MONOTONE CONVERGENCE THEOREMS FOR SEMIBOUNDED OPERATORS AND FORMS WITH APPLICATIONS

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ABSTRACT. Let H_n be a monotone sequence of nonnegative selfadjoint operators or relations in a Hilbert space. Then there exists a selfadjoint relation H_∞, such that H_n converges to H_∞ in the strong resolvent sense. This and related limit results are explored in detail and new simple proofs are presented. The corresponding statements for monotone sequences of semibounded closed forms are established as immediate consequences. Applications and examples, illustrating the general results, include sequences of multiplication operators, Sturm-Liouville operators with increasing potentials, forms associated with Kreĭn-Feller differential operators, singular perturbations of nonnegative selfadjoint operators, and the characterization of the Friedrichs and Kreĭn-von Neumann extensions of a nonnegative operator or relation.

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space and let $\mathbf{B}(\mathfrak{H})$ be the space of bounded everywhere defined linear operators on $\mathfrak{H}$. The following well-known fact on the strong limit of a uniformly bounded monotone increasing sequence of bounded nonnegative selfadjoint operators is one of the fundamental limit results in the theory of linear operators in Hilbert spaces; cf. [1], [15].

Theorem 1.1. *Let $H_n \in \mathbf{B}(\mathfrak{H})$ be a nondecreasing sequence of nonnegative selfadjoint operators in $\mathfrak{H}$ and assume that the sequence H_n is uniformly bounded from above, i.e., $H_n \leq M$ for some positive constant M and all $n \in \mathbb{N}$. Then there exists a nonnegative selfadjoint operator $H_\infty \in \mathbf{B}(\mathfrak{H})$ with $H_\infty \leq M$ and $H_n \leq H_\infty$ for all $n \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} H_n h = H_\infty h, \quad h \in \mathfrak{H}. \tag{1.1}$$

If there is no uniform upper bound, then the convergence in (1.1) has to be replaced by strong resolvent convergence and the strong resolvent limit H_∞ will in general be an unbounded nonnegative selfadjoint operator or a linear relation (multivalued operator).

Theorem 1.2. *Let $H_n \in \mathbf{B}(\mathfrak{H})$ be a nondecreasing sequence of nonnegative selfadjoint operators in $\mathfrak{H}$. Then there exists a nonnegative selfadjoint relation H_∞ with $H_n \leq H_\infty$ such that*

$$\lim_{n\to\infty} (H_n - \lambda)^{-1} h = (H_\infty - \lambda)^{-1} h, \quad h \in \mathfrak{H}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}. \tag{1.2}$$

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Furthermore, $\{ h \in \mathfrak{H} : \lim_{n\to\infty}(H_n h, h) < \infty \}$ *is equal to the domain of the square root of* H_∞*; it is dense if and only if* H_∞ *is an operator.*

Simple examples show that H_∞ in Theorem 1.2 is in general an unbounded operator or a linear relation. If, e.g., H is a nonnegative unbounded selfadjoint operator in $\mathfrak{H}$ with a spectral decomposition $H = \int_0^\infty t\, dE(t)$ and the sequence H_n is defined by $H_n = \int_0^n t\, dE(t)$, $n \in \mathbb{N}$, then H_n converges to the selfadjoint limit $H_\infty = H$ in the strong resolvent sense. As a further example, consider a nonnegative selfadjoint operator $H \in \mathbf{B}(\mathfrak{H})$ and let P be a nontrivial orthogonal projection. Then the sequence $H_n \in \mathbf{B}(\mathfrak{H})$ defined by $H_n = H + nP$, $n \in \mathbb{N}$, is increasing, and converges in strong resolvent sense to the orthogonal sum

$$H_\infty = (I-P)H{\upharpoonright} \ker P \;\oplus\; \{\{0,h\} : h \in \operatorname{ran} P\}. \tag{1.3}$$

For applications in mathematical physics it is necessary to allow the operators H_n in the sequence to be unbounded operators themselves, e.g., when considering sequences of differential operators and singular perturbations of unbounded operators; cf. [2], [14]. In this situation it is convenient to deal also with the corresponding sequence of densely defined closed nonnegative forms

$$\mathfrak{t}_n[h,k] = (H_n^{\frac{1}{2}} h, H_n^{\frac{1}{2}} k), \qquad \operatorname{dom} \mathfrak{t}_n = \operatorname{dom} H_n^{\frac{1}{2}}, \qquad n \in \mathbb{N},$$

and the corresponding limit form $\mathfrak{t}_\infty$.

Theorem 1.3. *Let* H_n *be a nondecreasing sequence of nonnegative selfadjoint operators in* $\mathfrak{H}$ *and let* $\mathfrak{t}_n$ *be the corresponding closed nonnegative forms. Then there exists a nonnegative selfadjoint relation* H_∞ *with* $H_n \le H_\infty$ *such that* (1.2) *holds. Furthermore,* H_∞ *is the representing relation for the closed nonnegative form*

$$\mathfrak{t}_\infty[h,k] = \lim_{n\to\infty} \mathfrak{t}_n[h,k], \quad h,k \in \operatorname{dom} \mathfrak{t}_\infty = \left\{ h \in \bigcap_{n=1}^\infty \operatorname{dom} \mathfrak{t}_n : \lim_{n\to\infty} \mathfrak{t}_n[h] < \infty \right\},$$

and H_∞ *is an operator if and only if* $\mathfrak{t}_\infty$ *is densely defined.*

A version of this theorem was given by T. Kato [13, Chapter VIII, Theorem 3.13a]: if $\mathfrak{t}_n$ is a nondecreasing sequence of closed forms which are semibounded from below, then the pointwise limit $\mathfrak{t}_\infty$ defines a closed form which is semibounded from below. Under the extra assumption that $\mathfrak{t}_\infty$ is densely defined, which implies that all $\mathfrak{t}_n$ are densely defined, the rest of the theorem in [13] is proved in the semibounded case. Similar results, even without Kato's density condition, were stated by B. Simon [16], [17] (see also [14]) and by V.A. Derkach and M.M. Malamud [7], [8].

In the present paper the general limit results are stated for a sequence of nondecreasing semibounded selfadjoint relations, see Theorems 3.1 and 3.5. The proofs presented here are particularly simple; 'improper extensions' of forms and Helly type arguments are not needed. The convergence theorems for forms are obtained immediately from the general limit results for semibounded selfadjoint relations, see Theorem 4.2. The closedness of the limit form is a direct consequence. Theorems 3.5 and 4.2 contain all the results in Theorems 1.2 and 1.3. One further essential advantage when dealing with linear relations is that results for nonincreasing sequences of nonnegative selfadjoint operators and relations can be obtained from previous results by taking formal inverses, see Theorem 3.7. This procedure also leads to a corresponding result for a nonincreasing sequence of nonnegative forms, see Theorem 4.3.

The abstract results on limits of monotone sequences of operators, relations, and forms are illustrated with a number of examples and applications in Sections 3 - 5. These include nondecreasing sequences of multiplication operators, Sturm-Liouville operators with increasing potentials, forms associated with Kreĭn-Feller differential operators, singular perturbations of unbounded nonnegative selfadjoint operators, and the characterization of the Friedrichs and the Kreĭn-von Neumann extensions of a nonnegative operator or relation originally going back to T. Ando and K. Nishio [3]. Furthermore, already the finite-dimensional version of the main result (Corollary 3.6) has an important consequence in the spectral theory of singular canonical differential equations: it can be used to determine the number of square-integrable solutions of such a system; cf. [4].

2. Preliminaries

2.1. **Linear relations.** A linear *relation* H in a Hilbert space $\mathfrak{H}$ is a linear subspace H of the product space $\mathfrak{H} \times \mathfrak{H}$, which is said to be closed if its graph is closed as a subset of $\mathfrak{H} \times \mathfrak{H}$. The domain, range, kernel, and multivalued part of H are denoted by $\operatorname{dom} H$, $\operatorname{ran} H$, $\ker H$, and $\operatorname{mul} H$, respectively. If $\operatorname{mul} H = \{0\}$, then H is (the graph of) a linear operator. The inverse of H is defined by $H^{-1} = \{\{f', f\} : \{f, f'\} \in H\}$. The relation $H - \lambda$, $\lambda \in \mathbb{C}$, is defined as $H - \lambda = \{\, \{h, h' - \lambda h\} : \{h, h'\} \in H\,\}$. The *resolvent set* $\rho(H)$ and the *spectrum* $\sigma(H)$ (in $\mathbb{C}$) of H are defined by

$$\rho(H) = \{\, \lambda \in \mathbb{C} : (H - \lambda)^{-1} \in \mathbf{B}(\mathfrak{H})\,\} \quad \text{and} \quad \sigma(H) = \mathbb{C} \setminus \rho(H).$$

It is known that the resolvent set is an open subset of $\mathbb{C}$. The resolvent operator $(H - \lambda)^{-1}$ of a closed relation H satisfies the resolvent identity and, moreover,

$$\ker(H - \lambda)^{-1} = \operatorname{mul} H, \quad \lambda \in \rho(H). \tag{2.1}$$

The adjoint H^* of H is the closed linear relation defined by

$$H^* = \{\, \{k, k'\} \in \mathfrak{H} \times \mathfrak{H} : (h', k) = (h, k'),\ \{h, h'\} \in H\,\}.$$

The following identities are useful:

$$(\operatorname{dom} H)^{\perp} = \operatorname{mul} H^*, \quad (\operatorname{ran} H)^{\perp} = \ker H^*. \tag{2.2}$$

A relation H is said to be *symmetric* or *selfadjoint* if $H \subset H^*$ or $H = H^*$, respectively. If the relation H is selfadjoint, it follows from (2.2) that $(\operatorname{mul} H)^{\perp} = \overline{\operatorname{dom}}\, H$, where $\overline{\operatorname{dom}}\, H$ stands for the closure of $\operatorname{dom} H$ in $\mathfrak{H}$. Hence, a selfadjoint relation H in $\mathfrak{H}$ can be decomposed as a componentwise orthogonal sum

$$H = H_s \oplus H_{\rm mul}, \tag{2.3}$$

where $H_{\rm s} = \{\{f, f'\} \in H : f' \in \overline{\operatorname{dom}}\, H\}$ is a selfadjoint operator in the Hilbert space $\overline{\operatorname{dom}}\, H$ and $H_{\rm mul} = \{0\} \times \operatorname{mul} H$ is a selfadjoint relation in the Hilbert space $\operatorname{mul} H$. Clearly, $\operatorname{dom} H_s = \operatorname{dom} H$ and $\rho(H_s) = \rho(H)$. Moreover,

$$\|(H - \lambda)^{-1}\| \le \frac{1}{|\operatorname{Im} \lambda|}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.4}$$

The resolvent of a selfadjoint relation H has the representation

$$(H - \lambda)^{-1} = \int_{\mathbb{R}} \frac{dE(t)}{t - \lambda}, \quad \lambda \in \rho(H), \tag{2.5}$$

where $E(t)$ is the orthogonal sum of the spectral family of H_s in $\mathfrak{H} \ominus \operatorname{mul} H$ and the null operator in $\operatorname{mul} H$; cf. (2.1). Note that $H_s = \int_{\mathbb{R}} t \, dE(t)$.

2.2. **Nonnegative selfadjoint relations.** A linear relation H in $\mathfrak{H}$ is said to be *nonnegative*, denoted by $H \geq 0$, if $(f', f) \geq 0$ for all $\{f, f'\} \in H$. If the relation H is selfadjoint, then $H \geq 0$ if and only if $H_s \geq 0$, so that

$$H \geq 0 \quad \text{if and only if} \quad \sigma(H) \subset [0, \infty). \tag{2.6}$$

If $H = H^* \geq 0$, then H has a unique nonnegative selfadjoint square root $H^{\frac{1}{2}}$ in the sense of relations:

$$H^{\frac{1}{2}} = (H_s)^{\frac{1}{2}} \oplus H_{\mathrm{mul}},$$

where $(H_s)^{\frac{1}{2}}$ is the nonnegative square root of the densely defined nonnegative selfadjoint operator H_s in the Hilbert space $\overline{\operatorname{dom}}\, H = \mathfrak{H} \ominus \operatorname{mul} H$. Thus $H^{\frac{1}{2}}$ and H have the same multivalued part and $(H^{\frac{1}{2}})_s = (H_s)^{\frac{1}{2}}$. Moreover, equivalent are:

$$\operatorname{dom} H \text{ closed}; \quad \operatorname{dom} H^{\frac{1}{2}} \text{ closed}; \quad \operatorname{dom} H = \operatorname{dom} H^{\frac{1}{2}}, \tag{2.7}$$

with similar statements for the ranges since H^{-1} is also a nonnegative selfadjoint relation. If $H = H^* \geq 0$, then the following identity is not difficult to check

$$(H^{-1} + x)^{-1} = \frac{1}{x} - \frac{1}{x^2} \left(H + \frac{1}{x} \right)^{-1}, \quad x > 0. \tag{2.8}$$

Here each resolvent operator belongs to $\mathbf{B}(\mathfrak{H})$ by (2.6).

Proposition 2.1. *Let H be a nonnegative selfadjoint relation in a Hilbert space $\mathfrak{H}$. Then for $h \in \mathfrak{H}$ and $x > 0$,*

$$\lim_{x \downarrow 0} \big((H^{-1} + x)^{-1} h, h \big) = \begin{cases} \|(H^{\frac{1}{2}})_s h\|^2, & h \in \operatorname{dom} H^{\frac{1}{2}}, \\ \infty, & \text{otherwise}. \end{cases} \tag{2.9}$$

Proof. Let P be the orthogonal projection from $\mathfrak{H}$ onto $\overline{\operatorname{dom}}\, H$. Then it follows from (2.1) and (2.8) that for each $x > 0$ and $h \in \mathfrak{H}$:

$$\big((H^{-1} + x)^{-1} h, h \big) = \frac{1}{x} \|(I - P)h\|^2 + \frac{1}{x} \|Ph\|^2 - \frac{1}{x^2} \left(\left(H + \frac{1}{x} \right)^{-1} Ph, Ph \right).$$

Let $E(t)$ be the spectral family belonging to H, so that $H_s = \int_0^\infty t \, dE(t)$. Then the above formula can be rewritten as

$$\big((H^{-1} + x)^{-1} h, h \big) = \frac{1}{x} \|(I - P)h\|^2 + \int_0^\infty \frac{t}{xt + 1} \, d(E(t)Ph, Ph), \quad x > 0. \tag{2.10}$$

By the nonnegativity of the terms the limit as $x \downarrow 0$ is finite if and only if the limit of each of the terms on the righthand side of (2.10) is finite. The first limit is finite if and only if $(I - P)h = 0$, i.e., if $h \in \overline{\operatorname{dom}}\, H$. By the monotone convergence theorem the limit of the second term is equal to $\int_0^\infty t \, d(E(t)h, h)$, which is finite and equal to $\|(H^{\frac{1}{2}})_s h\|^2$ if and only if $h \in \operatorname{dom} H^{\frac{1}{2}}$. □

2.3. **Ordering of nonnegative and semibounded selfadjoint relations.** Let H_1 and H_2 be nonnegative selfadjoint relations in $\mathfrak{H}$. Then H_1 and H_2 are said to satisfy the inequality $H_1 \geq H_2$, if

$$0 \leq (H_1 + x)^{-1} \leq (H_2 + x)^{-1} \quad \text{for some } x > 0. \tag{2.11}$$

In order to translate this definition in terms of square roots of the nonnegative selfadjoint relations, observe that if $H = H^* \geq 0$ then for each $x > 0$,

$$\operatorname{dom}(H + x)^{\frac{1}{2}} = \operatorname{dom} H^{\frac{1}{2}}. \tag{2.12}$$

Since $\operatorname{dom} H$ is a core for $H_s^{\frac{1}{2}}$ it follows that

$$\|(H_s + x)^{\frac{1}{2}} h\|^2 = \|(H^{\frac{1}{2}})_s h\|^2 + x\|h\|^2, \quad h \in \operatorname{dom} H^{\frac{1}{2}}, \quad x > 0. \tag{2.13}$$

The next result extends well-known facts for densely defined nonnegative selfadjoint operators; cf. [13, Ch. VI, §2.6]. A simple, but detailed, proof is given in [10, Lemma 3.2, 3.3].

Proposition 2.2. *Let H_1 and H_2 be nonnegative selfadjoint relations. The following statements are equivalent:*

(i) $H_1 \geq H_2$;
(ii) $H_2^{-1} \geq H_1^{-1}$;
(iii) $(H_1 + x)^{-1} \leq (H_2 + x)^{-1}$ *for every* $x > 0$;
(iv) $\operatorname{dom} H_1^{\frac{1}{2}} \subset \operatorname{dom} H_2^{\frac{1}{2}}$ *and* $\|(H_1^{\frac{1}{2}})_s h\| \geq \|(H_2^{\frac{1}{2}})_s h\|$ *for all* $h \in \operatorname{dom} H_1^{\frac{1}{2}}$.

A linear relation H in $\mathfrak{H}$ is said to be *semibounded from below* if there exists $\gamma \in \mathbb{R}$ such that $H - \gamma$ is nonnegative, i.e., $(h', h) \geq \gamma(h, h)$ for all $\{h, h'\} \in H$. The supremum of all such γ is called the *lower bound* of H. Let H_1 and H_2 be selfadjoint relations in $\mathfrak{H}$ which are semibounded from below by γ_1 and γ_2, respectively. Then H_1 and H_2 are said to satisfy the inequality $H_1 \geq H_2$, if

$$0 \leq (H_1 + x)^{-1} \leq (H_2 + x)^{-1} \quad \text{for some } x > -\gamma_j, \quad j = 1, 2. \tag{2.14}$$

Clearly with $y \in \mathbb{R}$, $H_j + y$ is semibounded from below by $\gamma_j + y$ and, in particular, by $y - x$ if $x > -\gamma_j$. Hence, (2.14) is equivalent to

$$0 \leq ((H_1 + y) + (x - y))^{-1} \leq (H_2 + y + (x - y))^{-1},$$

which shows the following basic shifting property: $H_1 \geq H_2$ if and only if $H_1 + y \geq H_2 + y$ for some or, equivalently, for all $y \in \mathbb{R}$. If in the present definition H_2 is the zero operator on $\mathfrak{H}$, the inequality (2.14) means that $0 \leq x(H_1 + x)^{-1} \leq I$, $x > 0$, reflecting nonnegativity of H_1. With obvious modifications Proposition 2.2 remains true for semibounded selfadjoint relations. This implies immediately, for instance, the transitivity property for the ordering: $H_1 \geq H_2$ and $H_2 \geq H_3 \Rightarrow H_1 \geq H_3$.

2.4. **Convergence of selfadjoint relations.** Let H_n be a sequence of linear relations in a Hilbert space $\mathfrak{H}$. The *strong graph limit* of the sequence H_n is the relation which consists of all $\{h, h'\} \in \mathfrak{H} \times \mathfrak{H}$ for which there exists a sequence $\{h_n, h'_n\} \in H_n$ such that $\{h_n, h'_n\} \to \{h, h'\}$ in $\mathfrak{H} \times \mathfrak{H}$. Clearly, if Γ is the strong graph limit of the sequence H_n, then Γ^{-1} is the strong graph limit of the sequence H_n^{-1}. The following result goes back to [14, Theorem VIII.26] for the operator case.

Proposition 2.3. *Let H_n and H_∞ be selfadjoint relations in a Hilbert space $\mathfrak{H}$. Then the sequence H_n converges to H_∞ in the strong resolvent sense*

$$(H_n - \lambda)^{-1} h \to (H_\infty - \lambda)^{-1} h, \quad h \in \mathfrak{H},$$

for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, if and only if H_∞ is the strong graph limit of the sequence H_n.

Proof. ($\Rightarrow$) Assume that H_n converges to H_∞ in the strong resolvent sense for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let Γ be the strong graph limit of the sequence H_n. Let $\{h, h'\} \in H_\infty$, then the sequence

$$\big\{(H_n - \lambda)^{-1}(h' - \lambda h), (I + \lambda(H_n - \lambda)^{-1})(h' - \lambda h)\big\} \in H_n$$

converges to

$$\big\{(H_\infty - \lambda)^{-1}(h' - \lambda h), (I + \lambda(H_\infty - \lambda)^{-1})(h' - \lambda h)\big\} = \{h, h'\}.$$

Hence $\{h, h'\} \in \Gamma$ and consequently $H_\infty \subset \Gamma$. Conversely, let $\{h, h'\} \in \Gamma$ and let $\{h_n, h_n'\} \in H_n$ be such that $\{h_n, h_n'\} \to \{h, h'\}$. Then

$$\begin{aligned}(H_\infty - \lambda)^{-1}&(h_n' - \lambda h_n) - h_n \\ &= (H_\infty - \lambda)^{-1}(h_n' - \lambda h_n) - (H_n - \lambda)^{-1}(h_n' - \lambda h_n) \\ &= \big[(H_\infty - \lambda)^{-1} - (H_n - \lambda)^{-1}\big]\big((h_n' - \lambda h_n) - (h' - \lambda h)\big) \\ &\quad + \big[(H_\infty - \lambda)^{-1} - (H_n - \lambda)^{-1}\big](h' - \lambda h),\end{aligned}$$

and the terms on the righthand side tend to 0 as $n \to \infty$ due to the uniform bound given in (2.4) and the strong resolvent convergence. Hence,

$$(H_\infty - \lambda)^{-1}(h' - \lambda h) = h,$$

so that $\{h, h'\} \in H_\infty$. This shows that $\Gamma \subset H_\infty$.

($\Leftarrow$) Let H_∞ be the strong graph limit of the sequence H_n and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let $h \in \mathfrak{H}$, then, since H_∞ is selfadjoint, there is an element $\{f, f'\} \in H_\infty$ with $f' - \lambda f = h$, so that $(H_\infty - \lambda)^{-1}h = f$. By the assumption there exists a sequence $\{f_n, f_n'\} \in H_n$ converging to $\{f, f'\}$. Therefore,

$$\begin{aligned}(H_n - \lambda)^{-1}h &- (H_\infty - \lambda)^{-1}h \\ &= (H_n - \lambda)^{-1}\big((f' - \lambda f) - (f_n' - \lambda f_n)\big) \\ &\quad + (H_n - \lambda)^{-1}(f_n' - \lambda f_n) - (H_\infty - \lambda)^{-1}(f' - \lambda f) \\ &= (H_n - \lambda)^{-1}\big((f' - \lambda f) - (f_n' - \lambda f_n)\big) + f_n - f.\end{aligned}$$

Here the righthand side tends to 0 as $n \to \infty$ due to the uniform bound given in (2.4). ☐

The following version of Proposition 2.3 for semibounded selfadjoint relations is useful.

Proposition 2.4. *Let H_n and H_∞ be selfadjoint relations in a Hilbert space $\mathfrak{H}$ semibounded from below by some common constant $\mu \in \mathbb{R}$. Then the sequence H_n converges to H_∞ in the strong resolvent sense if and only if*

$$(H_n + y)^{-1}h \to (H_\infty + y)^{-1}h, \quad h \in \mathfrak{H},$$

for some, and hence for all, $y > -\mu$. Furthermore, these statements are equivalent to

$$((H_n - \mu)^{-1} + x)^{-1}h \to ((H_\infty - \mu)^{-1} + x)^{-1}h, \quad h \in \mathfrak{H},$$

for some, and hence for all, $x > 0$.

Proof. A slight modification of the proof of Proposition 2.3 shows that H_∞ is the graph limit of the sequence H_n if and only if $(H_n + y)^{-1}h \to (H_\infty + y)^{-1}h$, $h \in \mathfrak{H}$, for some, and hence for all, $y > -\mu$. Hence, the first part follows from Proposition 2.3. The second part is an immediate consequence of (2.8). □

3. Monotone sequences of semibounded selfadjoint operators and relations

3.1. **Nondecreasing sequences of nonnegative selfadjoint operators and relations.** The situation of a nondecreasing sequence of nonnegative selfadjoint operators or relations is described in the following theorem.

Theorem 3.1. *Let H_n be a nondecreasing sequence of nonnegative selfadjoint operators or relations in a Hilbert space $\mathfrak{H}$. Then there exists a nonnegative selfadjoint relation H_∞ with $H_n \le H_\infty$, such that H_∞ is the limit of the sequence H_n in the strong resolvent sense. Furthermore,*

$$\operatorname{dom} H_\infty^{\frac{1}{2}} = \left\{ h \in \bigcap_{n=1}^{\infty} \operatorname{dom} H_n^{\frac{1}{2}} : \lim_{n\to\infty} \|(H_n^{\frac{1}{2}})_s h\| < \infty \right\} \tag{3.1}$$

and

$$\|(H_\infty^{\frac{1}{2}})_s h\| = \lim_{n\to\infty} \|(H_n^{\frac{1}{2}})_s h\|, \quad h \in \operatorname{dom} H_\infty^{\frac{1}{2}}. \tag{3.2}$$

If, in particular, the sequence H_n in Theorem 3.1 consists of nonnegative selfadjoint operators, then $(H_n^{\frac{1}{2}})_s$ in (3.1) and (3.2) can be replaced by $H_n^{\frac{1}{2}}$ and the theorem is equivalent to Theorem 1.3. If, moreover, $H_n \in \mathbf{B}(\mathfrak{H})$, then Theorem 3.1 contains Theorem 1.2.

The existence of the strong resolvent limit H_∞ in Theorem 3.1 is easily derived from the basic limit Theorem 1.1, while the proof of the formulas (3.1) and (3.2) is based on the following elementary lemma about the interchange of "space and time" limits for monotone sequences of real functions; also a proof of this lemma is included to emphasize the simplicity of the full proof.

Lemma 3.2. *Let f_n be a nondecreasing sequence of nonincreasing functions defined on some open interval (a, b) and let $f_n(a) = \lim_{x\downarrow a} f_n(x)$ be finite. Assume that for all $x \in (a, b)$ the limit $f_\infty(x) = \lim_{n\to\infty} f_n(x)$ is also finite. Then the limit function f_∞ is nonincreasing on (a, b) and at the endpoint a one has the equality*

$$\lim_{x\downarrow a} f_\infty(x) = \lim_{n\to\infty} f_n(a). \tag{3.3}$$

In particular, both limits in (3.3) are finite or infinite simultaneously.

Proof. Clearly, f_∞ is nonincreasing and $f_\infty(x) \ge f_n(x)$ for all $x \in (a, b)$, $n \in \mathbb{N}$. Hence, $\lim_{x\downarrow a} f_\infty(x) \ge \lim_{n\to\infty} f_n(a)$. If $\lim_{x\downarrow a} f_\infty(x) > \lim_{n\to\infty} f_n(a)$, then $f_\infty(x) > \delta + \lim_{n\to\infty} f_n(a) \ge \delta + \lim_{n\to\infty} f_n(x) = \delta + f_\infty(x)$ for some $x \in (a, b)$ and $\delta > 0$; a contradiction which proves (3.3). □

Proof of Theorem 3.1. By assumption $H_n \le H_m$ for $m \ge n$. Therefore,

$$0 \le (H_m + x)^{-1} \le (H_n + x)^{-1}, \quad x > 0. \tag{3.4}$$

It follows from the analog of Theorem 1.1, when applied to the nonincreasing sequence $(H_n + x)^{-1} \ge 0$, that for any fixed $x > 0$ there exists a nonnegative operator

$L_x \in \mathbf{B}(\mathfrak{H})$, such that $(H_n+x)^{-1}h \to L_x h$, $h \in \mathfrak{H}$. Define the closed linear relation H_∞ by

$$H_\infty = \{\, \{L_x h, (I - xL_x)h\} : h \in \mathfrak{H}\,\}, \quad x > 0, \tag{3.5}$$

so that $L_x = (H_\infty + x)^{-1} \geq 0$. Then H_∞ is selfadjoint, $H_\infty + x \geq 0$, and moreover $-x \in \rho(H_\infty)$. Therefore, by Proposition 2.4 the sequence H_n converges to H_∞ in the strong resolvent sense. The inequalities $0 \leq (H_\infty + x)^{-1} \leq (H_n + x)^{-1} \leq 1/x$ mean that $0 \leq H_n \leq H_\infty$. Since H_∞ is also the strong graph limit of the sequence H_n (see Proposition 2.3), it is clear that the definition of H_∞ in (3.5) does not actually depend on $x > 0$.

It remains to prove (3.1) and (3.2). Since H_n converges to H_∞ in the strong resolvent sense it follows from Proposition 2.4 that

$$\big((H_n+I)^{-1}+x)^{-1}h, h\big) \to \big((H_\infty+I)^{-1}+x)^{-1}h, h\big), \quad h \in \mathfrak{H}, \quad x > 0. \tag{3.6}$$

Now, with $h \in \mathfrak{H}$ fixed, define the functions f_n and f_∞ on $(0,\infty)$ by

$$f_n(x) = \big(((H_n+I)^{-1}+x)^{-1}h, h\big), \quad f_\infty(x) = \big(((H_\infty+I)^{-1}+x)^{-1}h, h\big). \tag{3.7}$$

Clearly, each of the functions f_n and f_∞ is continuous and nonincreasing for $x > 0$. Furthermore, the sequence f_n is monotonically nondecreasing with f_∞ as pointwise limit. By applying Proposition 2.1, (2.12), and (2.13) one gets for $n \in \mathbb{N} \cup \{\infty\}$

$$f_n(0) = \lim_{x \downarrow 0} \big(((H_n+I)^{-1}+x)^{-1}h, h\big) = \begin{cases} \|(H_n^{\frac{1}{2}})_s h\|^2 + \|h\|^2, & h \in \operatorname{dom} H_n^{\frac{1}{2}}, \\ \infty, & \text{otherwise}. \end{cases} \tag{3.8}$$

Hence, $h \in \bigcap_{n=1}^\infty \operatorname{dom} H_n^{\frac{1}{2}}$ if and only if $f_n(0)$ in (3.8) is finite for every $n \in \mathbb{N}$. Therefore, by Lemma 3.2, h belongs to the righthand side of (3.1) if and only if

$$f_\infty(0) = \lim_{n\to\infty} f_n(0) \tag{3.9}$$

is finite, which means that $h \in \operatorname{dom} H_\infty^{\frac{1}{2}}$; see (3.8) with $n = \infty$. This proves (3.1) and, finally, (3.2) follows from (3.8) and (3.9). □

3.2. **Some properties of the limit relation H_∞.** The limit H_∞ of a sequence of operators H_n in Theorem 3.1 need not be bounded and it can be multivalued. However, the limit H_∞ may have an operator part $(H_\infty)_s$ which is bounded even if each H_n is unbounded; see Example 3.4 below.

Proposition 3.3. *Let H_n be a nondecreasing sequence of nonnegative selfadjoint operators in a Hilbert space $\mathfrak{H}$ converging to the selfadjoint relation H_∞ as in Theorem 3.1. Then:*

(i) *if for some $M \geq 0$ and every $0 < \varepsilon < 1$ there exists $n_\varepsilon \in \mathbb{N}$, such that $(M+\varepsilon, M+\varepsilon^{-1}) \subset \rho(H_n)$ for all $n \geq n_\varepsilon$, then $\|(H_\infty)_s\| \leq M$ holds;*

(ii) *if the operators H_n are unbounded and the operator part $(H_\infty)_s$ of H_∞ is bounded, then* $\operatorname{mul} H_\infty$ *is infinite-dimensional.*

Proof. (i) Let $M \geq 0$ satisfy the given condition. As H_∞ is a nonnegative relation it suffices to show that $(M,\infty) \subset \rho(H_\infty)$ for some $M \geq 0$, since then $\sigma((H_\infty)_s) \subset [0, M]$ and $\|(H_\infty)_s\| \leq M$. Suppose that $(M,\infty) \not\subset \rho(H_\infty)$. Then choose some $\mu \in (M,\infty) \cap \sigma(H_\infty)$. It follows from [13, Theorem VIII.1.14] that every open interval around μ contains a point of $\sigma(H_n)$ for sufficiently large n. Then there exists

$0 < \varepsilon < 1$, such that $\mu \in (M+\varepsilon, M+\varepsilon^{-1})$ and hence $(M+\varepsilon, M+\varepsilon^{-1}) \not\subset \rho(H_n)$ for all sufficiently large n, a contradiction.

(ii) Let $(H_\infty)_s$ be bounded. Then $H_n \le H_\infty$ and Proposition 2.2 imply that

$$\overline{\operatorname{dom}}\, H_\infty = \operatorname{dom} H_\infty = \operatorname{dom} H_\infty^{\frac{1}{2}} \subset \operatorname{dom} H_n^{\frac{1}{2}}. \tag{3.10}$$

If, moreover, $\operatorname{mul} H_\infty$ is finite-dimensional, then the set on the lefthand side of (3.10) has finite codimension in $\mathfrak{H}$, due to (2.2). As the set on the righthand side of (3.10) is dense in $\mathfrak{H}$ this implies that $\operatorname{dom} H_n^{\frac{1}{2}} = \mathfrak{H}$ and hence $H_n^{\frac{1}{2}}$ and H_n are bounded, a contradiction. □

The converse assertion in Proposition 3.3 (i) is in general not true; an extreme situation appears in the next example, which also illustrates Proposition 3.3 (ii).

Example 3.4. *Let $\Delta \subset \mathbb{R}$ be an interval and let $V_n : \Delta \to \mathbb{R}$ be a nondecreasing sequence of measurable nonnegative functions. Then the multiplication operators*

$$H_n h = V_n h, \quad \operatorname{dom} H_n = \left\{ h \in L^2(\Delta) : V_n h \in L^2(\Delta) \right\},$$

form a nondecreasing sequence of nonnegative selfadjoint operators in $L^2(\Delta)$. Let

$$\delta := \left\{ t \in \Delta : \lim_{n\to\infty} V_n(t) < \infty \right\}$$

and denote the pointwise limit of V_n on δ by V_δ. Let H_δ be the corresponding multiplication operator in $L^2(\delta)$, i.e.,

$$H_\delta h = V_\delta h, \quad \operatorname{dom} H_\delta = \left\{ h \in L^2(\delta) : V_\delta h \in L^2(\delta) \right\}.$$

Then the sequence H_n converges in the strong resolvent sense to the nonnegative selfadjoint relation H_∞ given by

$$H_\infty = H_\delta \oplus \left\{ \{0, h\} : h \in L^2(\delta^c) \right\}, \quad \delta^c := \Delta \backslash \delta,$$

with respect to the space decomposition $L^2(\Delta) = L^2(\delta) \oplus L^2(\delta^c)$. In fact,

$$(H_\infty + x)^{-1} = (H_\delta + x)^{-1} \oplus \left\{ \{h, 0\} : h \in L^2(\delta^c) \right\}, \quad x > 0,$$

which implies

$$\begin{aligned} &\|(H_\infty + x)^{-1}h - (H_n + x)^{-1}h\|^2 \\ &\quad = \int_\delta \left| \left((V_\delta(t) + x)^{-1} - (V_n(t) + x)^{-1} \right) h(t) \right|^2 dt + \int_{\delta^c} \left| (V_n(t) + x)^{-1} h(t) \right|^2 dt \end{aligned}$$

for all $h \in L^2(\Delta)$. Since $\lim_{n\to\infty} V_n(t) = V_\delta(t)$, $t \in \delta$, the first integral on the righthand side tends to 0 for $n \to \infty$ and, since $\lim_{n\to\infty}(V_n(t)+x)^{-1} = 0$, $t \in \delta^c$, also the second integral tends to zero by the monotone convergence theorem.

Now consider the multiplication operators H_n on $L^2(0,\infty)$ determined by

$$V_n(t) = nt, \quad t \in [0, \infty), \quad n \in \mathbb{N}.$$

Here $\lim_{n\to\infty} V_n(t) = \infty$ for all $t > 0$, and hence $\delta = \{0\}$ and $L^2(\delta^c) = L^2(0,\infty)$. Consequently,

$$\sigma(H_n) = [0, \infty) \text{ for all } n \in \mathbb{N}, \quad \sigma((H_\infty)_s) = \emptyset,$$

i.e., the constant spectrum $\sigma(H_n) = [0,\infty)$ disappears in the limit from the finite complex plane and formally ∞ is the only spectral point of H_∞. If H is an arbitrary bounded (or unbounded) nonnegative selfadjoint operator on a Hilbert space $\mathfrak{H}$, then the sequence $H \oplus H_n$ in $\mathfrak{H} \oplus L^2(0,\infty)$ converses in the strong resolvent *sense to* $H_\infty = H \oplus (\{0\} \times L^2(0,\infty))$*, where* $H = (H_\infty)_s$*. Hence, a sequence of unbounded selfadjoint operators may converge to a selfadjoint relation with a bounded (or unbounded) operator part.*

Assertions (i) and (ii) in Proposition 3.3 also hold if the sequence consists of selfadjoint relations. In fact, if H_n is a nondecreasing sequence of nonnegative selfadjoint relations with unbounded operators parts, H_n converges to H_∞ and the operator part of H_∞ is bounded, then $\mathrm{mul}\, H_\infty \ominus \mathrm{mul}\, H_n$ is infinite-dimensional for every n.

3.3. **Nondecreasing sequences of semibounded selfadjoint operators or relations.** The next theorem is an immediate extension of Theorem 3.1 to the semibounded situation; for this purpose note that with a lower bound γ,

$$((H_\infty - \gamma)^{\frac{1}{2}})_s = ((H_\infty - \gamma)_s)^{\frac{1}{2}} = ((H_\infty)_s - \gamma)^{\frac{1}{2}}.$$

Theorem 3.5. *Let* H_n *be a nondecreasing sequence of selfadjoint relations bounded from below by* γ *in a Hilbert space* $\mathfrak{H}$*. Then there exists a selfadjoint relation* H_∞ *bounded from below by* $\gamma \le H_n \le H_\infty$*, such that* H_∞ *is the limit of the sequence* H_n *in the strong resolvent sense. Furthermore,*

$$\mathrm{dom}\,(H_\infty - \gamma)^{\frac{1}{2}} = \left\{ h \in \bigcap_{n=1}^{\infty} \mathrm{dom}\,(H_n - \gamma)^{\frac{1}{2}} : \lim_{n\to\infty} \|((H_n)_s - \gamma)^{\frac{1}{2}} h\| < \infty \right\} \tag{3.11}$$

and

$$\|((H_\infty)_s - \gamma)^{\frac{1}{2}} h\|^2 = \lim_{n\to\infty} \|((H_n)_s - \gamma)^{\frac{1}{2}} h\|^2, \quad h \in \mathrm{dom}\,(H_\infty - \gamma)^{\frac{1}{2}}. \tag{3.12}$$

If H_n is a nondecreasing sequence of bounded selfadjoint operators bounded from below by γ in a Hilbert space $\mathfrak{H}$, then (3.11) and (3.12) are simplified as follows:

$$\mathrm{dom}\,(H_\infty - \gamma)^{\frac{1}{2}} = \{\, h \in \mathfrak{H} : \lim_{n\to\infty} (H_n h, h) < \infty \,\}$$

and

$$\|((H_\infty)_s - \gamma)^{\frac{1}{2}} h\|^2 = \lim_{n\to\infty} (H_n h, h) - \gamma \|h\|^2, \quad h \in \mathrm{dom}\,(H_\infty - \gamma)^{\frac{1}{2}}.$$

Moreover, if the operator part $(H_\infty)_s$ is bounded, then

$$((H_\infty)_s h, h) = \lim_{n\to\infty} (H_n h, h), \quad h \in \mathrm{dom}\, H_\infty.$$

Next a finite-dimensional version of Theorem 3.5 is given. The last statement of the following result also contains the converse of Proposition 3.3 (i). It holds in the finite-dimensional case, since strong convergence of operators is in that case equivalent to convergence in the operator norm.

Corollary 3.6. *If* H_n *is a nondecreasing sequence of symmetric matrices in a finite-dimensional Hilbert space* $\mathfrak{H}$ *and if* γ *is the smallest eigenvalue of* H_1*, then*

$$(H_n + x)^{-1} \to (H_\infty + x)^{-1}, \qquad x > -\gamma.$$

Furthermore, $\mathrm{dom}\, H_\infty = \{\, h \in \mathfrak{H} : \lim_{n\to\infty} (H_n h, h) < \infty \,\}$ *and*

$$((H_\infty)_s h, h) = \lim_{n\to\infty} (H_n h, h), \quad h \in \mathrm{dom}\, H_\infty.$$

Moreover, for every $0 < \varepsilon < 1$ *there exists* $n_\varepsilon \in \mathbb{N}$ *such that*

$$(\|(H_\infty)_s\| + \varepsilon, \|(H_\infty)_s\| + \varepsilon^{-1}) \subset \rho(H_n), \quad n \ge n_\varepsilon.$$

For an application of Corollary 3.6 in the spectral theory of singular canonical differential equations see [4].

3.4. **Nonincreasing sequences of nonnegative selfadjoint operators or relations.** Since the strong resolvent convergence of the sequence H_n is equivalent to the strong resolvent convergence of the sequence of inverses H_n^{-1} (see Proposition 2.4), Theorem 3.1 can be translated into a result for nonincreasing sequences of nonnegative selfadjoint relations, giving a description in terms of ranges instead of domains.

Theorem 3.7. *Let H_n be a nonincreasing sequence of nonnegative selfadjoint operators or relations in a Hilbert space $\mathfrak{H}$. Then there exists a nonnegative selfadjoint relation H_∞ with $H_\infty \le H_n$, such that H_∞ is the limit of the sequence H_n in the strong resolvent sense. Furthermore,*

$$\operatorname{ran} H_\infty^{\frac{1}{2}} = \left\{ h \in \bigcap_{n=1}^{\infty} \operatorname{ran} H_n^{\frac{1}{2}} : \lim_{n\to\infty} \|(H_n^{-\frac{1}{2}})_s h\| < \infty \right\} \tag{3.13}$$

and

$$\|(H_\infty^{-\frac{1}{2}})_s h\| = \lim_{n\to\infty} \|(H_n^{-\frac{1}{2}})_s h\|, \quad h \in \operatorname{ran} H_\infty^{\frac{1}{2}}. \tag{3.14}$$

Proof. The sequence H_n^{-1} is nondecreasing, so by Theorem 3.1 there exists a nonnegative selfadjoint relation, say, H_∞^{-1}, such that H_∞^{-1} is the limit of the sequence H_n^{-1} in the strong resolvent sense and $H_n^{-1} \le H_\infty^{-1}$. Then $H_\infty \le H_n$ and H_∞ is the strong resolvent limit of the sequence H_n by Proposition 2.4. The rest of the statements is a direct translation of similar statements in Theorem 3.1. □

4. Monotone sequences of semibounded closed forms

4.1. **Semibounded forms.** Let $\mathfrak{t} = \mathfrak{t}[\cdot,\cdot]$ be a symmetric form in the Hilbert space $\mathfrak{H}$ with domain $\operatorname{dom}\mathfrak{t}$. The notation $\mathfrak{t}[h]$ will be used to denote $\mathfrak{t}[h,h]$, $h \in \operatorname{dom}\mathfrak{t}$. The symmetric form $\mathfrak{t}$ is said to be *semibounded from below*, in short *semibounded*, if there exists $\gamma \in \mathbb{R}$ such that $\mathfrak{t}[h] \ge \gamma\|h\|^2$ for all $h \in \operatorname{dom}\mathfrak{t}$; cf. [13]. The inclusion $\mathfrak{t}_1 \subset \mathfrak{t}_2$ for semibounded forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ is defined by

$$\operatorname{dom}\mathfrak{t}_1 \subset \operatorname{dom}\mathfrak{t}_2, \quad \mathfrak{t}_1[h] = \mathfrak{t}_2[h], \quad h \in \operatorname{dom}\mathfrak{t}_1.$$

The semibounded form $\mathfrak{t}$ is *closed* if

$$h_n \to h, \quad \mathfrak{t}[h_n - h_m] \to 0, \quad h_n \in \operatorname{dom}\mathfrak{t}, \quad h \in \mathfrak{H}, \quad m,n \to \infty, \tag{4.1}$$

imply that $h \in \operatorname{dom}\mathfrak{t}$ and $\mathfrak{t}[h_n - h] \to 0$. The semibounded form $\mathfrak{t}$ is *closable* if it has a closed extension; in this case the closure of $\mathfrak{t}$ is the smallest closed extension of $\mathfrak{t}$. The inequality $\mathfrak{t}_1 \ge \mathfrak{t}_2$ for semibounded forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ is defined by

$$\operatorname{dom}\mathfrak{t}_1 \subset \operatorname{dom}\mathfrak{t}_2, \quad \mathfrak{t}_1[h] \ge \mathfrak{t}_2[h], \quad h \in \operatorname{dom}\mathfrak{t}_1. \tag{4.2}$$

In particular, $\mathfrak{t}_1 \subset \mathfrak{t}_2$ implies $\mathfrak{t}_1 \ge \mathfrak{t}_2$. If the forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closable, the inequality $\mathfrak{t}_1 \ge \mathfrak{t}_2$ is preserved by their closures.

There is a one-to-one correspondence between all closed semibounded (nonnegative) forms $\mathfrak{t}$ in $\mathfrak{H}$ and all semibounded (nonnegative, respectively) selfadjoint relations H in $\mathfrak{H}$ via $\operatorname{dom} H \subset \operatorname{dom}\mathfrak{t}$ and

$$\mathfrak{t}[h,k] = (H_s h, k), \quad h \in \operatorname{dom} H, \quad k \in \operatorname{dom}\mathfrak{t}. \tag{4.3}$$

This one-to-one correspondence can also be expressed as follows

$$\mathfrak{t}[h,k] = (h',k), \quad \{h,h'\} \in H, \quad k \in \operatorname{dom} \mathfrak{t}, \tag{4.4}$$

since $(h',k) = (h',Pk) = (H_s h, k)$, where P is the orthogonal projection from $\mathfrak{H}$ onto $\overline{\operatorname{dom}}\, \mathfrak{t} = (\operatorname{mul} H)^\perp$. Let the closed form $\mathfrak{t}$ be bounded from below by γ and let the semibounded selfadjoint relation H be connected to $\mathfrak{t}$ via (4.3) or (4.4), then it follows from (2.12), (2.13) that $\operatorname{dom} \mathfrak{t} = \operatorname{dom} (H_s - \gamma)^{\frac{1}{2}}$ and

$$\mathfrak{t}[h,k] = ((H_s-\gamma)^{\frac{1}{2}} h, (H_s-\gamma)^{\frac{1}{2}} k) + \gamma(h,k), \quad h,k \in \operatorname{dom} \mathfrak{t}. \tag{4.5}$$

The formulas (4.4), (4.5) are analogs of Kato's representation theorems for, in general, nondensely defined closed semibounded forms; cf. [10]. In the case of nonnegative relations and forms the following result, which is a generalization of [13, Theorem VI.2.21], can be found in [10, Theorem 4.3]. It is immediate to obtain the result in the present context.

Theorem 4.1. *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be closed semibounded forms and let H_1 and H_2 be the corresponding semibounded selfadjoint relations. Then*

$$\mathfrak{t}_1 \geq \mathfrak{t}_2 \quad \textit{if and only if} \quad H_1 \geq H_2. \tag{4.6}$$

4.2. **Nondecreasing sequences of semibounded closed forms.** Theorem 4.1 makes it possible to translate Theorem 3.5 to the context of a nondecreasing sequence of semibounded closed forms.

Theorem 4.2. *Let $\mathfrak{t}_n$ be a nondecreasing sequence of closed forms bounded from below by γ in a Hilbert space $\mathfrak{H}$. Then there exists a closed form $\mathfrak{t}_\infty$ bounded from below by γ, such that*

$$\operatorname{dom} \mathfrak{t}_\infty = \left\{ h \in \bigcap_{n=1}^{\infty} \operatorname{dom} \mathfrak{t}_n : \lim_{n\to\infty} \mathfrak{t}_n[h] < \infty \right\} \tag{4.7}$$

and

$$\mathfrak{t}_\infty[h,k] = \lim_{n\to\infty} \mathfrak{t}_n[h,k], \quad h,k \in \operatorname{dom} \mathfrak{t}_\infty. \tag{4.8}$$

Moreover, the representing relations H_n of the forms $\mathfrak{t}_n$ converge in the strong resolvent sense to the representing relation H_∞ of the form $\mathfrak{t}_\infty$.

Proof. Let H_n and H_∞ be the semibounded selfadjoint relations in $\mathfrak{H}$ associated to $\mathfrak{t}_n$ and $\mathfrak{t}_\infty$, respectively; see (4.4), (4.5). Then by Theorem 4.1 H_n defines a nondecreasing sequence of selfadjoint relations bounded from below by γ. By Theorem 3.5 the strong resolvent limit of the sequence H_n is a semibounded selfadjoint relation H_∞ such that $\gamma \leq H_n \leq H_\infty$. It is clear from (4.5) and the formulas (3.11) and (3.12) in Theorem 3.5 (by polarization) that the limit form $\mathfrak{t}_\infty$ defined in (4.7), (4.8) is the closed semibounded form corresponding to the semibounded selfadjoint relation H_∞. □

Theorem 3.5 and Theorem 4.2 show that for nondecreasing sequences of semibounded selfadjoint relations H_n the strong resolvent convergence is equivalent to the pointwise convergence of the associated closed forms $\mathfrak{t}_n$. It is clear from Example 3.4 which involves a multivalued limit relation H_∞ that the limit form $\mathfrak{t}_\infty$ in (4.8) need not be densely defined, even if $\mathfrak{t}_n$ is a sequence of densely defined or bounded everywhere defined forms. This phenomenon can appear also in concrete applications, like boundary value problems for differential operators; see e.g. Kreĭn-Feller differential operators treated below in Section 5.2.

4.3. **Nonincreasing sequences of nonnegative closed forms.** Theorem 3.7 can be translated directly into a statement for nonincreasing sequences of nonnegative closed forms, yielding a description of the range $\operatorname{ran} H_\infty^{\frac{1}{2}}$. Here also descriptions of $\operatorname{dom} H_\infty^{\frac{1}{2}}$ and of the form $\mathfrak{t}_\infty$ associated to the limit H_∞ are given in the case of nonincreasing sequences H_n. Recall that any form $\mathfrak{t}$ has a regular part $\mathfrak{t}_{\rm reg}$, which is the largest closable form majorized by $\mathfrak{t}$; i.e., if $\tilde{\mathfrak{t}}$ is a closable form with $\tilde{\mathfrak{t}} \leq \mathfrak{t}$, then $\tilde{\mathfrak{t}} \leq \mathfrak{t}_{\rm reg}$. The regular part of a form has a monotonicity property: if $\mathfrak{s} \leq \mathfrak{t}$, then $\mathfrak{s}_{\rm reg} \leq \mathfrak{t}_{\rm reg}$; cf. [17], see also [11]. The following result goes back to Simon [17], see also [14]. Again, the present version allows nondensely defined forms.

Theorem 4.3. *Let $\mathfrak{t}_n$ be a nonincreasing sequence of closed nonnegative forms in a Hilbert space $\mathfrak{H}$ with corresponding nonnegative selfadjoint relations H_n and let $\mathfrak{t}_\infty$ be the closed nonnegative form corresponding to the strong resolvent limit H_∞ of the sequence H_n. Moreover, let the form $\mathfrak{t}$ be defined by*

$$\operatorname{dom} \mathfrak{t} = \bigcup_{n=1}^{\infty} \operatorname{dom} \mathfrak{t}_n, \quad \mathfrak{t}[h,k] = \lim_{n\to\infty} \mathfrak{t}_n[h,k], \quad h,k \in \operatorname{dom} \mathfrak{t}.$$

Then the form $\mathfrak{t}$ is related to the form $\mathfrak{t}_\infty$ via

$$\mathfrak{t}_\infty = \operatorname{clos} \mathfrak{t}_{\rm reg}. \tag{4.9}$$

In particular, the form $\mathfrak{t}$ is not necessarily closable: $\mathfrak{t}$ is closable if and only if $\mathfrak{t} \subset \mathfrak{t}_\infty$, and $\mathfrak{t}$ is closed if and only if $\mathfrak{t} = \mathfrak{t}_\infty$.

Proof. It follows from Theorem 3.7 that $\mathfrak{t}_\infty \leq \mathfrak{t}_n$. Now the definition of the form $\mathfrak{t}$ implies that $\mathfrak{t}_\infty \leq \mathfrak{t}$, since $\operatorname{dom} \mathfrak{t}_n \subset \operatorname{dom} \mathfrak{t}_\infty$ for all $n \in \mathbb{N}$ and $\mathfrak{t}_\infty[h,h] \leq \inf \mathfrak{t}_n[h,h] = \lim_{n\to\infty} \mathfrak{t}_n[h]$ for all $h \in \operatorname{dom} \mathfrak{t}_\infty$. The form $\mathfrak{t}_\infty$ is closed, so the inequality $\mathfrak{t}_\infty \leq \mathfrak{t}$ leads to $\mathfrak{t}_\infty \leq \mathfrak{t}_{\rm reg}$ (due to the monotonicity property of the regular part) and, since inequalities are preserved by closures, this yields

$$\mathfrak{t}_\infty \leq \operatorname{clos} \mathfrak{t}_{\rm reg}.$$

To obtain the reverse inequality, observe that $\mathfrak{t} \leq \mathfrak{t}_n$. As above this implies $\mathfrak{t}_{\rm reg} \leq \mathfrak{t}_n$ and $\operatorname{clos} \mathfrak{t}_{\rm reg} \leq \mathfrak{t}_n$. Taking limits leads to

$$\operatorname{clos} \mathfrak{t}_{\rm reg} \leq \mathfrak{t}_\infty.$$

Thus (4.9) is shown to hold. If $\mathfrak{t} \subset \mathfrak{t}_\infty$, then, clearly, $\mathfrak{t}$ is closable; if $\mathfrak{t}$ is closable, then $\mathfrak{t} = \mathfrak{t}_{\rm reg} \subset \operatorname{clos} \mathfrak{t}_{\rm reg} = \mathfrak{t}_\infty$. Likewise, if $\mathfrak{t} = \mathfrak{t}_\infty$, then $\mathfrak{t}$ is closed; if $\mathfrak{t}$ is closed, then $\mathfrak{t} = \operatorname{clos} \mathfrak{t}_{\rm reg} = \mathfrak{t}_\infty$. □

Theorem 4.3 shows that for nonincreasing sequences of nonnegative selfadjoint relations H_n the strong resolvent convergence and the pointwise convergence of the associated closed forms $\mathfrak{t}_n$ may yield, in general, different limit forms $\mathfrak{t}_\infty$ and $\mathfrak{t}$; see (4.9). This is illustrated by the following two examples.

Example 4.4. *In the Hilbert space $\mathfrak{H} = L^2[0,1]$ the operators H_n defined by*

$$H_n = -D^2, \quad \operatorname{dom} H_n = \left\{ h \in W_2^2[0,1] : \, Dh(0) = \frac{1}{n} h(0), \, h(1) = 0 \right\}$$

are selfadjoint and nonnegative. Here $W_2^k[0,1]$ denotes the usual Sobolev space of k^{th} order. The corresponding nonnegative closed forms $\mathfrak{t}_n$ are given by

$$\mathfrak{t}_n[h] = \int_0^1 |Dh(t)|^2 \, dt + \frac{1}{n} |h(0)|^2, \quad h \in \operatorname{dom} \mathfrak{t}_n,$$

$$\operatorname{dom} \mathfrak{t}_n = \{ h \in W_2^1[0,1] : h(1) = 0 \}.$$

The sequence of operators H_n *or, equivalently, the sequence of forms* $\mathfrak{t}_n$ *is nonincreasing. By Theorem 3.7 there is a nonnegative selfadjoint limit* H_∞ *and it can be identified as the selfadjoint realization corresponding to the boundary conditions*

$$Dh(0) = 0, \quad h(1) = 0.$$

Therefore, the corresponding form $\mathfrak{t}_\infty$ *is given by*

$$\mathfrak{t}_\infty[h] = \int_0^1 |Dh(t)|^2 \, dt, \quad \operatorname{dom} \mathfrak{t}_\infty = \{ h \in W_2^1[0,1] : h(1) = 0 \}.$$

Since all H_n *are uniformly bounded away from* 0*, Theorem 3.7 shows that* $\operatorname{ran} H_\infty = \mathfrak{H}$*. Of course, this is also clear by direct inspection.*

According to Theorem 4.3 the nonincreasing sequence of nonnegative closed forms $\mathfrak{t}_n$ *gives rise to the following limit* $\mathfrak{t}$*:*

$$\mathfrak{t}[h] = \int_0^1 |Dh(t)|^2 \, dt, \quad \operatorname{dom} \mathfrak{t} = \{ h \in W_2^1[0,1] : h(1) = 0 \}.$$

Therefore, $\mathfrak{t}$ *is a closed form and* $\mathfrak{t} = \mathfrak{t}_\infty$ *by Theorem 4.3, or by direct comparison.*

Example 4.5. *Consider a slight modification of the previous differential operators; cf.* [14, Ch. VI, Example 3.10]*. Let* H_n *be the nonnegative selfadjoint operator in* $\mathfrak{H} = L^2[0,1]$ *defined by*

$$H_n = -\frac{1}{n} D^2, \quad \operatorname{dom} H_n = \left\{ h \in W_2^2[0,1] : \, Dh(0) = nh(0), \, h(1) = 0 \right\}.$$

The corresponding closed form $\mathfrak{t}_n$ *is given by*

$$\mathfrak{t}_n[h] = \frac{1}{n} \int_0^1 |Dh(t)|^2 \, dt + |h(0)|^2, \quad h \in \operatorname{dom} \mathfrak{t}_n,$$

$$\operatorname{dom} \mathfrak{t}_n = \{ h \in W_2^1[0,1] : h(1) = 0 \}.$$

The sequence H_n *is nonincreasing and by Theorem 3.7 it has a nonnegative selfadjoint limit* H_∞*. In order to determine this limit, observe that* $\operatorname{ran} H_n = \mathfrak{H}$ *and that* $(1/n)H_n^{-1}$ *converges strongly to the resolvent* R *of the selfadjoint operator* $-D^2$ *in* $L^2[0,1]$ *with the boundary conditions* $h(0) = h(1) = 0$*. According to Theorem 3.7*

$$\operatorname{ran} H_\infty^{\frac{1}{2}} = \left\{ h \in \mathfrak{H} : \lim_{n \to \infty} \|H_n^{-\frac{1}{2}} h\| = \lim_{n \to \infty} (H_n^{-1} h, h)^{\frac{1}{2}} < \infty \right\} = \{0\},$$

since $(Rh, h) > 0$ *for any nontrivial* $h \in \mathfrak{H}$*. Hence* $\operatorname{ran} H_\infty \subset \operatorname{ran} H_\infty^{\frac{1}{2}} = \{0\}$*, so that* $H_\infty = \mathfrak{H} \times \{0\}$ *(cf.* (2.7)*). Therefore,* $\mathfrak{t}_\infty$ *is the zero form on* $\operatorname{dom} \mathfrak{t}_\infty = \mathfrak{H}$*.*

As described in Theorem 4.3 the nonincreasing sequence of nonnegative closed forms $\mathfrak{t}_n$ *gives rise to the following limit form* $\mathfrak{t}$*:*

$$\mathfrak{t}[h] = |h(0)|^2, \quad \operatorname{dom} \mathfrak{t} = \{ h \in W_2^1[0,1] : h(1) = 0 \}.$$

The form $\mathfrak{t}$ *is not closable; in fact,* $\mathfrak{t}$ *is singular. In other words, if* $\mathfrak{s}$ *is a nonnegative form which is majorized by* $\mathfrak{t}$ *and by the inner product in* $\mathfrak{H} = L^2[0,1]$*, then* $\mathfrak{s} = 0$*, see, e.g.* [11]*. To see this, let* $h \in \operatorname{dom} \mathfrak{t}$*, and decompose* $h = h_1 + h_2$*, where* $h_1, h_2 \in \operatorname{dom} \mathfrak{t}$ *with* $h_1(0) = 0$ *and* $h_2(0) = h(0)$*. By Cauchy-Schwarz's inequality* $\mathfrak{s}[h] = \mathfrak{s}[h_2] \le \|h_2\|^2$*, which can be made arbitrarily small. This shows that* $\mathfrak{s} = 0$*. Since* $\mathfrak{t}$ *is singular,* $\mathfrak{t}_{\rm reg} = 0$ *(on* $\operatorname{dom} \mathfrak{t}$*) and, therefore, by Theorem 4.3,* $\mathfrak{t}_\infty = \operatorname{clos} \mathfrak{t}_{\rm reg} = 0$ *with* $\operatorname{dom} \mathfrak{t}_\infty = \mathfrak{H}$*.*

5. Applications to differential operators, singular perturbations, and nonnegative extensions

5.1. **Sturm-Liouville operators with increasing potentials.** Let $(a,b) \subset \mathbb{R}$ be a bounded interval and let $V_n : (a,b) \to \mathbb{R}$ be a nondecreasing sequence of nonnegative bounded continuous functions. It will be assumed that the pointwise limit of the sequence V_n is finite on some interval $(\alpha,\beta) \subset (a,b)$ and infinite on the intervals $(a,\alpha]$ and $[\beta,b)$, i.e.,

$$(\alpha,\beta) = \left\{ t \in (a,b) : \lim_{n\to\infty} V_n(t) \in \mathbb{R} \right\}.$$

Denote by $V : (\alpha,\beta) \to \mathbb{R}$ the pointwise limit of V_n on (α,β). Note that V is bounded on any closed subset of (α,β) and hence $V \in L^1_{\mathrm{loc}}(\alpha,\beta)$.

In [13, Examples VI.1.36 and VI.2.16] it is shown that the nonnegative forms

$$\begin{aligned}
\mathfrak{t}_n[h,k] &= \int_a^b \left(Dh(t)\overline{Dk(t)} + V_n(t)h(t)\overline{k(t)} \right) dt, \\
\operatorname{dom} \mathfrak{t}_n &= \left\{ h \in L^2(a,b) : h \in AC(a,b),\, Dh \in L^2(a,b),\, h(a) = h(b) = 0 \right\},
\end{aligned}$$

are closed and densely defined. Moreover, the associated nonnegative selfadjoint operators H_n are given by

$$\begin{aligned}
H_n h &= -D^2 h + V_n h, \\
\operatorname{dom} H_n &= \left\{ h \in L^2(a,b) : h, Dh \in AC(a,b),\, D^2 h \in L^2(a,b),\, h(a) = h(b) = 0 \right\}.
\end{aligned}$$

Note that the domains $\operatorname{dom} \mathfrak{t}_n$ and $\operatorname{dom} H_n$ do not depend on $n \in \mathbb{N}$.

The sequence $\mathfrak{t}_n$ is nondecreasing and therefore, by Theorem 4.2, there exists a limit form $\mathfrak{t}_\infty$ which is closed, nonnegative, such that

$$\mathfrak{t}_\infty[h] = \lim_{n\to\infty} \mathfrak{t}_n[h], \quad \operatorname{dom} \mathfrak{t}_\infty = \left\{ h \in \operatorname{dom} \mathfrak{t}_1 : \lim_{n\to\infty} \mathfrak{t}_n[h] < \infty \right\}.$$

Observe that by the monotone convergence theorem

$$\lim_{n\to\infty} \int_a^b V_n(t)|h(t)|^2\,dt = \int_a^b \lim_{n\to\infty} V_n(t)|h(t)|^2\,dt, \quad h \in \operatorname{dom} \mathfrak{t}_n.$$

This limit is finite if and only if h vanishes on $(a,\alpha] \cup [\beta,b) =: (\alpha,\beta)^c$ and

$$\int_\alpha^\beta V(t)|h(t)|^2\,dt < \infty.$$

Therefore the domain of $\mathfrak{t}_\infty$ is given by

$$\operatorname{dom} \mathfrak{t}_\infty = \left\{ h \in L^2(a,b) : \begin{array}{c} h \in AC(a,b),\, Dh \in L^2(a,b) \\ h(a) = h(b) = 0,\, h|_{(\alpha,\beta)^c} = 0, \end{array} \int_\alpha^\beta V|h|^2\,dt < \infty \right\}.$$

Since $h \in \operatorname{dom} \mathfrak{t}_\infty$ vanishes on $(\alpha,\beta)^c$ it follows that $Dh|_{(a,\alpha)\cup(\beta,b)} = 0$. Hence,

$$\mathfrak{t}_\infty[h,k] = \lim_{n\to\infty} \mathfrak{t}_n[h,k] = \int_\alpha^\beta \left(Dh(t)\overline{Dk(t)} + V(t)h(t)\overline{k(t)} \right) dt, \quad h,k \in \operatorname{dom} \mathfrak{t}_\infty.$$

Furthermore, $\operatorname{dom} \mathfrak{t}_\infty$ is dense in $L^2(\alpha,\beta)$ and the same arguments as in [13, Ch. VI, §4.1] show that the nonnegative selfadjoint operator T_∞ associated with the restriction of $\mathfrak{t}_\infty$ to $L^2(\alpha,\beta)$ is

$$T_\infty h = -D^2 h + Vh, \quad h \in \operatorname{dom} T_\infty,$$

$$\operatorname{dom} T_\infty = \left\{ h \in L^2(\alpha,\beta) : \begin{array}{c} h, Dh \in AC(\alpha,\beta),\, h(\alpha)=h(\beta)=0 \\ Dh \in L^2(\alpha,\beta), -D^2h+Vh \in L^2(\alpha,\beta), \end{array} \int_\alpha^\beta V|h|^2\,dt < \infty \right\}.$$

Hence, with respect to the decomposition $L^2(a,b) = L^2(\alpha,\beta) \oplus L^2((a,\alpha)\cup(\beta,b))$ the nonnegative selfadjoint relation H_∞ associated with $\mathfrak{t}_\infty$ has the representation

$$H_\infty = T_\infty \,\oplus\, \big\{\{0,h\} : h \in L^2((a,\alpha)\cup(\beta,b))\big\},$$

and, in particular, $T_\infty = (H_\infty)_s$. It is emphasized that the selfadjoint operators H_n converge in the strong resolvent sense to H_∞, not to its operator part T_∞; cf. [17, Theorem 5.1].

5.2. **Kreĭn-Feller operators.** A sequence of Kreĭn-Feller differential operators is considered which gives rise to a nondensely defined limit form; see [12].

Let $m : [a,b] \to \mathbb{R}$ be a left-continuous strictly increasing function and assume that

$$m(a) < \lim_{t\downarrow a} m(t). \tag{5.1}$$

Let $L^2_m[a,b]$ be the space of all (equivalence classes of) functions which are measurable and square-integrable with respect to the Lebesgue-Stieltjes measure dm induced by the function m. The space $L^2_m[a,b]$ equipped with the scalar product $(h,k) = \int_{[a,b]} h\overline{k}\,dm$, $h,k \in L^2_m[a,b]$, is a Hilbert space. Note that due to (5.1) the characteristic function $\mathbf{1}_{\{a\}}$ of the set $\{a\}$ spans a one-dimensional subspace in $L^2_m[a,b]$, i.e., $0 \neq \mathbf{1}_{\{a\}} \in L^2_m[a,b]$. By means of the function m define a nondecreasing sequence of nonnegative forms by

$$\mathfrak{t}_n[h,k] = \int_{[a,b]} (Dh)(x)\overline{(Dk)(x)}\,dx \,+\, nh(a)\overline{k(a)}, \quad h,k \in \operatorname{dom}\mathfrak{t}_n,$$

$$\operatorname{dom}\mathfrak{t}_n = \big\{\, h \in L^2_m[a,b] :\, h \in AC[a,b],\, Dh \in L^2[a,b],\, h(b)=0 \,\big\}.$$

It can be shown that the forms $\mathfrak{t}_n$ are closed and densely defined in $L^2_m[a,b]$. By Theorem 4.2 there exists a limit form $\mathfrak{t}_\infty$ which is closed, nonnegative, and given by

$$\mathfrak{t}_\infty[h,k] = \int_{[a,b]} (Dh)(x)\overline{(Dk)(x)}\,dx, \quad h,k \in \operatorname{dom}\mathfrak{t}_\infty,$$

$$\operatorname{dom}\mathfrak{t}_\infty = \big\{\, h \in L^2_m[a,b] :\, h \in AC[a,b],\, Dh \in L^2[a,b],\, h(a)=0,\, h(b)=0 \,\big\}.$$

The domain $\operatorname{dom}\mathfrak{t}_\infty$ is not dense in $L^2_m[a,b]$; in fact, its orthogonal complement is spanned by the characteristic function $\mathbf{1}_{\{a\}}$.

Let H_n, $n \in \mathbb{N}$, and H_∞ be defined by

$$\begin{aligned} H_n = \{\, \{h,k\} \in L^2_m[a,b] \times L^2_m[a,b] :\, & h \in AC[a,b], \\ & Dh(x) - Dh(a) = \int_{[a,x]} k(t)\,dm(t),\; Dh(a) = nh(a),\; h(b) = 0 \,\}, \end{aligned}$$

and

$$H_\infty = \{ \{h,k\} \in L^2_m[a,b] \times L^2_m[a,b] : h \in AC[a,b],$$
$$Dh(x) - Dh(a) = \int_{[a,x]} k(t)\, dm(t),\ h(a) = 0,\ h(b) = 0 \}.$$

Then H_n is the graph of a nonnegative selfadjoint operator and H_∞ is a nonnegative selfadjoint relation with $\operatorname{mul} H_\infty = \operatorname{span}\{\mathbf{1}_{\{a\}}\}$. The sequence H_n converges to H_∞ in the strong resolvent sense. It can be shown that the forms associated to H_n and H_∞ are given by $\mathfrak{t}_n$ and $\mathfrak{t}_\infty$, respectively.

5.3. **Form-bounded perturbations of selfadjoint operators.** Let H be a nonnegative selfadjoint operator and let $\mathfrak{H}_{+1}(H) \subset \mathfrak{H} \subset \mathfrak{H}_{-1}(H)$ be the rigged space associated with H. Here $\mathfrak{H}_{+1}(H)$ stands for $\operatorname{dom} H^{\frac{1}{2}}$ equipped with the graph topology of $H^{\frac{1}{2}}$ and $\mathfrak{H}_{-1}(H)$ is the associated dual space. Recall that H can be continued to a bounded operator $\widetilde{H} : \mathfrak{H}_{+1}(H) \to \mathfrak{H}_{-1}(H)$. Then $V_+ := \widetilde{H} + I = (H+I)^\sim$ maps $\mathfrak{H}_{+1}(H)$ isometrically onto $\mathfrak{H}_{-1}(H)$; it is called the Riesz operator.

Let $\mathcal{H}$ be an auxiliary Hilbert space and let $G : \mathcal{H} \to \mathfrak{H}_{-1}(H)$ be a bounded operator with $\ker G = \{0\}$ and, for simplicity, assume that $\operatorname{ran} G$ is a closed subspace in $\mathfrak{H}_{-1}(H)$. Then G admits a bounded dual mapping G^+ from $\mathfrak{H}_{+1}(H)$ into $\mathcal{H}$ with a closed range. Indeed, if $G^* : \mathfrak{H}_{-1}(H) \to \mathcal{H}$ is the usual Hilbert space adjoint, then $G^+ = G^* V_+ : \mathfrak{H}_{+1}(H) \to \mathcal{H}$ satisfies

(5.2)
$$(f, Gh) = (V_+ f, Gh)_{-1} = (G^* V_+ f, h)_{\mathcal{H}} = (G^+ f, h)_{\mathcal{H}}, \quad f \in \mathfrak{H}_{+1}(H), h \in \mathcal{H},$$

where $(\cdot,\cdot)$ stands for the duality in $\mathfrak{H}_{+1}(H) \times \mathfrak{H}_{-1}(H)$ as the continuation of the original inner product $(\cdot,\cdot)_{\mathfrak{H}}$. Now, consider so-called *form-bounded perturbations* of H of the form

$$H_n = H + n\, GG^+, \quad n \in \mathbb{N}, \tag{5.3}$$

so the perturbation is allowed to be infinite-dimensional; cf. [2]. Here the operator $GG^+ : \mathfrak{H}_{+1}(H) \to \mathfrak{H}_{-1}(H)$ is bounded with $\operatorname{ran} GG^+ = \operatorname{ran} G$. Note, however, that the perturbation GG^+ is in general an unbounded operator with respect to the original Hilbert space topology on $\mathfrak{H}$. The precise interpretation of H_n in (5.3) is as the unique nonnegative selfadjoint operator that is associated with the closed nonnegative form $\mathfrak{t}_n$ defined by

$$\mathfrak{t}_n[f,g] = (H^{\frac{1}{2}} f, H^{\frac{1}{2}} g)_{\mathfrak{H}} + n\,(G^+ f, G^+ g)_{\mathcal{H}}, \quad f,g \in \operatorname{dom} \mathfrak{t}_n = \operatorname{dom} H^{\frac{1}{2}}. \tag{5.4}$$

To see that the form $\mathfrak{t}_n$ is closed, assume that $f_k \to f \in \mathfrak{H}$, $\mathfrak{t}_n[f_k - f_l] \to 0$ for $f_k, f_l \in \operatorname{dom} \mathfrak{t}_n$. Then $f \in \operatorname{dom} H^{\frac{1}{2}}$ and $\|H^{\frac{1}{2}}(f_k - f)\|_{\mathfrak{H}} \to 0$ and by continuity of G^+ also $\|G^+(f_k - f)\|_{\mathcal{H}} \to 0$, which proves the claim.

By Theorem 4.2 the nondecreasing sequence gives rise to a “limit perturbation” H_∞ which corresponds to the closed form $\mathfrak{t}_\infty = \lim_{n\to\infty} \mathfrak{t}_n$. The next result gives an expression for H_∞.

Proposition 5.1. *The strong resolvent limit H_∞ of the nondecreasing sequence of nonnegative selfadjoint operators H_n in* (5.3) *is the selfadjoint relation given by*

$$H_\infty = R^* R, \quad R = \{\{f, H^{\frac{1}{2}} f\} : f \in \operatorname{dom} H^{\frac{1}{2}} \cap \ker G^+\}$$

and it corresponds to the closed form

$$\mathfrak{t}_\infty[f] = \|Rf\|^2, \quad f \in \operatorname{dom} \mathfrak{t}_\infty = \operatorname{dom} H^{\frac{1}{2}} \cap \ker G^+.$$

Proof. The closed forms $\mathfrak{t}_n$ associated with H_n in (5.4) satisfy $\operatorname{dom}\mathfrak{t}_n = \operatorname{dom} H^{\frac{1}{2}}$ and

$$\mathfrak{t}_n[f] = \|H^{\frac{1}{2}} f\|_{\mathfrak{H}}^2 + n\,\|G^+ f\|_{\mathcal{H}}^2. \tag{5.5}$$

Hence, $\lim_{n\to\infty}\mathfrak{t}_n[f] < \infty$ if and only if $f \in \operatorname{dom} H^{\frac{1}{2}} \cap \ker G^+$, in which case $\lim_{n\to\infty}\mathfrak{t}_n[f] = \|H^{\frac{1}{2}} f\|_{\mathfrak{H}}^2 = \|Rf\|_{\mathfrak{H}}^2$. By Theorem 4.2, the limit $\mathfrak{t}_\infty$ coincides with the closed form associated with the strong resolvent limit H_∞ of the operators H_n. Since the form $\mathfrak{t}_\infty$ is closed precisely when R is closed, R^*R is a selfadjoint relation which by uniqueness coincides with the representing relation H_∞. □

The selfadjoint operators H_n can be interpreted as extensions of the following restriction of H:

$$A = H\restriction \ker G^+, \quad \operatorname{dom} A = \operatorname{dom} H \cap \ker G^+; \tag{5.6}$$

cf. (5.3). Clearly, A is a nonnegative operator in $\mathfrak{H}$. The operator A is closed in $\mathfrak{H}$, since $\ker G^+$ is closed in $\mathfrak{H}_{+1}(H)$. Indeed, if $f_n \in \operatorname{dom} A$ and $f_n \to f$, $Af_n \to f'$, then $f \in \operatorname{dom} H$ and

$$\|H^{\frac{1}{2}}(f_n - f)\|_{\mathfrak{H}}^2 = (H(f_n - f), f_n - f)_{\mathfrak{H}} \to 0,$$

which implies that $f_n \to f$ in the topology of $\mathfrak{H}_{+1}(H)$ and thus $f_n \to f \in \ker G^+$ as $f_n \in \ker G^+$. The Friedrichs extension A_F of A is defined as the selfadjoint relation associated to the closure of the nonnegative form $(A\cdot,\cdot)$ on $\operatorname{dom} A$ via (4.3) or (4.4). Note that A_F has a multivalued part if and only if A is not densely defined. Observe that the limit relation H_∞ in Proposition 5.1 is also a nonnegative selfadjoint extension of A. The connection between H_∞ and A_F will be further specified.

Theorem 5.2. *Let A be defined by (5.6), let $\widetilde{H} : \mathfrak{H}_{+1}(H) \to \mathfrak{H}_{-1}(H)$ be the rigged space continuation of H, and let H_∞ be as in Proposition 5.1. Moreover, let $\mathcal{L} = \operatorname{ran}(\widetilde{H} + I)^{-1}G$ and let $\operatorname{clos}\mathcal{L}$ be the closure of $\mathcal{L}$ in the original topology of $\mathfrak{H}$. Then:*

(i) *A is densely defined if and only if $\operatorname{clos}\mathcal{L} \cap \operatorname{dom} H = \{0\}$;*
(ii) *$H_\infty = A_F$ if and only if $\operatorname{clos}\mathcal{L} \cap \ker G^+ = \{0\}$;*
(iii) *if $\mathcal{L}$ is closed in $\mathfrak{H}$, i.e., $\mathcal{L} = \operatorname{clos}\mathcal{L}$, then $H_\infty = A_F$;*
(iv) *if, in particular, $\mathcal{L} = \operatorname{clos}\mathcal{L} \subset \operatorname{dom} H$, then*

$$H_\infty = A_F = A \,\widehat{+}\, (\{0\} \times \operatorname{ran} G). \tag{5.7}$$

In (5.7) the operator part of H_∞ is given by $(H_\infty)_s = (I - P)A$, where P is the orthogonal projection onto $\operatorname{ran} G$.

Proof. (i) Let $f \in \operatorname{dom} A$ and $h \in \mathcal{H}$. Then (5.2) yields

$$0 = (G^+ f, h)_{\mathcal{H}} = (f, Gh) = ((A + I)f, (\widetilde{H} + I)^{-1}Gh)_{\mathfrak{H}},$$

which shows that $\operatorname{ran}(A+I) \subset \mathcal{L}^\perp$. The converse inclusion is obtained by reversing the given steps and using $\operatorname{ran}(H+I) = \mathfrak{H}$; here the orthogonal complement is with respect to the original inner product on $\mathfrak{H}$. Hence

$$\ker(A^* + I) = \operatorname{clos}\mathcal{L}, \tag{5.8}$$

see (2.2). Recall that every selfadjoint extension $\widetilde{A}$ of A satisfies

$$\ker(A^* - \lambda) \cap \operatorname{dom}\widetilde{A} = (\widetilde{A} - \lambda)^{-1}(\operatorname{mul} A^*), \quad \lambda \in \rho(\widetilde{A}), \tag{5.9}$$

(see e.g. [6, Proposition 4.20]). Since H is an operator extension of A, this yields

$$\operatorname{clos}\mathcal{L} \cap \operatorname{dom} H = \{0\} \quad \text{if and only if} \quad \operatorname{mul} A^* = \{0\},$$

which is equivalent to $\overline{\operatorname{dom}}\, A = \mathfrak{H}$.

(ii) Recall that $H_\infty = A_F$ if and only if

$$\ker(A^* + I) \cap \operatorname{dom} H_\infty^{\frac{1}{2}} = \{0\}, \tag{5.10}$$

see [10, Proposition 2.4]. Hence, the assertion follows from (5.8) and the description of $\operatorname{dom} H_\infty^{\frac{1}{2}} = \operatorname{dom} \mathfrak{t}_\infty$ in Proposition 5.1.

(iii) Let $\mathcal{L}$ be closed and assume that $f \in \mathcal{L} \cap \ker G^+$. Then there exists $h \in \mathcal{H}$ such that $f = (\widetilde{H} + I)^{-1} G h$ and

$$0 = G^+ f = G^+ (\widetilde{H} + I)^{-1} G h = G^* G h,$$

since $G^+ = G^* V_+$. Because $\ker G = \{0\}$, this implies that $h = 0$ and $f = 0$. Hence, $\mathcal{L} \cap \ker G^+ = \{0\}$ and the statement follows from (ii).

(iv) Assume that $\mathcal{L} = \ker(A^* + I) \subset \operatorname{dom} H$. Then $H_\infty = A_F$, $\operatorname{ran} G \subset \mathfrak{H}$, and $\mathcal{L} = \operatorname{ran} (H + I)^{-1} G$. Moreover, (5.9) shows that

$$\mathcal{L} = \ker(A^* + I) \cap \operatorname{dom} H = (H + I)^{-1} (\operatorname{mul} A^*),$$

and, hence, $\operatorname{ran} G = \operatorname{mul} A^*$. Now it is easy to check that

$$A^* = H \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_{-1}(A^*) = H \mathbin{\widehat{+}} (\{0\} \times \operatorname{ran} G),$$

cf. (5.14) for the definition of $\widehat{\mathfrak{N}}_{-1}(A^*)$. In this case, $\operatorname{dom} A^* = \operatorname{dom} H$,

$$\operatorname{dom} H_\infty = \operatorname{dom} A_F = \operatorname{dom} A^* \cap \operatorname{dom} H_\infty^{\frac{1}{2}} = \operatorname{dom} H \cap \ker G^+ = \operatorname{dom} A,$$

and thus formula (5.7) follows.

Note that $\operatorname{mul} H_\infty = \operatorname{ran} G$ and, therefore, the selfadjoint operator part of H_∞ in (5.7) is given by $(H_\infty)_s = (I - P)A$. □

Note that $\mathcal{L}$ in Theorem 5.2 is automatically closed in $\mathfrak{H}$, if it is finite-dimensional, so that, the singular perturbations in (5.3) are of finite rank (cf. Section 5.2). It is also closed in $\mathfrak{H}$ if, for instance, the unperturbed operator H in (5.3) is bounded, in which case the rigging collapses: $\mathfrak{H}_{+1}(H) = \mathfrak{H} = \mathfrak{H}_{-1}(H)$ and the corresponding topologies are equal. In this case Theorem 5.2 (iv) gives the precise meaning for the limit (1.3) described in the example in the introduction; an infinite-dimensional perturbation is obtained also via multiplication operators in Example 3.4 and potentials in Section 5.1. In the case that $\operatorname{ran} G$ is infinite-dimensional, $\mathcal{L}$ need not be a closed subspace of $\mathfrak{H}$ and it may happen that A is not densely defined even if $\operatorname{ran} G \cap \mathfrak{H} = \{0\}$.

5.4. **Limit characterization of the Friedrichs and Kreĭn-von Neumann extensions of a nonnegative relation.** As an application of the monotone convergence theorems the Friedrichs and Kreĭn-von Neumann extensions of a nonnegative relation A in a Hilbert space $\mathfrak{H}$ are characterized as the strong resolvent limits of a sequence of semibounded selfadjoint extensions of A. Recall that the Friedrichs extension A_F and the Kreĭn-von Neumann extension A_K are nonnegative selfadjoint extensions of A having the following extremality property:

$$A_K \leq \widetilde{A} \leq A_F \tag{5.11}$$

for every nonnegative selfadjoint extension $\widetilde{A}$ of A. Note that, together with A, also A^{-1} is a nonnegative relation. Using (5.11) and the equivalence of (i) and (ii) in Proposition 2.2, it follows that

$$(A^{-1})_K = (A_F)^{-1}, \quad (A^{-1})_F = (A_K)^{-1}. \tag{5.12}$$

If, in particular, the lower bound of A is positive, then $A_K = A \,\widehat{+}\, (\ker A^* \times \{0\})$, and, similarly, if A is a bounded operator, then $A_F = A \,\widehat{+}\, (\{0\} \times \operatorname{mul} A^*)$. In addition,

$$(A - x)_F = A_F - x, \quad (A - x)_K \le A_K - x, \quad x < 0. \tag{5.13}$$

By means of the defect spaces $\widehat{\mathfrak{N}}_x(A^*) = \{\, \{f_x, x f_x\} : f_x \in \ker(A^* - x)\}$ define the extensions A_x of A by

$$A_x = A \,\widehat{+}\, \widehat{\mathfrak{N}}_x(A^*), \quad x < 0. \tag{5.14}$$

Clearly, A_x is selfadjoint and bounded from below by x, i.e., $A_x - x \ge 0$. Since $A \ge 0$ and $x < 0$, $A - x$ has a positive lower bound and hence

$$(A - x)_K = (A - x) \,\widehat{+}\, (\ker(A - x)^* \times \{0\}) = (A - x)_0 = A_x - x, \quad x < 0. \tag{5.15}$$

If $x_1 \le x_2 (< 0)$ then $A_{x_1} - x_1 \ge 0$ and $A_{x_2} - x_1 \ge 0$ are both selfadjoint extensions of $A - x_1 \ge 0$. Thus, (5.11) and (5.15) yield

$$A_{x_1} - x_1 = (A - x_1)_K \le A_{x_2} - x_1.$$

This shows that A_x is nondecreasing with respect to $x < 0$:

$$A_{x_1} \le A_{x_2}, \quad x_1 \le x_2 < 0. \tag{5.16}$$

The following result in the case that the Friedrichs and the Kreĭn-von Neumann extensions exist as densely defined selfadjoint operators goes back to [3]; see [8] for the case when A is not necessarily a densely defined operator, and [9] for the general case. The present proof is based on a direct application of Theorem 3.5.

Proposition 5.3. *Let A be a nonnegative relation in a Hilbert space $\mathfrak{H}$. Then the strong resolvent limits of the selfadjoint extensions A_x in (5.14) as $x \uparrow 0$ and $x \downarrow -\infty$ are the Kreĭn-von Neumann extension A_K and the Friedrichs extension A_F of A, respectively:*

$$(A_K - \lambda)^{-1} h = \lim_{x \uparrow 0} (A_x - \lambda)^{-1} h, \quad (A_F - \lambda)^{-1} h = \lim_{x \downarrow -\infty} (A_x - \lambda)^{-1} h, \quad h \in \mathfrak{H}.$$

Proof. First the statement concerning the limit with $x \to 0$ is shown. By Theorem 3.5 and monotonicity of A_x the strong resolvent limit of A_x as $x \uparrow 0$ exists and is a nonnegative selfadjoint relation, since A_x has a lower bound $x \to 0$; denote this limit by A_0. Now A_0 is the strong graph limit of A_x as $x \uparrow 0$. Since $A \subset A_x$ for all $x < 0$, this implies that $A \subset A_0$. Thus A_0 is a nonnegative selfadjoint extension of A and, hence, $A_K \le A_0$ by (5.11). It follows from (5.15) and (5.13) that $A_x \le A_K$ for all $x < 0$. Hence, by Proposition 2.2 $(A_K + I)^{-1} \le (A_x + I)^{-1}$ for all $-1 < x < 0$. Letting $x \to 0$ one gets the inequality $(A_K + I)^{-1} \le (A_0 + I)^{-1}$, i.e., $A_0 \le A_K$. Therefore, $A_0 = A_K$.

For the assertion concerning the limit when $x \downarrow -\infty$, observe that A^{-1} is also a closed nonnegative relation and that $(A_x)^{-1} = (A^{-1})_{1/x}$, $x < 0$. Therefore,

$$\lim_{x \downarrow -\infty} ((A_x)^{-1} - \lambda)^{-1} h = \lim_{y \uparrow 0} ((A^{-1})_y - \lambda)^{-1} h = ((A^{-1})_K - \lambda)^{-1} h, \quad h \in \mathfrak{H},$$

by the first part of the proof. Hence, by Proposition 2.4 and (5.12) A_x tends in the strong resolvent sense to $((A^{-1})_K)^{-1} = A_F$ as $x \downarrow -\infty$. □

Note that the relations A_x as $x \to -\infty$ do not have a common lower bound and hence Theorem 3.7 can not be applied directly to A_x with $x \to -\infty$ to obtain the limit description for the Friedrichs extension A_F.

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