)$, $f_x \in \operatorname{dom}(S^* - xI)$, and $x < \kappa$, then it is clear that

$$S_{K,x} f = S f_S + x f_x, \quad f = f_S + f_x, \ f_S \in \operatorname{dom}(S), \ f_x \in \ker(S^* - xI). \tag{2.3}$$

Lemma 2.2. *For $x < \kappa$, the operator $S_{K,x}$ is self-adjoint in $\mathfrak{H}$ and*

$$\sigma(S_{K,x}) \cap (-\infty, \kappa) = \{x\}. \tag{2.4}$$

In particular, $S_{K,x}$ is semibounded from below with lower bound x.

Proof. First we show that $S_{K,x}$ is symmetric. For $f = f_S + f_x \in \operatorname{dom}(S_{K,x})$ one has

$$\begin{aligned} \operatorname{Im}((S_{K,x} f, f)) &= \operatorname{Im}\big((Sf_S, f_S) + (Sf_S, f_x) + (S_{K,x} f_x, f_S) + (x f_x, f_x)\big) \\ &= \operatorname{Im}\big((Sf_S, f_x) + (S_{K,x} f_x, f_S)\big) \end{aligned} \tag{2.5}$$

and together with $(S_{K,x} f_x, f_S) = (f_x, S f_S)$ one concludes that $\operatorname{Im}((S_{K,x} f, f)) = 0$. Hence, $S_{K,x} \subseteq S_{K,x}^*$. In order to verify the inclusion $S_{K,x}^* \subseteq S_{K,x}$, consider $g \in \operatorname{dom}(S_{K,x})^*$. Then

$$(S_{K,x} f, g) = (S f_S + x f_x, g) = (f_S + f_x, S_{K,x}^* g) \tag{2.6}$$

holds for all $f = f_S + f_x \in \operatorname{dom}(S_{K,x})$. As $(S f_S, g) = (f_S, S_{K,x}^* g)$ one obtains

$$(f_x, (S_{K,x}^* - xI) g) = 0 \tag{2.7}$$

[1]Explicitly, $S \geq \kappa I$, but for all $\varepsilon > 0$, $S \not\geq (\kappa + \varepsilon) I$.

for all $f_x \in \ker(S^* - xI)$. Therefore, $(S^*_{K,x} - xI)g \in (\ker(S^* - xI))^\perp = \operatorname{ran}(S - xI)$ and hence there exists $h \in \operatorname{dom}(S)$ such that

$$(S^*_{K,x} - xI)g = (S - xI)h. \tag{2.8}$$

It follows that $g - h \in \ker(S^*_{K,x} - xI) \subseteq \ker(S^* - xI)$ and we conclude the decomposition

$$g = h + (g - h) \in \operatorname{dom}(S) \dot{+} \ker(S^* - xI) = \operatorname{dom}(S_{K,x}). \tag{2.9}$$

As $S_{K,x}$ is symmetric, it is clear that $S^*_{K,x}$ is an extension of $S_{K,x}$ and hence $S_{K,x}g = S^*_{K,x}g$. Thus, we have shown the inclusion $S^*_{K,x} \subseteq S_{K,x}$, and hence it follows that $S_{K,x}$ is self-adjoint.

It is clear from the definition of $S_{K,x}$ in (2.2) that x is an eigenvalue with corresponding eigenspace $\ker(S^* - xI)$. Since $S_{K,x}$ is self-adjoint, the closed subspace $(\ker(S^* - xI))^\perp = \operatorname{ran}(S - xI)$ reduces $S_{K,x}$ to a self-adjoint operator

$$\begin{aligned} &S'_{K,x} = S_{K,x} \restriction \operatorname{ran}(S - xI), \\ &\operatorname{dom}(S'_{K,x}) = \big(\operatorname{dom}(S) \dot{+} \ker(S^* - xI)\big) \cap \operatorname{ran}(S - xI), \end{aligned} \tag{2.10}$$

in $\operatorname{ran}(S - xI)$, and we claim that $S'_{K,x}$ is semibounded from below by κ. In fact, for $h = h_S + h_x = (S - xI)g \in \operatorname{dom}(S'_{K,x})$ we have $0 = (h_x, h) = (h_x, h_S) + (h_x, h_x)$ as $h \in \operatorname{ran}(S - xI)$. Therefore, using $S \geq \kappa I$ one concludes

$$\begin{aligned} \big((S'_{K,x} - \kappa I)h, h\big) &= \big((S - \kappa I)h_S, h_S\big) + \big((S - \kappa I)h_S, h_x\big) + \big((x - \kappa)h_x, h\big) \\ &= \big((S - \kappa I)h_S, h_S\big) + \big(h_S, (x - \kappa)h_x\big) \\ &= \big((S - \kappa I)h_S, h_S\big) + (\kappa - x)(h_x, h_x) \geq 0. \end{aligned} \tag{2.11}$$

Hence $S'_{K,x} \geq \kappa I$ and, in particular, $\sigma(S'_{K,x}) \subseteq [\kappa, \infty)$. It follows that

$$\sigma(S_{K,x}) \backslash \{x\} = \sigma(S'_{K,x}) \subseteq [\kappa, \infty), \tag{2.12}$$

implying (2.4). It is also clear that the lower bound of $S_{K,x}$ is x. □

The next proposition shows that $S_{K,x}$ is the smallest semibounded extension of S with lower bound x.

Proposition 2.3. *Let H be a semibounded self-adjoint extension of S and let $x < \kappa$. Then the following are equivalent:*

(*i*) $H \geq xI$.

(*ii*) $S_{K,x} \leq H$*, that is,* $(H - \eta I)^{-1} \leq (S_{K,x} - \eta I)^{-1}$ *for any* $\eta < x$.

In particular,

$$S_{K,x} \leq S_{K,y}, \quad x \leq y < \kappa. \tag{2.13}$$

Proof. The implication $(ii) \Rightarrow (i)$ is clear: In fact, since $S_{K,x} \geq xI$, it follows from $S_{K,x} \leq H$ that also $H \geq xI$; see [18, Lemma 5.2.2, Theorem 5.2.4].

To verify $(i) \Rightarrow (ii)$ it will be shown that

$$\big(\big[(S_{K,x} - \eta I)^{-1} - (H - \eta I)^{-1}\big]h, h\big) \geq 0, \quad h \in \mathfrak{H}. \tag{2.14}$$

For this purpose we write $h = (S - xI)g + g_x$ with $g \in \operatorname{dom}(S)$ and $g_x \in \operatorname{ran}(S - xI)^\perp = \ker(S^* - xI)$. As $S_{K,x}$ and H are both extensions of S one has

$$(S_{K,x} - \eta I)^{-1}(S - \eta I)g = g = (H - \eta I)^{-1}(S - \eta I)g, \tag{2.15}$$

and hence the left-hand side of (2.14) simplifies and can be rewritten as

$$\begin{aligned}
&\big(\big[(S_{K,x}-\eta I)^{-1}-(H-\eta I)^{-1}\big]g_x,g_x\big)\\
&\quad=\frac{1}{x-\eta}\big(\big[I-(x-\eta)(H-\eta I)^{-1}\big]g_x,g_x\big)\\
&\quad=\frac{1}{x-\eta}\big(\big[I-(x-\eta)(H-\eta I)^{-1}\big]g_x,\big[I-(x-\eta)(H-\eta I)^{-1}\big]g_x\big)\\
&\qquad+\frac{1}{x-\eta}\big(\big[I-(x-\eta)(H-\eta I)^{-1}\big]g_x,(x-\eta)(H-\eta I)^{-1}g_x\big).
\end{aligned} \tag{2.16}$$

Thus it suffices to show that the last term satisfies

$$\big(\big[I-(x-\eta)(H-\eta I)^{-1}\big]g_x,(H-\eta I)^{-1}g_x\big)\geq 0,\quad g_x\in\ker(S^*-xI). \tag{2.17}$$

In fact, as $I-(x-\eta)(H-\eta I)^{-1}=(H-xI)(H-\eta I)^{-1}$ and $H\geq xI$, this is clear. Thus (2.14) holds and $S_{K,x}\leq H$ follows. □

Due to the monotonicity of the family of self-adjoint operators $S_{K,x}$ in $\mathfrak{H}$ observed in Proposition 2.3 there is a self-adjoint limit $S_{K,\kappa}$ in $\mathfrak{H}$ in the strong resolvent sense as $x\uparrow\kappa$, that is,

$$S_{K,\kappa}=\text{sr-lim}_{x\uparrow\kappa}S_{K,x}; \tag{2.18}$$

see [18, Theorem 5.2.11 and p. 712]. It follows from [18, Corollary 5.2.12 (*ii*)] that $S_{K,\kappa}$ is also an extension of S and from the monotonicity of $S_{K,x}$ one concludes that $S_{K,\kappa}$ is bounded from below by κ. Moreover, [18, Corollary 1.9.6 (*i*)] shows that $S_{K,\kappa}$ is the strong graph limit of the family $S_{K,x}$ as $x\uparrow\kappa$. The next proposition extends the equivalence in Proposition 2.3 to the limit operator $S_{K,\kappa}$ and hence the latter is the smallest semibounded self-adjoint extension of S with lower bound κ.

Proposition 2.4. *Let $S_{K,\kappa}$ be defined as the strong resolvent limit in* (2.18) *and let H be a semibounded self-adjoint extension of S. Then the following are equivalent:*

- (*i*) $H\geq\kappa I$.
- (*ii*) $S_{K,\kappa}\leq H$, *that is,* $(H-\eta I)^{-1}\leq(S_{K,\kappa}-\eta I)^{-1}$ *for any* $\eta<\kappa$.

Proof. The proof of (*ii*) ⇒ (*i*) is the same as in Proposition 2.3 and the implication (*i*) ⇒ (*ii*) follows from [18, Corollary 5.2.12 (*i*)] in conjunction with Proposition 2.3. □

We shall now explore in which way the (formal) Krein-type extension

$$S^*\upharpoonright\big(\operatorname{dom}(S)+\ker(S^*-\kappa I)\big) \tag{2.19}$$

is related to the self-adjoint operator $S_{K,\kappa}$ in (2.18); note that the same argument as in the beginning of the proof of Lemma 2.2 shows that the operator in (2.19) is symmetric and that the sum of $\operatorname{dom}(S)$ and $\ker(S^*-\kappa I)$ in (2.19) is not necessarily direct. The statement in the next lemma is contained in [18, Lemma 5.4.1]; here we provide a simple direct and self-contained proof.

Lemma 2.5. *Let $S_{K,\kappa}$ be defined as the strong resolvent limit in* (2.18)*. Then*

$$S^*\upharpoonright\big(\operatorname{dom}(S)+\ker(S^*-\kappa I)\big)\subset S_{K,\kappa} \tag{2.20}$$

and

$$\ker(S_{K,\kappa}-\kappa I)=\ker(S^*-\kappa I). \tag{2.21}$$

Proof. Recall first that the limit $S_{K,\kappa}$ is a self-adjoint extension of S and hence it follows that $\ker(S_{K,\kappa} - \kappa I) \subset \ker(S^* - \kappa I)$. It will be shown in the following that there is equality, which also implies that (2.20) is valid. The proof is based on an adaption of the procedure in [18, Lemma 1.4.10]. Define for each $h \in \mathfrak{H}$ the element $h_x \in \mathfrak{H}$, $x < \kappa$, by

$$h_x = (I + (x - \kappa)(S_{K,\kappa} - xI)^{-1})h. \tag{2.22}$$

For all $\varphi \in \operatorname{dom}(S_{K,\kappa})$ one sees that

$$(h_x, (S_{K,\kappa} - xI)\varphi) = (h, (S_{K,\kappa} - \kappa I)\varphi). \tag{2.23}$$

In particular, this holds for all $\varphi \in \operatorname{dom}(S)$; thus if $h \in \ker(S^* - \kappa I)$ it follows that $h_x \in \ker(S^* - xI)$.

Now assume in (2.22) that $h \in \ker(S^* - \kappa I) \ominus \ker(S_{K,\kappa} - \kappa I)$. Therefore, by what has been shown above $h_x \in \ker(S^* - xI)$ and, since $h \perp \ker(S_{K,\kappa} - \kappa I)$, it follows that

$$(x - \kappa)(S_{K,\kappa} - xI)^{-1}h \underset{x\uparrow\kappa}{\longrightarrow} 0 \text{ strongly.} \tag{2.24}$$

To see (2.24), let $E_{S_{K,\kappa}}(\lambda)$, $\lambda \in \mathbb{R}$, be the spectral family corresponding to $S_{K,\kappa}$. Then one has

$$\|(x - \kappa)(S_{K,\kappa} - xI)^{-1}h\|^2 = \int_\kappa^\infty \left(\frac{\kappa - x}{\lambda - x}\right)^2 d(E_{S_{K,\kappa}}(\lambda)h, h) \underset{x\uparrow\kappa}{\longrightarrow} (P_\kappa h, h), \tag{2.25}$$

where P_κ is the orthogonal projection onto $\ker(S_{K,\kappa} - \kappa I)$. Hence, (2.24) is clear. Therefore, from (2.22) and (2.24) one obtains $h_x \to h$ in $\mathfrak{H}$ as $x \to \kappa$ and, consequently, $\{h_x, xh_x\} \to \{h, \kappa h\}$ in $\mathfrak{H} \times \mathfrak{H}$ as $x \to \kappa$. Since $S_{K,\kappa}$ is the limit of $S_{K,x}$, $x < \kappa$, in the strong graph sense, it follows that $h \in \ker(S_{K,\kappa} - \kappa I)$. The condition $h \perp \ker(S_{K,\kappa} - \kappa I)$ thus leads to $h = 0$. Therefore $\ker(S^* - \kappa I) = \ker(S_{K,\kappa} - \kappa I)$ has been shown and the inclusion (2.20) is established. □

The following result from [18, Corollary 5.4.5] shows in which situations there is equality in (2.20).

Proposition 2.6. *Let $S_{K,\kappa}$ be defined as the strong resolvent limit in* (2.18). *Then*

$$S^* \upharpoonright \big(\operatorname{dom}(S) + \ker(S^* - \kappa I)\big) = S_{K,\kappa} \tag{2.26}$$

if and only if

$$\operatorname{ran}(S - \kappa I) = \overline{\operatorname{ran}(S - \kappa I)} \cap \operatorname{ran}(S^* - \kappa I). \tag{2.27}$$

In particular, if $\operatorname{ran}(S - \kappa I)$ *is closed, then* (2.26) *holds.*

Proof. Let us consider the symmetric operator

$$H = S^* \upharpoonright \big(\operatorname{dom}(S) + \ker(S^* - \kappa I)\big) \tag{2.28}$$

and determine its adjoint H^* and $\operatorname{ran}(H^* - \kappa I)$. Note first, that $S \subset H$ implies $H^* \subset S^*$ and hence $g \in \operatorname{dom}(H^*) \subset \operatorname{dom}(S^*)$ if and only if

$$(Hh, g) = (h, S^*g) \tag{2.29}$$

holds for all $h \in \operatorname{dom}(H) = \operatorname{dom}(S) + \ker(S^* - \kappa I)$; in this case it is clear that $H^*g = S^*g$. Now decompose $h = h_S + h_\kappa$ with $h_S \in \operatorname{dom}(S)$ and $h_\kappa \in \ker(S^* - \kappa I)$. Using $(Sh_S, g) = (h_S, S^*g)$ it follows that (2.29) reduces to $(\kappa h_\kappa, g) = (h_\kappa, S^*g)$. Thus, $g \in \operatorname{dom}(H^*)$ if and only if $(h_\kappa, (S^* - \kappa I)g) = 0$ for all $h_\kappa \in \ker(S^* - \kappa I)$, and hence we conclude that H^* is the restriction of S^* to all $g \in \operatorname{dom}(S^*)$ for which

$$(S^* - \kappa I)g \in (\ker(S^* - \kappa I))^\perp = \overline{\operatorname{ran}(S - \kappa I)}. \tag{2.30}$$

Therefore, we have

$$\operatorname{ran}(H^* - \kappa I) = \overline{\operatorname{ran}(S - \kappa I)} \cap \operatorname{ran}(S^* - \kappa I). \tag{2.31}$$

Now it will be shown that (2.26) and (2.27) are equivalent. In fact, if (2.26) holds, then $H = S_{K,\kappa}$ is self-adjoint and (2.31) takes the form

$$\operatorname{ran}(H - \kappa I) = \overline{\operatorname{ran}(S - \kappa I)} \cap \operatorname{ran}(S^* - \kappa I). \tag{2.32}$$

It is clear that $\operatorname{ran}(H - \kappa I) = \operatorname{ran}(S - \kappa I)$, and hence (2.27) follows. Conversely, if (2.27) holds, then $\operatorname{ran}(H - \kappa I) = \operatorname{ran}(S - \kappa I)$ implies (2.32) and therefore $\operatorname{ran}(H - \kappa I) = \operatorname{ran}(H^* - \kappa I)$ by (2.31). We claim that this implies $H = H^*$. In fact, $H \subset H^*$ is clear and to see the inclusion $H^* \subset H$ consider some $f \in \operatorname{dom}(H^*)$. By assumption $(H^* - \kappa I)f = (H - \kappa I)h$ for some $h \in \operatorname{dom}(H)$ and this yields $f - h \in \ker(H^* - \kappa I) \subset \ker(S^* - \kappa I)$, which then gives

$$f = h + (f - h) \in \operatorname{dom}(S) + \ker(S^* - \kappa I), \tag{2.33}$$

so that $f \in \operatorname{dom}(H)$. Eventually, as $H \subset S_{K,\kappa}$ by Lemma 2.5 and both operators are self-adjoint we conclude (2.26). □

According to [83, Satz 10.22 (b)] the restriction of $S - \kappa I$ onto $(\ker(S - \kappa I))^\perp \cap \operatorname{dom}(S)$ is boundedly invertible if there exists a self-adjoint extension H of S such that $\kappa \notin \sigma_{ess}(H)$. Thus, $\operatorname{ran}(S - \kappa I)$ is closed in this situation and Proposition 2.6 leads to the following statement.

Corollary 2.7. *If there exists a self-adjoint extension H of S such that one has $\kappa \notin \sigma_{ess}(H)$, then* (2.26) *holds.*

Under the assumptions (2.34) in the next corollary one has that the extension $S^* \upharpoonright (\operatorname{dom}(S) + \ker(S^* - \kappa I))$ of S is self-adjoint and hence equal to $S_{K,\kappa}$ by Lemma 2.5.

Corollary 2.8. *If S has deficiency indices (r, r) with $r \in \mathbb{N}$, while*

$$\dim(\ker(S - \kappa I)) = 0 \quad \textit{and} \quad \dim(\ker(S^* - \kappa I)) = r, \tag{2.34}$$

then (2.26) *holds.*

Remark 2.9. The classical Krein–von Neumann extension of a nonnegative symmetric operator is defined as the smallest nonnegative self-adjoint extension. Therefore, in the present situation, if $\kappa \geq 0$ then $S_{K,0}$ is the Krein–von Neumann extension of S by Proposition 2.3 and Proposition 2.4. In the case $\kappa > 0$ one has

$$S_{K,0} = S^* \upharpoonright \operatorname{dom}(S_{K,0}), \quad \operatorname{dom}(S_{K,0}) = \operatorname{dom}(S) \dot{+} \ker(S^*), \tag{2.35}$$

and in the case $\kappa = 0$ the Krein–von Neumann extension of S can be obtained via the strong resolvent limit $S_{K,0} = \text{sr-lim}_{x\uparrow 0} S_{K,x}$; see (2.18). For some contributions dealing with Krein–von Neumann extensions and related issues we refer the reader, for instance, to [1, Sect. 109], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [18, Sect. 5.4], [23], [25], [26], [28, Part III], [31], [32, Sect. 3.3], [38], [39, App. D.3], [42], [43], [44], [45], [46], [61, Ch. 3], [62], [63], [68], [71], [74, Sects. 13.3, 14.8], [75], [76], [79], [80], [81], and the references cited therein. ⋄

2.2. **Boundary triplets and Weyl–Titchmarsh functions.** Let S be a densely defined, closed, symmetric operator in a Hilbert space $\mathfrak{H}$. In the following we recall the notion of boundary triplets and Weyl–Titchmarsh functions very briefly. For our purposes the characterization (2.38) of all self-adjoint extensions of S via self-adjoint operators and relations in the boundary space is particularly important. We refer the reader to the monograph [18] for more details and further references.

Definition 2.10. A triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *boundary triplet* for S^* if $\mathcal{G}$ is a Hilbert space and the linear mappings $\Gamma_0, \Gamma_1 : \operatorname{dom}(S^*) \to \mathcal{G}$ satisfy the abstract Green identity

$$(S^* f, g) - (f, S^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \operatorname{dom}(S^*), \tag{2.36}$$

and $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom}(S^*) \to \mathcal{G} \times \mathcal{G}$ is onto.

We recall that a boundary triplet exists if and only if the deficiency indices of S coincide, that is, if and only if S admits self-adjoint extensions in $\mathfrak{H}$, and that a boundary triplet is not unique (if $S \neq S^*$). Assuming that a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^* is given, one infers

$$\operatorname{dom}(S) = \ker(\Gamma_0) \cap \ker(\Gamma_1) \tag{2.37}$$

and the mapping $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom}(S^*) \to \mathcal{G} \times \mathcal{G}$ is continuous if $\operatorname{dom}(S^*)$ is equipped with the graph norm (and the product Hilbert space $\mathcal{G} \times \mathcal{G}$ carries the natural norm). Furthermore, there is a one-to-one correspondence between the self-adjoint relations in $\mathcal{G}$ and the self-adjoint extensions A_Θ of S in $\mathfrak{H}$ given by

$$\Theta \mapsto A_\Theta, \quad \operatorname{dom}(A_\Theta) = \big\{ f \in \operatorname{dom}(S^*) \,|\, \{\Gamma_0 f, \Gamma_1 f\} \in \Theta \big\}, \tag{2.38}$$

and in the special case that Θ is (the graph of) a self-adjoint operator in $\mathcal{G}$ one has $\operatorname{dom}(A_\Theta) = \ker(\Gamma_1 - \Theta\Gamma_0)$. It follows that

$$B_0 := S^* \upharpoonright \ker(\Gamma_0) \tag{2.39}$$

is a self-adjoint extension of S in $\mathfrak{H}$ and the domain decomposition

$$\operatorname{dom}(S^*) = \operatorname{dom}(B_0) \dotplus \ker(S^* - zI) = \ker(\Gamma_0) \dotplus \ker(S^* - zI) \tag{2.40}$$

holds for $z \in \rho(B_0)$. This decomposition also implies that for any $z \in \rho(B_0)$ the operators

$$\gamma(z)\Gamma_0 f_z = f_z \ \text{ and } \ M(z)\Gamma_0 f_z = \Gamma_1 f_z, \quad f_z \in \ker(S^* - zI), \tag{2.41}$$

are well defined. It turns out that $\gamma(z) = (\Gamma_0 \upharpoonright \ker(S^* - zI))^{-1}$, $z \in \rho(B_0)$, is a bounded and everywhere defined operator from $\mathcal{G}$ to $\mathfrak{H}$ and that the identities

$$\gamma(z) = \big(1 + (z - z')(B_0 - zI)^{-1}\big)\gamma(z'), \quad z, z' \in \rho(B_0), \tag{2.42}$$

and

$$\gamma(z)^* = \Gamma_1 (B_0 - \bar{z} I)^{-1}, \quad z \in \rho(B_0), \tag{2.43}$$

hold. Moreover,

$$M(z) = \Gamma_1 \gamma(z), \quad z \in \rho(B_0), \tag{2.44}$$

is a bounded and everywhere defined operator in $\mathcal{G}$ and the function $z \mapsto M(z)$ is an (operator-valued) Nevanlinna–Herglotz (resp., Riesz–Herglotz) function on $\rho(B_0)$. In particular, for $x \in \rho(B_0) \cap \mathbb{R}$ one has self-adjointness, $M(x) = M(x)^*$, and if B_0 is semibounded from below by κ, then $x \mapsto M(x)$ is a monotone increasing (operator-valued) function on $(-\infty,\kappa)$, and hence there exists a self-adjoint limit $M(\kappa)$ in $\mathcal{G}$ in the strong resolvent sense as $x \uparrow \kappa$,

$$M(\kappa) = \text{sr-lim}_{x\uparrow\kappa} M(x), \tag{2.45}$$

see [18, Theorem 5.2.11]. Here the limit $M(\kappa)$ can be multivalued and is therefore regarded as a self-adjoint relation in $\mathcal{G}$. In the same way as for $S_{K,\kappa}$, one also has that $M(\kappa)$ is the strong graph limit of the family $M(x)$ as $x \uparrow \kappa$.

We assume from now on that the densely defined, closed, symmetric operator S is semibounded from below with the lower bound $\kappa \in \mathbb{R}$. In the next lemma we identify the self-adjoint parameters in $\mathcal{G}$ that correspond to the self-adjoint Krein-type extensions $S_{K,x}$, $x < \kappa$, in (2.2).

Lemma 2.11. *Let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a boundary triplet for S^*, let $B_0 = S^* \upharpoonright \ker(\Gamma_0)$, and suppose that M is the corresponding Weyl–Titchmarsh function. Then for $x < \kappa$ such that $x \in \rho(B_0)$ one has*

$$S_{K,x} = A_{M(x)} \tag{2.46}$$

and, in particular, if B_0 is the Friedrichs extension of S, then (2.46) *holds for all $x < \kappa$.*

Proof. One notes that $S_{K,x}$, $x < \kappa$, is a self-adjoint extension of S by Lemma 2.2 and for $x \in \rho(B_0) \cap \mathbb{R}$ the extension $A_{M(x)} = S^* \upharpoonright \ker(\Gamma_1 - M(x)\Gamma_0)$ is self-adjoint by (2.38) and the fact that $M(x)$ is a self-adjoint operator in $\mathcal{G}$. Hence, it suffices to check that $S_{K,x} \subseteq A_{M(x)}$. For this purpose consider $f = f_S + f_x \in \operatorname{dom}(S_{K,x})$ and note that $\Gamma_0 f_S = \Gamma_1 f_S = 0$ by (2.37) as $f_S \in \operatorname{dom}(S)$. Therefore,

$$(\Gamma_1 - M(x)\Gamma_0)f = \Gamma_1 f_x - M(x)\Gamma_0 f_x = 0 \tag{2.47}$$

by the definition of $M(x)$. Thus, $\operatorname{dom}(S_{K,x}) \subseteq \operatorname{dom}(A_{M(x)})$ and hence (2.46) follows. □

Next it will be shown that (2.46) extends to the limits $S_{K,\kappa}$ and $M(\kappa)$. To show this, we shall use that $S_{K,\kappa}$ is the strong graph limit of $S_{K,x}$ and $M(\kappa)$ is the strong graph limit of $M(x)$ as $x \uparrow \kappa$.

Proposition 2.12. *Let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a boundary triplet for S^* with corresponding Weyl–Titchmarsh function M and assume that the self-adjoint extension $B_0 = S^* \upharpoonright \ker(\Gamma_0)$ is bounded from below by κ. Then*

$$S_{K,\kappa} = A_{M(\kappa)}, \tag{2.48}$$

in particular, if B_0 is the Friedrichs extension of S then (2.48) *holds.*

Proof. First, we recall that $S_{K,\kappa}$ in (2.18) is a self-adjoint extension of S. Furthermore, $M(\kappa)$ in (2.45) is a self-adjoint relation in $\mathcal{G}$ and hence

$$\begin{aligned} &A_{M(\kappa)} = S^* \upharpoonright \operatorname{dom}(A_{M(\kappa)}), \\ &\operatorname{dom}(A_{M(\kappa)}) = \{f \in \operatorname{dom}(S^*) \,|\, \{\Gamma_0 f, \Gamma_1 f\} \in M(\kappa)\}, \end{aligned} \tag{2.49}$$

is also a self-adjoint extension of S. We will verify the inclusion $S_{K,\kappa} \subseteq A_{M(\kappa)}$, which then implies the equality (2.48) as both $S_{K,\kappa}$ and $A_{M(\kappa)}$ are self-adjoint. For this purpose we shall use that $S_{K,\kappa}$ is the strong graph limit of the family $S_{K,x}$ as $x \uparrow \kappa$. Consider $f \in \operatorname{dom}(S_{K,\kappa})$ and choose a sequence $f_{x_n} \in \operatorname{dom}(S_{K,x_n})$ as $x_n \uparrow \kappa$ such that $f_{x_n} \to f$ and $S_{K,x_n} f_{x_n} \to S_{K,\kappa} f$. We note that $S_{K,x_n} = A_{M(x_n)}$ by (2.46), and hence $(\Gamma_1 - M(x_n)\Gamma_0) f_{x_n} = 0$. Since $(\Gamma_0,\Gamma_1)^\top : \operatorname{dom}(S^*) \to \mathcal{G} \times \mathcal{G}$

is continuous with respect to the graph norm and $M(\kappa)$ is the strong graph limit of $M(x_n)$ as $x_n \uparrow \kappa$ one concludes

$$\{\Gamma_0 f, \Gamma_1 f\} = \lim_{n\to\infty} \{\Gamma_0 f_{x_n}, \Gamma_1 f_{x_n}\} = \lim_{n\to\infty} \{\Gamma_0 f_{x_n}, M(x_n)\Gamma_0 f_{x_n}\} \in M(\kappa), \quad (2.50)$$

that is, $f \in \operatorname{dom}(A_{M(\kappa)})$. It follows that $S_{K,\kappa} \subseteq A_{M(\kappa)}$ and hence (2.48) holds. $\square$

In the case where the strong resolvent limit $S_{K,\kappa}$ of $S_{K,x}$ as $x \uparrow \kappa$ has the form (2.26), it turns out that the strong resolvent limit $M(\kappa)$ of $M(x)$ as $x \uparrow \kappa$ has a particular simple structure, and vice versa.

Lemma 2.13. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with corresponding Weyl–Titchmarsh function M and assume that the self-adjoint extension $B_0 = S^* \upharpoonright \ker(\Gamma_0)$ is bounded from below by κ. Then $S_{K,\kappa}$ in* (2.18) *has the form*

$$S_{K,\kappa} = S^* \upharpoonright \big(\operatorname{dom}(S) \dot{+} \ker(S^* - \kappa I)\big) \quad (2.51)$$

if and only if $M(\kappa)$ in (2.45) *has the form*

$$M(\kappa) = \big\{\{\Gamma_0 f_\kappa, \Gamma_1 f_\kappa\} \,|\, f_\kappa \in \ker(S^* - \kappa I)\big\}. \quad (2.52)$$

In particular, if $\kappa \notin \sigma_{ess}(B_0)$, then (2.52) *holds.*

Proof. According to Proposition 2.12 one has $S_{K,\kappa} = A_{M(\kappa)}$ and, as $S_{K,\kappa}$ is self-adjoint in $\mathfrak{H}$, it follows from (2.38) that

$$M(\kappa) = \big\{\{\Gamma_0 f, \Gamma_1 f\} \,|\, f \in \operatorname{dom}(S_{K,\kappa})\big\}. \quad (2.53)$$

Therefore, if (2.51) holds, then (2.37) implies that the right-hand side in (2.53) coincides with the right-hand side in (2.52). Conversely, if the self-adjoint relation $M(\kappa)$ in (2.45) has the form (2.52), then one obtains

$$\big\{\{\Gamma_0 f_\kappa, \Gamma_1 f_\kappa\} \,|\, f_\kappa \in \ker(S^* - \kappa I)\big\} = \big\{\{\Gamma_0 f, \Gamma_1 f\} \,|\, f \in \operatorname{dom}(S_{K,\kappa})\big\} \quad (2.54)$$

by (2.53). Thus, for $f \in \operatorname{dom}(S_{K,\kappa})$ there exists $f_\kappa \in \ker(S^* - \kappa I)$ such that $\{\Gamma_0 f, \Gamma_1 f\} = \{\Gamma_0 f_\kappa, \Gamma_1 f_\kappa\}$ and hence (2.37) implies $f - f_\kappa \in \operatorname{dom}(S)$, so that $f = (f - f_\kappa) + f_\kappa \in \operatorname{dom}(S) \dot{+} \ker(S^* - \kappa I)$. Hence, we have shown the inclusion $\subseteq$ in (2.51); the inclusion $\supseteq$ is clear from Lemma 2.5.

Finally, if $\kappa \notin \sigma_{ess}(B_0)$, then Corollary 2.7 shows that $S_{K,\kappa}$ has the form (2.51) and hence (2.52) holds. $\square$

In the next corollary the special nonnegative situation is considered and the classical Krein–von Neumann extension is identified.

Corollary 2.14. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* with corresponding Weyl–Titchmarsh function M and assume that S and the self-adjoint extension $B_0 = S^* \upharpoonright \ker(\Gamma_0)$ are both nonnegative, that is, $\kappa \geq 0$. Then the Krein–von Neumann extension $S_{K,0}$ of S satisfies*

$$S_{K,0} = A_{M(0)}. \quad (2.55)$$

In the special case $\kappa > 0$ one has $A_{M(0)} = S^* \upharpoonright \ker(\Gamma_1 - M(0)\Gamma_0)$.

3. Relatively Form Compact Perturbations and Spectral Instability

Let again S be a densely defined, closed, symmetric operator in $\mathfrak{H}$ which is semibounded from below with lower bound κ. In this section we shall study perturbations of the Krein-type extension $S_{K,x}$ of S and we recall a recent result from [16] on the instability of the lower bound $x \leq \kappa$ under arbitrary small negative perturbations αV.

Hypothesis 3.1. We shall assume that $V \geq 0$ is self-adjoint in $\mathfrak{H}$ and relatively form compact with respect to $S_{K,x}$, that is, for some $\nu_0 < x \leq \kappa$,

$$\operatorname{dom}\big((S_{K,x}-\nu_0 I)^{1/2}\big) \subseteq \operatorname{dom}\big(V^{1/2}\big) \quad \text{and} \quad V^{1/2}(S_{K,x}-\nu_0 I)^{-1/2} \in \mathcal{B}_\infty(\mathfrak{H}). \tag{3.1}$$

We note that if (3.1) holds for some $\nu_0 < x$, then (3.1) holds for all $z \in \rho(S_{K,x})$. In fact, $(S_{K,x} - zI)^{1/2}$ is a normal operator in $\mathfrak{H}$ defined via the functional calculus of the self-adjoint operator $S_{K,x}$, see, for instance, [74, Chapter 5.3]. Therefore, one has

$$\operatorname{dom}\big((S_{K,x} - \nu_0 I)^{1/2}\big) = \operatorname{dom}\big((S_{K,x} - zI)^{1/2}\big), \quad z \in \rho(S_{K,x}), \tag{3.2}$$

and for $z \in \rho(S_{K,x})$ it is clear that

$$\begin{aligned} &V^{1/2}(S_{K,x} - zI)^{-1/2} \\ &\quad = \big[V^{1/2}(S_{K,x} - \nu_0 I)^{-1/2}\big]\big[(S_{K,x} - \nu_0 I)^{1/2}(S_{K,x} - zI)^{-1/2}\big] \in \mathcal{B}_\infty(\mathfrak{H}) \end{aligned} \tag{3.3}$$

as $(S_{K,x} - \nu_0 I)^{1/2}(S_{K,x} - zI)^{-1/2} \in \mathcal{B}(\mathfrak{H})$. In addition, one observes that

$$\big((S_{K,x} - zI)^{-1/2}V^{1/2}\big)^* = V^{1/2}(S_{K,x} - \bar{z}I)^{-1/2} \in \mathcal{B}_\infty(\mathfrak{H}), \tag{3.4}$$

and hence

$$\overline{(S_{K,x} - zI)^{-1/2}V^{1/2}} \in \mathcal{B}_\infty(\mathfrak{H}), \quad z \in \rho(S_{K,x}). \tag{3.5}$$

Next we define the Birman–Schwinger operator family by

$$K(z) := \overline{V^{1/2}(S_{K,x} - zI)^{-1}V^{1/2}}, \quad z \in \rho(S_{K,x}), \tag{3.6}$$

which will play an important role in the following considerations. One observes that by (3.1) and (3.4),

$$K(z) = V^{1/2}(S_{K,x} - zI)^{-1/2}\overline{(S_{K,x} - zI)^{-1/2}V^{1/2}} \in \mathcal{B}_\infty(\mathfrak{H}) \tag{3.7}$$

for all $z \in \rho(S_{K,x})$. We also note that for $\nu < \nu' < x$ one has $0 \leq K(\nu) \leq K(\nu')$ and

$$\lim_{\nu \downarrow -\infty} \|K(\nu)\|_{\mathcal{B}(\mathfrak{H})} = 0; \tag{3.8}$$

see [16, Lemma 3.4]. In particular, it follows from (3.8) that for any $\alpha \in \mathbb{R}\backslash\{0\}$ there exists $\nu_\alpha \leq x$ such that $-1/\alpha \in \rho(K(\nu))$ for all $\nu < \nu_\alpha$.

It follows that under Hypothesis 3.1, Hypothesis 2.1 in [35] is satisfied, in particular, this permits one to define a self-adjoint operator $T_x(\alpha)$, $\alpha \in \mathbb{R}\backslash\{0\}$, in $\mathfrak{H}$ via its resolvent as in [35, Theorem 2.3], [52, 60]; the operator $T_x(\alpha)$ in the next proposition is then referred to as a relatively form compact perturbation of the Krein-type extension $S_{K,x}$, where $x \leq \kappa$.

Proposition 3.2. *Let $x \leq \kappa$, $\alpha \in \mathbb{R}\backslash\{0\}$, and assume Hypothesis 3.1. Then there exists a self-adjoint operator $T_x(\alpha)$ in $\mathfrak{H}$ such that*

$$\begin{aligned} &(T_x(\alpha) - zI)^{-1} \\ &\quad = (S_{K,x} - zI)^{-1} - \alpha\overline{(S_{K,x} - zI)^{-1}V^{1/2}}[I + \alpha K(z)]^{-1}V^{1/2}(S_{K,x} - zI)^{-1} \end{aligned} \tag{3.9}$$

for all $z \in \{\zeta \in \rho(S_{K,x}) \,|\, -1/\alpha \in \rho(K(\zeta))\}$*, and* $T_x(\alpha)$ *is a semibounded, self-adjoint extension of the symmetric operator* $S_{K,x} + \alpha V$ *defined on the (not necessarily dense) set* $\operatorname{dom}(S_{K,x}) \cap \operatorname{dom}(V)$.

It follows, in particular, that

$$\big[(T_x(\alpha) - zI)^{-1} - (S_{K,x} - zI)^{-1}\big] \in \mathcal{B}_\infty(\mathfrak{H}) \tag{3.10}$$

and hence $\sigma_{ess}(T_x(\alpha)) = \sigma_{ess}(S_{K,x})$. Furthermore, one has

(*i*) If $\alpha \in (0, \infty)$, then $T_x(\alpha) \geq xI$ and, in particular, $\sigma(T_x(\alpha)) \cap (-\infty, x) = \emptyset$.

(*ii*) If $\alpha \in (-\infty, 0)$, then $\sigma(T_x(\alpha)) \cap (-\infty, x)$ is either empty or consists of discrete eigenvalues.

Remark 3.3. In the case that $V \geq 0$ is self-adjoint in $\mathfrak{H}$ and relatively compact with respect to $S_{K,x}$, that is, for some $\nu_0 < x \leq \kappa$,

$$\operatorname{dom}(S_{K,x} - \nu_0 I) \subseteq \operatorname{dom}(V) \quad \text{and} \quad V(S_{K,x} - \nu_0 I)^{-1} \in \mathcal{B}_\infty(\mathfrak{H}), \tag{3.11}$$

it follows that V is also relatively form compact with respect to $S_{K,x}$, see, for instance, [36, Theorem 3.5 (i)]. In this situation the semibounded self-adjoint operator $T_x(\alpha)$ in Proposition 3.2 has the form

$$T_x(\alpha) = S_{K,x} + \alpha V, \quad \operatorname{dom}(T_x(\alpha)) = \operatorname{dom}(S_{K,x}). \tag{3.12}$$

$\diamond$

Next, consider the interesting case $\alpha < 0$ in the situation $x < \kappa$, where one automatically has $\ker(S_{K,x} - xI) = \ker(S^* - xI) \neq \{0\}$ (see Definition 2.1) and in the case $x = \kappa$, where it is assumed that $\ker(S_{K,\kappa} - \kappa I) = \ker(S^* - \kappa I) \neq \{0\}$. The next theorem is a variant of [15, Theorem 2.2] and [16, Theorem 3.9].

Theorem 3.4. *Let* $x \leq \kappa$*, assume Hypothesis 3.1, and denote by* $T_x(\alpha)$ *the relatively form compact perturbation of the Krein-type extension* $S_{K,x}$ *in Proposition 3.2. Then the following assertions hold:*

(*i*) *If* $x < \kappa$ *and* $\ker(S_{K,x} - xI) \not\subseteq \ker(V)$*, then*

$$\sigma(T_x(\alpha)) \cap (-\infty, x) \neq \emptyset \ \textit{ for } \textbf{any } \ \alpha < 0. \tag{3.13}$$

If, in addition, V *is relatively compact with respect to* $S_{K,x}$*, then* $T_x(\alpha) = S_{K,x} + \alpha V$ *and*

$$\sigma(S_{K,x} + \alpha V) \cap (-\infty, x) \neq \emptyset \ \textit{ for } \textbf{any } \ \alpha < 0. \tag{3.14}$$

(*ii*) *If* $x = \kappa$*,* $\ker(S_{K,\kappa} - \kappa I) \neq \{0\}$*, and* $\ker(S_{K,\kappa} - \kappa I) \not\subseteq \ker(V)$*, then*

$$\sigma(T_\kappa(\alpha)) \cap (-\infty, \kappa) \neq \emptyset \ \textit{ for } \textbf{any } \ \alpha < 0. \tag{3.15}$$

If, in addition, V *is relatively compact with respect to* $S_{K,\kappa}$*, then* $T_\kappa(\alpha) = S_{K,\kappa} + \alpha V$ *and*

$$\sigma(S_{K,\kappa} + \alpha V) \cap (-\infty, \kappa) \neq \emptyset \ \textit{ for } \textbf{any } \ \alpha < 0. \tag{3.16}$$

In the case that S has finite deficiency indices and $x < \kappa$, it is clear that x is an isolated eigenvalue of finite multiplicity of $S_{K,x}$. In this case the spectral instability would already follow from well-known results in asymptotic perturbation theory, see, for instance, [51, Theorem 8] or [53, Theorem VIII.4.9], see also [15, Remark 3.2].

4. Regular Sturm–Liouville Operators

In this section we consider a Sturm–Liouville differential expression of the form

$$\ell = \frac{1}{r}\left[-\frac{d}{dt}p\frac{d}{dt} + q\right] \tag{4.1}$$

on a compact interval $[a,b] \subset \mathbb{R}$, where it is assumed that the coefficients p, q, and r are a.e. real-valued functions on (a,b) which satisfy the conditions

$$\begin{cases} p(t) > 0, \ r(t) > 0 \ \text{ for a.e. } t \in (a,b), \\ 1/p, q, r \in L^1((a,b)). \end{cases} \tag{4.2}$$

The nonnegative function r will serve as a weight function and the corresponding Hilbert space $L^2((a,b); rdt)$ is denoted by $L^2_r((a,b))$ in the following. The minimal operator corresponding to ℓ in $L^2_r((a,b))$ is then given by

$$\begin{aligned} &Sf = \ell f, \\ &f \in \operatorname{dom}(S) = \big\{ g \in \mathcal{D}_{max} \,\big|\, g(a) = (pg')(a) = g(b) = (pg')(b) = 0 \big\}, \end{aligned} \tag{4.3}$$

where

$$\mathcal{D}_{max} = \big\{ g \in L^2_r((a,b)) \,\big|\, g, pg' \in AC([a,b]);\ \ell g \in L^2_r((a,b)) \big\} \tag{4.4}$$

is the usual maximal domain and $AC([a,b])$ denotes the space of absolutely continuous functions on $[a,b]$. One recalls that $g \in AC([a,b])$ if and only if there exists $h \in L^1((a,b))$ such that

$$g(y) - g(x) = \int_x^y h(s)\,ds, \quad a \le x \le y \le b; \tag{4.5}$$

in this case g is differentiable almost everywhere on (a,b) and $g' = h$ a.e. on (a,b).

The minimal operator S is a densely defined, closed, symmetric operator in $L^2_r((a,b))$ with deficiency indices $(2,2)$, semibounded from below with lower bound $\kappa \in \mathbb{R}$, and its adjoint coincides with the maximal operator

$$S^* f = \ell f, \quad f \in \operatorname{dom}(S^*) = \mathcal{D}_{max}. \tag{4.6}$$

For $z \in \mathbb{C}$ we will fix the solutions $u_1(\,\cdot\,, z)$ and $u_2(\,\cdot\,, z)$ of $\ell u = zu$ by the conditions

$$u_1(a,z) = (pu_2')(a,z) = 1 \quad \text{and} \quad u_2(a,z) = (pu_1')(a,z) = 0. \tag{4.7}$$

Since ℓ is a regular Sturm–Liouville expression the solutions $u_1(\,\cdot\,,z)$ and $u_2(\,\cdot\,,z)$ belong to the space $L^2_r((a,b))$ and hence

$$\ker(S^* - zI) = \text{lin.span}\,\{u_1(\,\cdot\,,z), u_2(\,\cdot\,,z)\}. \tag{4.8}$$

In the next proposition we provide a possible choice for a boundary triplet such that B_0 is the Friedrichs extension, $B_0 = S_F$, of S; the proof is straightforward and can be found, for instance, in [18, Proposition 6.3.1].

Proposition 4.1. *Consider the regular Sturm–Liouville expression in* (4.1) *and let S be the corresponding minimal operator in $L^2_r((a,b))$. Then $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, where*

$$\Gamma_0 f = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f = \begin{pmatrix} (pf')(a) \\ -(pf')(b) \end{pmatrix}, \quad f \in \mathcal{D}_{max}, \tag{4.9}$$

is a boundary triplet for S^* such that

$$B_0 f = \ell f, \quad f \in \operatorname{dom}(B_0) = \big\{ g \in \mathcal{D}_{max} \,|\, g(a) = g(b) = 0 \big\}, \tag{4.10}$$

coincides with the Friedrichs extension of S, $B_0 = S_F$, and the corresponding Weyl–Titchmarsh function M is given by

$$M(z) = \frac{1}{u_2(b,z)} \begin{pmatrix} -u_1(b,z) & 1 \\ 1 & -(pu_2')(b,z) \end{pmatrix}, \quad z \in \rho(B_0). \tag{4.11}$$

One notes that the resolvent of B_0 is a compact operator in $L^2_r((a,b))$ and that $\kappa = \min(\sigma(B_0))$ is a simple eigenvalue with corresponding eigenfunction $u_2(\,\cdot\,,\kappa)$. For our further discussion one observes that for

$$f_z = c_1 u_1(\,\cdot\,,z) + c_2 u_2(\,\cdot\,,z) \in \ker(S^* - zI), \quad c_1, c_2 \in \mathbb{C}, \tag{4.12}$$

one has

$$\Gamma_0 f_z = \begin{pmatrix} c_1 \\ c_1 u_1(b,z) + c_2 u_2(b,z) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f_z = \begin{pmatrix} c_2 \\ -c_1 (pu_1')(b,z) - c_2 (pu_2')(b,z) \end{pmatrix}, \tag{4.13}$$

and taking the constancy of the Wronskian of u_1 and u_2 into account,

$$W(u_1(\cdot,z), u_2(\cdot,z))(b) = u_1(b,z)(pu_2')(b,z) - u_2(b,z)(pu_1')(b,z) = 1, \tag{4.14}$$

it follows that

$$M(z) = \big\{\{\Gamma_0 f_z, \Gamma_1 f_z\} \,\big|\, f_z \in \ker(S^* - zI)\big\}, \quad z \in \rho(B_0), \tag{4.15}$$

takes on the form (4.11). To find the strong resolvent limit of $M(x)$ as $x \uparrow \kappa$ we use Lemma 2.13, which together with (4.13) yields

$$M(\kappa) = \left\{ \left\{ \begin{pmatrix} c_1 \\ c_1 u_1(b,\kappa) \end{pmatrix}, \begin{pmatrix} c_2 \\ -c_1(pu_1')(b,\kappa) - c_2(pu_2')(b,\kappa) \end{pmatrix} \right\} \,\middle|\, c_1, c_2 \in \mathbb{C} \right\}, \tag{4.16}$$

which, in general, is a multivalued operator.

Next, we shall specify the operator domains of the Krein-type extensions $S_{K,x}$, $x \le \kappa$, using the general results from Section 2. First, one notes that

$$\operatorname{dom}(S_{K,x}) = \operatorname{dom}(S) \dot{+} \ker(S^* - xI), \quad x \le \kappa, \tag{4.17}$$

by (2.2) and Corollary 2.7; here, for $x = \kappa$, the sum is indeed direct as $\ker(S - \kappa I) = \{0\}$. The next corollary is a consequence of Lemma 2.11, equations (4.11) and (2.38), and Proposition 2.12.

Corollary 4.2. *The Krein-type extensions $S_{K,x}$, $x < \kappa$, of the minimal regular Sturm–Liouville operator S in $L^2_r((a,b))$ are of the form*

$$\begin{aligned} &S_{K,x} f = \ell f, \\ &f \in \operatorname{dom}(S_{K,x}) = \left\{ g \in \mathcal{D}_{max} \,\middle|\, \begin{matrix} g(b) = u_1(b,x)g(a) + u_2(b,x)(pg')(a), \\ (pg')(b) = (pu_1')(b,x)g(a) + (pu_2')(b,x)(pg')(a) \end{matrix} \right\}, \end{aligned} \tag{4.18}$$

and for $x = \kappa$ one has

$$\begin{aligned} &S_{K,\kappa} f = \ell f, \\ &f \in \operatorname{dom}(S_{K,\kappa}) = \left\{ g \in \mathcal{D}_{max} \,\middle|\, \begin{matrix} g(b) = u_1(b,\kappa)g(a), \\ (pg')(b) = (pu_1')(b,\kappa)g(a) + (pu_2')(b,\kappa)(pg')(a) \end{matrix} \right\}. \end{aligned} \tag{4.19}$$

Proof. For $x < \kappa$ it follows that $f \in \operatorname{dom}(S_{K,x})$ if and only if $\Gamma_1 f = M(x)\Gamma_0 f$, that is,

$$\begin{aligned} u_2(b,x)(pf')(a) &= -u_1(b,x)f(a) + f(b), \\ -u_2(b,x)(pf')(b) &= f(a) - (pu_2')(b,x)f(b). \end{aligned} \tag{4.20}$$

Substituting $f(b) = u_2(b,x)(pf')(a) + u_1(b,x)f(a)$ in the second equation and using $1 - u_1(b,x)(pu_2')(b,x) = -u_2(b,x)(pu_1')(b,x)$ leads to (4.18). For $x = \kappa$ one obtains (4.19) from (4.16). $\square$

It is easy to see that both solutions $u_1(\,\cdot\,,x)$ and $u_2(\,\cdot\,,x)$ of $\ell u = xu$ satisfy the boundary conditions in (4.18), and in the same way both solutions $u_1(\,\cdot\,,\kappa)$ and $u_2(\,\cdot\,,\kappa)$ of $\ell u = \kappa u$ satisfy the boundary conditions in (4.19). One also notes that in the underlying regular case, under the standard assumptions (4.2), one has for $\nu_0 < x$,

$$\operatorname{dom}\big((S_{K,x} - \nu_0 I)^{1/2}\big) = \big\{g \in L_r^2((a,b)) \,\big|\, g \in AC([a,b]), \sqrt{p}g' \in L^2((a,b))\big\}, \quad (4.21)$$

see, for instance, [18, Section 6.8].

Example 4.3. Assume that $p = r = 1$ and $q = 0$. Then the Sturm–Liouville differential expression in (4.1) reduces to the unperturbed Schrödinger differential expression $\ell = -d^2/dt^2$ and one obtains explicitly

$$u_1(t,z) = \begin{cases} \cos[\sqrt{z}\,(t-a)], & z > 0, \\ 1, & z = 0, \\ \cosh[\sqrt{-z}\,(t-a)], & z < 0, \end{cases} \quad u_2(t,z) = \begin{cases} \frac{\sin[\sqrt{z}\,(t-a)]}{\sqrt{z}}, & z > 0, \\ t-a, & z = 0, \\ \frac{\sinh[\sqrt{-z}\,(t-a)]}{\sqrt{-z}}, & z < 0, \end{cases} \quad t \in [a,b], \quad (4.22)$$

and the eigenvalues of

$$B_0 f = -f'', \quad f \in \operatorname{dom}(B_0) = \{g \in \mathcal{D}_{max} \,|\, g(a) = g(b) = 0\}, \quad (4.23)$$

are given by

$$\sigma(B_0) = \sigma_d(B_0) = \left\{\frac{(k\pi)^2}{(b-a)^2}\right\}_{k\in\mathbb{N}}. \quad (4.24)$$

In particular, the lower bound of B_0 and the minimal operator S is $\kappa = \pi^2/(b-a)^2$ and one obtains the explicit form of the Krein-type extensions for $x \leq \pi^2/(b-a)^2$ of S from (4.18) and (4.19). We also note that the usual Krein–von Neumann extension $S_K = S_0$ of S is given by

$$\begin{aligned} &S_0 f = -f'', \\ &f \in \operatorname{dom}(S_0) = \{g \in \mathcal{D}_{max} \,|\, (b-a)g'(a) = g(b) - g(a) = (b-a)g'(b)\}, \end{aligned} \quad (4.25)$$

and that the Krein-type extension at $\kappa = \pi^2/(b-a)^2$ is given by

$$\begin{aligned} &S_{\pi^2/(b-a)^2} f = -f'', \\ &f \in \operatorname{dom}\big(S_{\pi^2/(b-a)^2}\big) = \{g \in \mathcal{D}_{max} \,|\, g(a) + g(b) = 0 = g'(a) + g'(b)\}. \end{aligned} \quad (4.26)$$

Below we shall consider relatively form compact perturbations of the Krein-type extensions $S_{K,x}$, $x \leq \kappa$, and apply Theorem 3.4. The next preparatory lemma will ensure that the assumption $\ker(S_{K,x} - xI) \not\subseteq \ker(V)$ is satisfied. The lemma is stated in a slightly more general form for locally integrable functions V.

Lemma 4.4. *Let $x \leq \kappa$ and assume that $V \in L^1_{loc}((a,b))\backslash\{0\}$. Then there exists $f \in \ker(S_{K,x} - xI)$ such that $Vf \neq 0$.*

Proof. Consider the solution $u_2(\,\cdot\,,x) \in \ker(S_{K,x} - xI)$ from (4.7) and observe that in the case $x < \kappa$ there exists some $\vartheta \in \mathbb{R}$ such that

$$u_2(a,x) = 0 \quad \text{and} \quad \vartheta u_2(b,x) = (pu_2')(b,x), \quad (4.27)$$

and hence $u_2(\,\cdot\,,x)$ is an eigenfunction corresponding to the eigenvalue x of a semibounded self-adjoint extension of S with separated boundary conditions of the form (4.27). Therefore, the Sturm Oscillation Theorem (see, e.g., [39, Theorem 8.2.4]) implies that $u_2(\,\cdot\,,x)$ has at most finitely many zeros in (a,b) and thus $Vu_2(\,\cdot\,,x)\neq 0$ as otherwise $V=0$. The same argument shows that in the case $x=\kappa$ the function $u_2(\,\cdot\,,\kappa)$ has no zeros in (a,b) and hence $Vu_2(\,\cdot\,,\kappa)\neq 0$ as otherwise $V=0$. □

As a consequence of Lemma 4.4 and Theorem 3.4 we conclude spectral instability of the Krein-type extensions $S_{K,x}$, $x\leq\kappa$, in the next corollary (see also Remark 3.3).

Corollary 4.5. *Let $x\leq\kappa$ and assume that $0\leq V\in L^1_{loc}((a,b))\backslash\{0\}$ is such that V is a relatively form compact perturbation of $S_{K,x}$. Then*

$$\sigma(T_x(\alpha))\cap(-\infty,x)\neq\emptyset \text{ for } \textbf{any } \alpha<0. \tag{4.28}$$

In particular, if $V\neq 0$ is a relatively compact perturbation of $S_{K,x}$, then $T_x(\alpha)=S_{K,x}+\alpha V$ and

$$\sigma(S_{K,x}+\alpha V)\cap(-\infty,x)\neq\emptyset \text{ for } \textbf{any } \alpha<0. \tag{4.29}$$

A concrete realization of Corollary 4.5 is presented in the following theorem. First, we recall that the space $L^s_r((a,b))$, $s\in[1,\infty)$, consists of all complex-valued measurable functions f on (a,b) that satisfy

$$\int_a^b |f(t)|^s\, r(t)dt<\infty. \tag{4.30}$$

Theorem 4.6. *Assume that (4.2) holds, let $x\leq\kappa$, and suppose that $0\leq V\in L^1_{loc}((a,b))\backslash\{0\}$. If $V\in L^1_r((a,b))$, then*

$$\sigma(T_x(\alpha))\cap(-\infty,x)\neq\emptyset \text{ for } \textbf{any } \alpha<0. \tag{4.31}$$

In particular, if $V\in L^2_r((a,b))$, then $T_x(\alpha)=S_{K,x}+\alpha V$ and

$$\sigma(S_{K,x}+\alpha V)\cap(-\infty,x)\neq\emptyset \text{ for } \textbf{any } \alpha<0. \tag{4.32}$$

Proof. We start by verifying that $V\in L^1_r((a,b))$ is relatively form compact with respect to $S_{K,x}$. Indeed, for $m\in\mathbb{N}$ let

$$V_m(x)=\begin{cases}V(x) & \text{if } V(x)\leq m,\\ 0 & \text{if } V(x)>m,\end{cases} \tag{4.33}$$

and note that $V^{1/2},V_m^{1/2}\in L^2_r((a,b))$ (see (4.30)). Moreover,

$$\big\|V^{1/2}-V_m^{1/2}\big\|_{L^2_r((a,b))}\to 0,\quad m\to\infty, \tag{4.34}$$

and since $V_m^{1/2}$ is bounded and $(S_{K,x}-\nu_0 I)^{-1/2}$, $\nu_0<x$, is a compact operator in $L^2_r((a,b))$ it is clear that $V_m^{1/2}(S_{K,x}-\nu_0 I)^{-1/2}$ is also compact in $L^2_r((a,b))$. It follows from (4.21) that the functions in $\operatorname{dom}\big((S_{K,x}-\nu_0 I)^{1/2}\big)$ are absolutely continuous on $[a,b]$ (see [18, Lemma 6.8.1]) and hence bounded. Therefore, for $f\in L^2_r((a,b))$ we have $g=(S_{K,x}-\nu_0 I)^{-1/2}f\in L^\infty((a,b))\subset\operatorname{dom}\big(V^{1/2}\big)$ and

$$\begin{aligned}&\big\|V^{1/2}(S_{K,x}-\nu_0 I)^{-1/2}f-V_m^{1/2}(S_{K,x}-\nu_0 I)^{-1/2}f\big\|_{L^2_r((a,b))}\\&\quad\leq\big\|V^{1/2}-V_m^{1/2}\big\|_{L^2_r((a,b))}\|g\|_{L^\infty((a,b))}.\end{aligned} \tag{4.35}$$

Thus, $V^{1/2}(S_{K,x}-\nu_0 I)^{-1/2} \in \mathcal{B}\big(L^2_r((a,b))\big)$ by the closed graph theorem as $(S_{K,x}-\nu_0 I)^{-1/2} \in \mathcal{B}\big(L^2_r((a,b))\big)$ and $V^{1/2}$ is self-adjoint and hence closed, implying closedness of $V^{1/2}(S_{K,x}-\nu_0 I)^{-1/2}$.

Next, we use the second representation theorem to express the densely defined, semibounded, closed sesquilinear form $\mathfrak{t}_x[\cdot,\cdot]$ associated to $S_{K,x}$ by

$$\mathfrak{t}_x[h,k] = \big((S_{K,x}-\nu_0)^{1/2}h, (S_{K,x}-\nu_0)^{1/2}k\big)_{L^2_r((a,b))} + \nu_0(h,k)_{L^2_r((a,b))} \tag{4.36}$$

for $h,k \in \operatorname{dom}(\mathfrak{t}_x) = \operatorname{dom}((S_{K,x}-\nu_0 I)^{1/2})$. Making use of [18, Corollary 6.8.6] (with some $\varepsilon > 0$ and $C_\varepsilon > 0$) one estimates for $g = (S_{K,x}-\nu_0 I)^{-1/2} f \in \operatorname{dom}(\mathfrak{t}_x)$,

$$\begin{aligned} \|g\|^2_{L^\infty((a,b))} &\leq C_\varepsilon \|g\|^2_{L^2_r((a,b))} + \varepsilon \mathfrak{t}_x[g,g] \\ &= C_\varepsilon \big\|(S_{K,x}-\nu_0 I)^{-1/2} f\big\|^2_{L^2_r((a,b))} \\ &\quad + \varepsilon\big(\|f\|^2_{L^2_r((a,b))} + \nu_0 \big\|(S_{K,x}-\nu_0 I)^{-1/2} f\big\|^2_{L^2_r((a,b))}\big) \\ &\leq C'_\varepsilon \|f\|^2_{L^2_r((a,b))} \end{aligned} \tag{4.37}$$

with some $C'_\varepsilon > 0$. One then concludes from (4.35) that

$$\begin{aligned} &\big\|V^{1/2}(S_{K,x}-\nu_0 I)^{-1/2} f - V_m^{1/2}(S_{K,x}-\nu_0 I)^{-1/2} f\big\|_{L^2_r((a,b))} \\ &\quad \leq \sqrt{C'_\varepsilon}\big\|V^{1/2} - V_m^{1/2}\big\|_{L^2_r((a,b))} \|f\|_{L^2_r((a,b))} \end{aligned} \tag{4.38}$$

and it follows that the compact operators $V_m^{1/2}(S_{K,x}-\nu_0 I)^{-1/2}$ converge uniformly to $V^{1/2}(S_{K,x}-\nu_0 I)^{-1/2}$ in the unit ball of $L^2_r((a,b))$ and hence $V_m^{1/2}(S_{K,x}-\nu_0 I)^{-1/2}$ converge in operator norm to $V^{1/2}(S_{K,x}-\nu_0 I)^{-1/2}$, as $m \to \infty$, implying the latter is compact. Thus, V is relatively form compact with respect to $S_{K,x}$. Therefore, (4.31) follows from Corollary 4.5.

The same argument as above shows that $V \in L^2_r((a,b))$ is relatively compact with respect to $S_{K,x}$, and hence (4.32) follows from Corollary 4.5. □

Remark 4.7. A different Hilbert–Schmidt-type argument shows that for any self-adjoint realization $\widetilde{S}$ of the regular Sturm-Liouville expression ℓ in (4.1) and $\widetilde{\nu}_0 < \min\big(\sigma\big(\widetilde{S}\big)\big)$ one has $V^{1/2}\big(\widetilde{S} - \widetilde{\nu}_0 I\big)^{-1/2} \in \mathcal{B}_4\big(L^2_r((a,b))\big)$, and, with some extra effort, this can be improved to $V^{1/2}\big(\widetilde{S} - \widetilde{\nu}_0 I\big)^{-1/2} \in \mathcal{B}_2\big(L^2_r((a,b))\big)$; these facts will be discussed elsewhere [17]. ◇

5. Krein-type extensions of the Laplacian on a bounded Lipschitz domain

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and let ν be the unit normal vector field pointing outwards on $\partial\Omega$. We denote the L^2-based Sobolev spaces on Ω by $H^s(\Omega)$, $s \geq 0$. The minimal operator corresponding to $-\Delta$ in $L^2(\Omega)$ is given by

$$Sf = -\Delta f, \quad f \in \operatorname{dom}(S) = H^2_0(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^2(\Omega)}}. \tag{5.1}$$

Then S is a densely defined, closed, symmetric operator in $L^2(\Omega)$, with infinite deficiency indices, semibounded from below with positive lower bound κ that satisfies the Berezin-Li-Yau estimate [65] (see also [33])

$$\kappa \geq \frac{n}{n+2} 4\pi [\Gamma((n+2)/2)/|\Omega|]^{2/n} > 0. \tag{5.2}$$

The adjoint of S coincides with the maximal operator

$$S^*f = -\Delta f, \quad f \in \operatorname{dom}(S^*) = \mathcal{D}_{max} = \big\{g \in L^2(\Omega) \,\big|\, \Delta g \in L^2(\Omega)\big\}. \tag{5.3}$$

In the following we shall use that the Dirichlet trace mapping $C^\infty(\overline{\Omega}) \ni f \mapsto f|_{\partial\Omega}$ extends by continuity to a continuous surjective mapping

$$\tau_D : \big\{f \in H^{3/2}(\Omega) \,\big|\, \Delta f \in L^2(\Omega)\big\} \to H^1(\partial\Omega), \tag{5.4}$$

where $H^1(\partial\Omega)$ denotes the first order L^2-based Sobolev space on $\partial\Omega$, and that the Neumann trace mapping $C^\infty(\overline{\Omega}) \ni f \mapsto \nu \cdot \nabla f|_{\partial\Omega}$ extends by continuity to a continuous surjective mapping

$$\tau_N : \big\{f \in H^{3/2}(\Omega) \,\big|\, \Delta f \in L^2(\Omega)\big\} \to L^2(\partial\Omega); \tag{5.5}$$

see, for instance, [14, 38]. (As usual in the Lipschitz context, $L^2(\partial\Omega)$ is equipped with the standard surface measure $d^{n-1}\sigma$.)

For our purpose the self-adjoint Dirichlet and Neumann realization of the Laplacian are particularly important; they are given by

$$A_D f = -\Delta f, \quad f \in \operatorname{dom}(A_D) = \big\{g \in H^{3/2}(\Omega) \,\big|\, \Delta g \in L^2(\Omega),\, \tau_D g = 0\big\}, \tag{5.6}$$

and

$$A_N f = -\Delta f, \quad f \in \operatorname{dom}(A_N) = \big\{g \in H^{3/2}(\Omega) \,\big|\, \Delta g \in L^2(\Omega),\, \tau_N g = 0\big\}, \tag{5.7}$$

see [49, 50] and also [14, 37]. One notes that A_D coincides with the Friedrichs extension of S and hence has lower bound $\kappa > 0$; see (5.2). Both self-adjoint operators A_D and A_N have compact resolvents and hence their spectra are purely discrete. Furthermore, for $\beta \in \mathbb{R}$ we shall make use of the Robin Laplacian

$$A_R^{(\beta)} f = -\Delta f, \quad f \in \operatorname{dom}\big(A_R^{(\beta)}\big) = \big\{g \in H^{3/2}(\Omega) \,\big|\, \Delta g \in L^2(\Omega),\, \tau_N g = \beta \tau_D g\big\}, \tag{5.8}$$

which is a semibounded self-adjoint extension of S with compact resolvent and coincides with A_N if $\beta = 0$ and with A_D if (formally) $\beta = \infty$. The smallest eigenvalue of $A_R^{(\beta)}$ is simple and will be denoted by $\mu(\beta)$. It is important to note that for any $x < \kappa$ there exists $\beta_x \in \mathbb{R}$ such that $x = \mu(\beta_x)$.

For later purposes we first provide a result on the spectral instability of the Robin Laplacian $A_R^{(\beta)}$; see [15, Corollary 3.1 and Remark 3.3]. Here we make use of [15, Theorem 2.2] and we mention that asymptotic perturbation theory could also be applied (see [51, Theorem 8] or [53, Theorem VIII.4.9]).

Theorem 5.1. *Assume that $0 \le V \in L^p(\Omega)\backslash\{0\}$ with $p \ge 2$ if $n = 2$ and $p > 2n/3$ if $n \ge 3$, and let $\beta \in \mathbb{R}$. Then V is a relatively compact perturbation of $A_R^{(\beta)}$, for all $\alpha \in \mathbb{R}$ the operator*

$$(A_R^{(\beta)} + \alpha V)f = -\Delta f + \alpha V f, \quad \operatorname{dom}(A_R^{(\beta)} + \alpha V) = \operatorname{dom}(A_R^{(\beta)}), \tag{5.9}$$

is self-adjoint in $L^2(\Omega)$, and

$$\sigma(A_R^{(\beta)} + \alpha V) \cap (-\infty, \mu(\beta)) \neq \emptyset \text{ for } \textbf{\textit{any}}\ \alpha < 0. \tag{5.10}$$

Proof. First, we verify that V is a relatively compact perturbation of $A_R^{(\beta)}$, so that Hypothesis 3.1 is satisfied (see Remark 3.3). In fact, for $\lambda < \mu(\beta)$ one observes that $(A_R^{(\beta)} - \lambda)^{-1} : L^2(\Omega) \to H^{3/2}(\Omega)$ is closed and everywhere defined, and hence bounded. For $\varepsilon > 0$ sufficiently small we also have that

$$V : H^{3/2-\varepsilon}(\Omega) \to L^2(\Omega) \tag{5.11}$$

is bounded (see, e.g., the proof of [15, Corollary 3.1]) and since the embedding $H^{3/2}(\Omega) \hookrightarrow H^{3/2-\varepsilon}(\Omega)$ is compact, one concludes that

$$V\big(A_R^{(\beta)} - \lambda I\big)^{-1} : L^2(\Omega) \to L^2(\Omega) \tag{5.12}$$

is compact, that is, V is relatively compact with respect to $A_R^{(\beta)}$. Next, we note that the condition

$$\ker\big(A_R^{(\beta)} - \mu(\beta) I\big) \not\subseteq \ker(V) \tag{5.13}$$

is satisfied. In fact, it is no restriction to assume that the eigenfunction $f_{\mu(\beta)}$ corresponding to the simple eigenvalue $\mu(\beta)$ is nonnegative (namely an eigenfunction $f_{\mu(\beta)}$ can be assumed to be real and then also $|f_{\mu(\beta)}|$ is an eigenfunction) and now a consequence of the Harnack inequality stated in [40, Corollary 8.21] implies that $|f_{\mu(\beta)}|$ does not vanish inside Ω. Thus the same is true for $f_{\mu(\beta)}$ and hence (5.13) holds as otherwise $V = 0$. Thus, the assertion follows from [15, Theorem 2.2]. □

Next, we recall the definition of the Dirichlet-to-Neumann map and an extension of the Dirichlet and Neumann trace operators (5.4) and (5.5) to the maximal domain; see [14, 19] for more details. For $z \in \rho(A_D)$ the Dirichlet-to-Neumann map is defined as

$$D(z) : H^1(\partial\Omega) \to L^2(\partial\Omega), \quad \tau_D f_z \mapsto \tau_N f_z, \tag{5.14}$$

where $f_z \in H^{3/2}(\Omega)$ is such that $-\Delta f_z = z f_z$. In order to extend the Dirichlet and Neumann trace operators to the maximal domain $\operatorname{dom}(S^*) = \mathcal{D}_{max}$ consider the spaces

$$\mathcal{G}_D(\partial\Omega) := \big\{\tau_D f \,\big|\, f \in \operatorname{dom}(A_N)\big\} \quad \text{and} \quad \mathcal{G}_N(\partial\Omega) := \big\{\tau_N f \,\big|\, f \in \operatorname{dom}(A_D)\big\}, \tag{5.15}$$

and equip these spaces with the scalar products

$$\begin{aligned} (\varphi, \psi)_{\mathcal{G}_D(\partial\Omega)} &:= (\Sigma^{-1/2}\varphi, \Sigma^{-1/2}\psi)_{L^2(\partial\Omega)}, \quad \Sigma = \operatorname{Im}(D(i)^{-1}), \\ (\varphi, \psi)_{\mathcal{G}_N(\partial\Omega)} &:= (\Lambda^{-1/2}\varphi, \Lambda^{-1/2}\psi)_{L^2(\partial\Omega)}, \quad \Lambda = -\overline{\operatorname{Im}(D(i))}. \end{aligned} \tag{5.16}$$

The corresponding dual spaces of antilinear continuous functionals are denoted by $\mathcal{G}_D(\partial\Omega)'$ and $\mathcal{G}_N(\partial\Omega)'$, respectively, and consequently one obtains Gelfand triplets $\{\mathcal{G}_D(\partial\Omega), L^2(\partial\Omega), \mathcal{G}_D(\partial\Omega)'\}$ and $\{\mathcal{G}_N(\partial\Omega), L^2(\partial\Omega), \mathcal{G}_N(\partial\Omega)'\}$. We shall also use that there are isometric isomorphisms $\iota_+ : \mathcal{G}_N(\partial\Omega) \to L^2(\partial\Omega)$ and $\iota_- : \mathcal{G}_N(\partial\Omega)' \to L^2(\partial\Omega)$ such that

$$\langle \varphi, \psi \rangle_{\mathcal{G}_N(\partial\Omega)' \times \mathcal{G}_N(\partial\Omega)} = (\iota_- \varphi, \iota_+ \psi)_{L^2(\partial\Omega)}, \quad \varphi \in \mathcal{G}_N(\partial\Omega)', \, \psi \in \mathcal{G}_N(\partial\Omega); \tag{5.17}$$

we note that $\iota_+ = \Lambda^{-1/2}$ and ι_- is the extension of $\Lambda^{1/2}$ onto $\mathcal{G}_N(\partial\Omega)'$. It was shown in [14, 19] that the Dirichlet and Neumann trace operators in (5.4) and (5.5) admit unique extensions to continuous surjective operators

$$\widetilde{\tau}_D : \mathcal{D}_{max} \to \mathcal{G}_N(\partial\Omega)' \quad \text{and} \quad \widetilde{\tau}_N : \mathcal{D}_{max} \to \mathcal{G}_D(\partial\Omega)', \tag{5.18}$$

where $\mathcal{D}_{max} = \operatorname{dom}(S^*)$ is equipped with the graph norm. Furthermore,

$$\ker(\widetilde{\tau}_D) = \ker(\tau_D) = \operatorname{dom}(A_D) \quad \text{and} \quad \ker(\widetilde{\tau}_N) = \ker(\tau_N) = \operatorname{dom}(A_N). \tag{5.19}$$

With the extended Dirichlet and Neumann trace operators one also extends the Dirichlet-to-Neumann map $D(z)$, $z \in \rho(A_D)$, to a bounded operator

$$\widetilde{D}(z) : \mathcal{G}_N(\partial\Omega)' \to \mathcal{G}_D(\partial\Omega)', \quad \widetilde{\tau}_D f_z \mapsto \widetilde{\tau}_N f_z, \tag{5.20}$$

where $f_z \in \ker(S^* - zI)$. The next result can be found in [19], see also [18, Theorem 8.7.6]. For this fix $\eta < \kappa$ and recall the domain decomposition

$$\mathcal{D}_{max} = \mathrm{dom}\,(S^*) = \mathrm{dom}\,(A_D) \dot{+} \ker(S^* - \eta I); \tag{5.21}$$

see (2.40). We will also use that the extended Dirichlet-to-Neumann map $\widetilde{D}(\cdot)$ has the remarkable regularization property

$$\mathrm{ran}\,\big(\widetilde{D}(\eta) - \widetilde{D}(z)\big) \subseteq \mathcal{G}_N(\partial\Omega), \quad z \in \rho(A_D). \tag{5.22}$$

Theorem 5.2. *Consider the Laplacian on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, fix $\eta < \kappa$, and decompose $f \in \mathrm{dom}\,(S^*)$ according to (5.21) in the form $f = f_D + f_\eta$, where $f_D \in \mathrm{dom}\,(A_D)$ and $f_\eta \in \ker(S^* - \eta I)$. Then $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$, where*

$$\Gamma_0 f = \iota_- \widetilde{\tau}_D f \quad \text{and} \quad \Gamma_1 f = -\iota_+ \tau_N f_D, \quad f = f_D + f_\eta \in \mathcal{D}_{max}, \tag{5.23}$$

is a boundary triplet for S^* such that

$$B_0 f = -\Delta f, \quad f \in \mathrm{dom}\,(B_0) = \{g \in \mathcal{D}_{max} \,|\, \widetilde{\tau}_D g = 0\}, \tag{5.24}$$

coincides with the Friedrichs extension A_D of S and the corresponding Weyl–Titchmarsh function M is given by

$$M(z) = \iota_+ \big(\widetilde{D}(\eta) - \widetilde{D}(z)\big) \iota_-^{-1}, \quad z \in \rho(B_0). \tag{5.25}$$

We note that the resolvent of B_0 is a compact operator in $L^2(\Omega)$ and that the lower bound $\kappa = \min(\sigma(B_0))$ is a simple eigenvalue. In order to determine the boundary conditions of the Krein-type extensions $S_{K,x}$ for $x < \kappa$ we use Lemma 2.11 and observe that $\Gamma_1 f - M(x)\Gamma_0 f = 0$ holds for $f \in \mathcal{D}_{max}$ if and only if

$$\tau_N f_D + \big(\widetilde{D}(\eta) - \widetilde{D}(x)\big)\widetilde{\tau}_D f = 0, \quad f \in \mathcal{D}_{max}, \tag{5.26}$$

holds. Since $f = f_D + f_\eta$ and $f_D \in \ker(\widetilde{\tau}_D)$ we have $\widetilde{D}(\eta)\widetilde{\tau}_D f = \widetilde{D}(\eta)\widetilde{\tau}_D f_\eta = \widetilde{\tau}_N f_\eta$ and hence (5.26) takes the form

$$\tau_N f_D + \widetilde{\tau}_N f_\eta - \widetilde{D}(x)\widetilde{\tau}_D f = 0, \quad f \in \mathcal{D}_{max}, \tag{5.27}$$

which implies that the Krein-type extensions $S_{K,x}$, $x < \kappa$, of S are given by

$$S_{K,x} f = -\Delta f, \quad f \in \mathrm{dom}\,(S_{K,x}) = \big\{g \in \mathcal{D}_{max} \,\big|\, \widetilde{\tau}_N g = \widetilde{D}(x)\widetilde{\tau}_D g\big\} \tag{5.28}$$

and, in particular, the classical Krein–von Neumann extension has the form

$$S_0 f = -\Delta f, \quad f \in \mathrm{dom}\,(S_0) = \big\{g \in \mathcal{D}_{max} \,\big|\, \widetilde{\tau}_N g = \widetilde{D}(0)\widetilde{\tau}_D g\big\}; \tag{5.29}$$

such types of descriptions go back already to Višik [82] and Grubb [41] and were obtained for the Laplacian on Lipschitz domains in [13, 14].

In the final theorem of this section we now show that the Krein-type extensions $S_{K,x}$, $x \leq \kappa$, are spectrally unstable.

Theorem 5.3. *Let $x \leq \kappa$ and assume that $0 \leq V \in L^\infty(\Omega)\backslash\{0\}$. Then for all $\alpha \in \mathbb{R}$, the operator*

$$(S_{K,x} + \alpha V) f = -\Delta f + \alpha V f, \quad \mathrm{dom}\,(S_{K,x} + \alpha V) = \mathrm{dom}\,(S_{K,x}), \tag{5.30}$$

is self-adjoint in $L^2(\Omega)$ and

$$\sigma(S_{K,x} + \alpha V) \cap (-\infty, x) \neq \emptyset \ \textit{ for } \textbf{\textit{any}} \ \alpha < 0. \tag{5.31}$$

Proof. First, one observes that $V \in L^\infty(\Omega)$ is a self-adjoint bounded multiplication operator in $L^2(\Omega)$ and hence $S_{K,x} + \alpha V$, $x \leq \kappa$, is self-adjoint in $L^2(\Omega)$ for all $\alpha \in \mathbb{R}$.

In order to show (5.31) for $x < \kappa$ we choose $\beta_x \in \mathbb{R}$ such that the smallest eigenvalue $\mu(\beta_x)$ of the Robin Laplacian $A_R^{(\beta_x)}$ in (5.8) satisfies

$$x = \mu(\beta_x). \tag{5.32}$$

As $A_R^{(\beta_x)}$ is a semibounded self-adjoint extension of S and $A_R^{(\beta_x)} \geq xI$ it follows from Proposition 2.3 that $S_{K,x} \leq A_R^{(\beta_x)}$. Furthermore, from Theorem 5.1 we see that

$$\sigma(A_R^{(\beta_x)} + \alpha V) \cap (-\infty, x) \neq \emptyset \text{ for any } \alpha < 0, \tag{5.33}$$

and hence $\min \sigma(A_R^{(\beta_x)} + \alpha V) < x$ for any $\alpha < 0$. The inequality $S_{K,x} \leq A_R^{(\beta_x)}$ yields $S_{K,x} + \alpha V \leq A_R^{(\beta_x)} + \alpha V$ and, in particular,

$$\min \sigma(S_{K,x} + \alpha V) \leq \min \sigma(A_R^{(\beta_x)} + \alpha V)) < x \tag{5.34}$$

for any $\alpha < 0$. This implies (5.31) for $x < \kappa$.

It remains to discuss the case $x = \kappa$. Using Lemma 2.13 one observes that

$$M(\kappa) = \{\{\Gamma_0 f_\kappa, \Gamma_1 f_\kappa\} \,|\, f_\kappa \in \ker(S^* - \kappa I)\}, \tag{5.35}$$

and if we define the Cauchy data (Dirichlet-to-Neumann relation) at κ by

$$\widetilde{D}(\kappa) = \{\{\widetilde{\tau}_D f_\kappa, \widetilde{\tau}_N f_\kappa\} \,|\, f_\kappa \in \ker(S^* - \kappa I)\}, \tag{5.36}$$

then it turns out that the Krein-type extension $S_{K,\kappa}$ of S is given by

$$S_{K,\kappa} f = -\Delta f, \quad f \in \operatorname{dom}(S_{K,\kappa}) = \big\{ g \in \mathcal{D}_{max} \,\big|\, \{\widetilde{\tau}_D g, \widetilde{\tau}_N g\} \in \widetilde{D}(\kappa) \big\}. \tag{5.37}$$

We consider the Friedrich extension A_D and note that $\ker(A_D) \neq \{0\}$ as the lower bound κ is a simple eigenvalue. The same arguments as in the proof of [15, Corollary 3.1] and Theorem 5.1 show that V is a relatively compact perturbation of A_D and hence [15, Theorem 2.2] implies

$$\sigma(A_D + \alpha V) \cap (-\infty, \kappa) \neq \emptyset \text{ for } \textbf{any } \alpha < 0. \tag{5.38}$$

As above we have $S_{K,\kappa} \leq A_D$ (see Proposition 2.3) and thus $S_{K,\kappa} + \alpha V \leq A_D + \alpha V$. Together with (5.38) this implies (5.31) for $x = \kappa$. □

Remark 5.4. We note that in the proof of Theorem 5.3 the multiplication operator $\alpha V \in L^\infty(\Omega)$ is a relatively compact perturbation of the Robin Laplacian $A_R^{(\beta_x)}$, $x < \kappa$, by Theorem 5.1, but in general only a bounded additive perturbation of the Krein-type extension $S_{K,x}$. ⋄

Acknowledgments. We are indebted to Vladimir Lotoreichik for a very helpful discussion and to the anonymous referee for his very careful reading of our manuscript. J.B. is most grateful for a stimulating research stay at Baylor University, where some parts of this paper were written in May of 2025. F.G. and H.S. gratefully acknowledge kind invitations to the Institute of Applied Mathematics at the Graz University of Technology, Austria. This research was funded by the Austrian Science Fund (FWF) Grant-DOI: 10.55776/P33568.

References

[1] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space, Volume II*, Pitman, Boston, 1981.

[2] A. Alonso and B. Simon, *The Birman-Krein-Vishik theory of selfadjoint extensions of semibounded operators*, J. Operator Th. **4**, 251–270 (1980); Addenda: **6**, 407 (1981).

[3] T. Ando and K. Nishio, *Positive selfadjoint extensions of positive symmetric operators*, Tohoku Math. J. (2), **22**, 65–75 (1970).

[4] Yu. M. Arlinskii, *On m-accretive extensions and restrictions*, Meth. Funct. Anal. Top. **4**, 1-26 (1998).

[5] Yu. M. Arlinskii, *Abstract boundary conditions for maximal sectorial extensions of sectorial operators*, Math. Nachr. **209**, 5–36 (2000).

[6] Yu. M. Arlinskii, S. Hassi, Z. Sebestyén, and H. S. V. de Snoo, *On the class of extremal extensions of a nonnegative operator*, in *Recent Advances in Operator Theory and Related Topics*, L. Kérchy, C. Foias, I. Gohberg, and H. Langer (eds.), Operator Theory: Advances and Applications, Vol. 127, Birkhäuser, Basel, 2001, pp. 41–81.

[7] Yu. Arlinskiĭ and Yu. Kovalev, *Operators in divergence form and their Friedrichs and Krein extensions*, Opusc. Math. **31**, 501–517 (2011).

[8] Yu. M. Arlinskii and E. R. Tsekanovskii, *On the theory of nonnegative selfadjoint extensions of a nonnegative symmetric operator*, Rep. Nat. Acad. Sci. Ukraine **2002**, no. 11, 30–37.

[9] Yu. Arlinskiĭ and E. Tsekanovskiĭ, *M.Kreĭn's research on semibounded operators, its contemporary developments, and applications*, in *Modern Analysis and Applications. The Mark Krein Centenary Conference*, Vol. 1, V. Adamyan, Y. M. Berezansky, I. Gohberg, M. L. Gorbachuk, V. Gorbachuk, A. N. Kochubei, H. Langer, and G. Popov (eds.), Operator Theory: Advances and Applications, Vol. 190, Birkhäuser, Basel, 2009, pp. 65–112.

[10] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *The Krein–von Neumann extension and its connection to an abstract buckling problem*, Math. Nachr. **283**, 165–179 (2010).

[11] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *A survey on the Krein–von Neumann extension, the corresponding abstract buckling problem, and Weyl-type spectral asymptotics for perturbed Krein Laplacians in nonsmooth domains*, in *Mathematical Physics, Spectral Theory and Stochastic Analysis*, M. Demuth and W. Kirsch (eds.), Operator Theory: Advances and Applications, Vol. 232, Birkhäuser, Basel, 2013, pp. 1–106.

[12] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, and G. Teschl, *Spectral theory for perturbed Krein Laplacians in nonsmooth domains*, Adv. Math. **223**, 1372–1467 (2010).

[13] J. Behrndt, F. Gesztesy, T. Micheler, and M. Mitrea, *The Krein–von Neumann realization of perturbed Laplacians on bounded Lipschitz domains*, Operator Theory: Advances and Applications, Vol. 255, Birkhäuser, Basel, 2016, pp. 49–66.

[14] J. Behrndt, F. Gesztesy, and M. Mitrea, *Sharp boundary trace theory and Schrödinger operators on bounded Lipschitz domains*, Mem. Amer. Math. Soc. **307**, No. 1550 (2025).

[15] J. Behrndt, F. Gesztesy, and H.S.V. de Snoo, *Weak coupling and spectral instability for Neumann Laplacians*, Proc. Amer. Math. Soc. **143** (2025), 1675–1686.

[16] J. Behrndt, F. Gesztesy, and H.S.V. de Snoo, *Weak coupling and spectral instability for Krein-type extensions*, in preparation.

[17] J. Behrndt, F. Gesztesy, R. Nichols, and H.S.V. de Snoo, *On relative* (*form*) *Hilbert–Schmidt perturbations of regular Sturm–Liouville operators*, in preparation.

[18] J. Behrndt, S. Hassi, and H.S.V. de Snoo, *Boundary Value Problems, Weyl Functions, and Differential Operators*, Monographs in Mathematics, Vol. 108, Birkhäuser, Springer, 2020.

[19] J. Behrndt and T. Micheler, *Elliptic differential operators on Lipschitz domains and abstract boundary value problems*, J. Funct. Anal. **267** (2014), 3657–3709.

[20] R. Blankenbecler, M. L. Goldberger, and B. Simon, *The bound states of weakly coupled long-range one-dimensional quantum Hamiltonians*, Ann. Phys. **108** (1977), 69–78.

[21] D. Bollé, F. Gesztesy, and M. Klaus, *Scattering theory for one-dimensional systems with* $\int dxV(x) = 0$, J. Math. Anal. Appl. **122** (1987), 496–518.

[22] W. Bulla, F. Gesztesy, W. Renger, and B. Simon, *Weakly coupled bound states in quantum waveguides*, Proc. Amer. Math. Soc. **125** (1997), 1487–1495.

[23] E.A. Coddington and H. de Snoo, *Positive selfadjoint extensions of positive symmetric subspaces*, Math. Z. **159**, 203–214 (1978).

[24] J.-C. Cuenin and K. Merz, *Weak coupling limit for Schrödinger-type operators with degenerate kinetic energy for a large class of potentials*, Lett. Math. Phys. **111** (2021), Paper No. 46, 29 pp.

[25] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95**, 1–95 (1991).

[26] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. **73**, 141–242 (1995).

[27] P. Exner, S. Kondej, and V. Lotoreichik, *Bound states of weakly deformed soft waveguides*, Asymptot. Anal. **138** (2024), 151–174.

[28] W. G. Faris, *Self-Adjoint Operators*, Lecture Notes in Mathematics, Vol. 433, Springer, Berlin, 1975.

[29] S. Fassari and M. Klaus, *Coupling constant thresholds of perturbed periodic Hamiltonians*, J. Math. Phys. **39** (1998), 4369–4416.

[30] R. L. Frank, S. Morozov, and S. Vugalter, *Weakly coupled bound states of Pauli operators*, Calc. Var. Partial Diff. Eq. **40** (2011), 253–271.

[31] G. Fucci, F. Gesztesy, K. Kirsten, L. L. Littlejohn, R. Nichols, and J. Stanfill, *The Krein–von Neumann extension revisited*, Applicable Anal. **101**, 1593–1616 (2022).

[32] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd revised and extended ed., de Gruyter Studies in Math., Vol. 19, de Gruyter, Berlin, 2011.

[33] L. Geisinger, A. Laptev, and T. Weidl, *Geometrical versions of improved Berezin–Li–Yau inequalities*, J. Spectral Th. **1**, 87–109 (2011).

[34] F. Gesztesy and H. Holden, *A unified approach to eigenvalues and resonances of Schrödinger operators using Fredholm determinants*, J. Math. Anal. Appl. **123** (1987), 181–198; addendum **132** (1988), 309.

[35] F. Gesztesy, Y. Latushkin, M. Mitrea, and M. Zinchenko, *Nonself-adjoint operators, infinite determinants, and some applications*, Russian J. Math. Phys. **12** (2005), 443–471.

[36] F. Gesztesy, M. Malamud, M. Mitrea, and S. Naboko, *Generalized polar decompositions for closed operators in Hilbert spaces and some applications* Integral Equations Operator Theory **64** (2009), 83–113.

[37] F. Gesztesy and M. Mitrea, *Generalized Robin boundary conditions, Robin-to-Dirichlet maps, and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains*, in *Perspectives in Partial Differential Equations, Harmonic Analysis and Applications: A Volume in Honor of Vladimir G. Maz'ya's 70th Birthday*, D. Mitrea and M. Mitrea (eds.), Proceedings of Symposia in Pure Mathematics, Vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 105–173.

[38] F. Gesztesy and M. Mitrea, *A description of all self-adjoint extensions of the Laplacian and Krein-type resolvent formulas on non-smooth domains*, J. Analyse Math. **113** (2011), 53–172.

[39] F. Gesztesy, R. Nichols, and M. Zinchenko, *Sturm–Liouville Operators, Their Spectral Theory, and Some Applications*, Colloquium Publications, Vol. 67, Amer. Math. Soc., Providence, RI, 2024.

[40] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.

[41] G. Grubb, *A characterization of the non-local boundary value problems asso- ciated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa (3), **22**, 425–513 (1968).

[42] G. Grubb, *Spectral asymptotics for the "soft" selfadjoint extension of a symmetric elliptic differential operator*, J. Operator Th. **10**, 9–20 (1983).

[43] G. Grubb, *Krein-like extensions and the lower boundedness problem for elliptic operators*, J. Diff. Eq. **252**, 852–885 (2012).

[44] P. Hartman, Perturbation of spectra and Krein extensions, Rend. Circ. Mat. Palermo (2) **5**, 341–354 (1957).

[45] S. Hassi, M. Malamud, and H. de Snoo, *On Kreĭn's extension theory of nonnegative operators*, Math. Nachr. **274–275**, 40–73 (2004).

[46] S. Hassi and H. de Snoo, *Friedrichs and Krein type extensions in terms of representing maps*, Ann. Funct. Anal. 15 **4**, Paper No. 78 (2024).

[47] V. Hoang, D. Hundertmark, J. Richter, and S. Vugalter, *Quantitative bounds versus existence of weakly coupled bound states for Schrödinger type operators*, Ann. Henri Poincaré **24** (2023), 783–842.

[48] H. Holden, *On coupling constant thresholds in two dimensions*, J. Operator Th. **14** (1985), 263–276.

[49] D. Jerison and C. Kenig, *The Neumann problem in Lipschitz domains*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 203–207.

[50] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130** (1995), 161–219.

[51] T. Kato, *Perturbation theory of semi-bounded operators*, Math. Ann. **125** (1953), 435–447.

[52] T. Kato, *Wave operators and similarity for some non-self-adjoint operators*, Math. Ann. **162** (1966), 258–279.

[53] T. Kato, *Perturbation Theory for Linear Operators*, corr. printing of the 2nd ed., Springer, Berlin, 1980.

[54] M. Klaus, *On the bound state of Schrödinger operators in one dimension*, Ann. Phys. **108** (1977), 288–300.

[55] M. Klaus, *A remark about weakly coupled Schrödinger operators*, Helv. Phys. Acta **52** (1979), 223–229.

[56] M. Klaus, *Some applications of the Birman–Schwinger principle*, Helv. Phys. Acta **55** (1982), 49–68.

[57] M. Klaus and B. Simon, *Coupling constant thresholds in nonrelativistic quantum mechanics. I. Short-range two-body case*, Ann. Phys. **130** (1980), 251–281.

[58] M. Klaus and B. Simon, *Coupling constant thresholds in nonrelativistic quantum mechanics II. Two cluster thresholds in N-body systems*, Commun. Math. Phys. **78** (1980), 153–168.

[59] S. Kondej and V. Lotoreichik, *Weakly coupled bound state of 2-D Schrödinger operator with potential-measure*, J. Math. Anal. Appl. **420** (2014), 1416–1438.

[60] R. Konno and S. T. Kuroda, *On the finiteness of perturbed eigenvalues*, J. Fac. Sci., Univ. Tokyo, Sec. I, **13** (1966), 55–63.

[61] V. Koshmanenko, *Singular Quadratic Forms in Perturbation Theory*, Mathematics and its Applications, Vol. 474, Kluwer, Dordrecht, 1999.

[62] M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I*, Mat. Sbornik **20**, 431–495 (1947) (Russian).

[63] M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. II*, Mat. Sbornik **21**, 365–404 (1947) (Russian).

[64] S. N. Lakaev, *Discrete spectrum and resonances of a one-dimensional Schrödinger operator for small values of the coupling constants*, Theoret. Math. Phys. **44** (1980), 810–814.

[65] P. Li and S. T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983), 309–318.

[66] M. Melgaard, *On bound states for systems of weakly coupled Schrödinger equations in one space dimension*, J. Math. Phys. **43** (2002), 5365–5385.

[67] S. Molchanov and B. Vainberg, *Negative eigenvalues of non-local Schrödinger operators with sign-changing potentials*, Proc. Amer. Math. Soc. **151** (2023), 4757–4770.

[68] G. Nenciu, *Applications of the Kreĭn resolvent formula to the theory of self-adjoint extensions of positive symmetric operators*, J. Operator Th. **10**, 209–218 (1983).

[69] S. H. Patil, *T-matrix analysis of one-dimensional weakly coupled bound states*, Phys. Rev. A **22**(1980), 1655–1663.

[70] S. H. Patil, *Wave functions for weakly-coupled bound states*, Phys. Rev. A **25**(1982), 2467–2472.

[71] V. Prokaj and Z. Sebestyén, *On extremal positive operator extensions*, Acta Sci. Math. (Szeged) **62**, 485–491 (1996).

[72] J. Rauch, *Perturbation theory for eigenvalues and resonances of Schrödinger Hamiltonians*, J. Funct. Anal. **35** (1980), 304–315.

[73] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV: Analysis of Operators.* Academic Press, New York, 1978.

[74] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*, Grad. Texts in Math., Vol. 265, Springer, 2012.

[75] Z. Sebestyén and E. Sikolya, *On Krein–von Neumann and Friedrichs extensions*, Acta Sci. Math. (Szeged) **69**, 323–336 (2003).

[76] Z. Sebestyén and J. Stochel, *Restrictions of positive self-adjoint operators*, Acta Sci. Math. (Szeged) **55**, 149–154 (1991).

[77] B. Simon, *The bound states of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Phys. **97** (1976), 279–288.

[78] B. Simon, *On the absorption of eigenvalues by continuous spectrum in regular perturbation theory*, J. Funct. Anal. **25** (1977), 338–344.

[79] C. F. Skau, *Positive self-adjoint extensions of operators affiliated with a von Neumann algebra*, Math. Scand. **44**, 171-195 (1979).

[80] O. G. Storozh, *On the hard and soft extensions of a nonnegative operator*, J. Math. Sci. **79**, 1378–1380 (1996).

[81] E. R. Tsekanovskii, *Accretive extensions and problems on the Stieltjes operator-valued functions realizations*, in *Operator Theory and Complex Analysis*, T. Ando and I. Gohberg (eds.), Operator Theory: Advances and Applications, Vol. 59, Birkhäuser, Basel, 1992, pp. 328–347.

[82] M. I. Višik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obsc. **1**, 187–246 (1952) (Russian); Engl. transl. in Amer. Math. Soc. Transl. (2), **24**, 107–172 (1963).

[83] J. Weidmann, *Lineare Operatoren in Hilberträumen. Teil 1*, Teubner, 2000.

Technische Universität Graz, Institut für Angewandte Mathematik, Steyrergasse 30, 8010 Graz, Austria

Email address: behrndt@tugraz.at

URL: https://www.math.tugraz.at/~behrndt/

Department of Mathematics, Baylor University, Sid Richardson Bldg., 1410 S. 4th Street, Waco, TX 76706, USA

Email address: Fritz_Gesztesy@baylor.edu

URL: https://math.artsandsciences.baylor.edu/person/fritz-gesztesy-phd

Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, P.O. Box 407, 9700 AK Groningen, Netherlands

Email address: h.s.v.de.snoo@rug.nl