NONRELATIVISTIC LIMIT OF TWO-DIMENSIONAL DIRAC OPERATORS ON LIPSCHITZ DOMAINS

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We dedicate this paper to our friend and colleague Rainer Picard on the occasion of his 80th birthday.

Abstract. We consider a family of two-dimensional self-adjoint Dirac operators on Lipschitz domains subject to a class of quantum dot boundary conditions and prove that the nonrelativistic limit is the Dirichlet Laplacian. As an application a version of the classical Faber-Krahn inequality for Dirac operators is discussed.

1. Introduction

The Dirac equation provides a quantum mechanical description of the propagation of spin-1/2 particles and appears as an illustration in many of Rainer's works on abstract boundary value problems and evolutionary systems in the Hilbert space context, see, e.g., [47], [48], and the monograph [49]. The Dirac equation equipped with certain boundary conditions is also used in the desription of graphene [1, 42, 45] in the two-dimensional case and appears in the investigation of quarks in hadrons in the three-dimensional situation [22, 27, 29, 38]; for a more mathematical modern perspective we refer to [3, 5, 6, 15, 19, 20, 31, 46, 51] and the references therein.

The present paper is strongly inspired by the recent contributions [7, 12, 30], where the eigenvalue curves and the nonrelativistic limit of a family of self-adjoint Dirac operators on domains in the three-dimensional setting with so-called generalized MIT bag boundary conditions were studied. In fact, here we consider the two-dimensional counterparts of the operators in [7, 12, 30] with a class of quantum dot boundary conditions on a bounded Lipschitz domain $\Omega_+ \subset \mathbb{R}^2$ or on the unbounded Lipschitz domain $\Omega_- = \mathbb{R}^2 \setminus \overline{\Omega}_+$, which have the form

$$\begin{aligned} H_\kappa^{\Omega_\pm,c} &= -ic(\sigma\cdot\nabla) + \frac{c^2}{2}\sigma_3, \\ \operatorname{dom} H_\kappa^{\Omega_\pm,c} &= \big\{ f_\pm \in H^1(\Omega_\pm;\mathbb{C}^2) : \\ &\qquad f_\pm = \pm i(\sinh(\kappa)I_2 - \cosh(\kappa)\sigma_3)(\sigma\cdot\nu)f_\pm \text{ on } \partial\Omega_\pm \big\}, \end{aligned} \tag{1.1}$$

where $\kappa \in \mathbb{R}$ is a paramater, $c > 0$ is the speed of light, $\sigma\cdot\nabla = \sigma_1\partial_1 + \sigma_2\partial_2$ with the usual Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{C}^{2\times2}$, $I_2 \in \mathbb{C}^{2\times2}$ is the identity matrix, and $H^1(\Omega_\pm;\mathbb{C}^2)$ denotes the first order L^2-based Sobolev space. Observe that the operators $H_\kappa^{\Omega_\pm,c}$, $\kappa \in \mathbb{R}$, are unbounded and self-adjoint in $L^2(\Omega_\pm;\mathbb{C}^2)$, and their spectra are contained in $(-\infty, -c^2/2] \cup [c^2/2, \infty)$; cf. Proposition 2.2 for more details. Dirac operators with boundary conditions as in (1.1) are of particular interest since they appear (in the massless case) in the analysis of graphene quantum dots, see [19, 20] and also the very recent contribution [31] for further references.

Our main objective is to investigate the nonrelativistic limit of $H_\kappa^{\Omega_\pm,c}$, that is, we subtract the energy $c^2/2$ of the resting particle and compute the limit of the resolvent of $H_\kappa^{\Omega_\pm,c} - c^2/2$ as $c \to \infty$, which turns out to be the resolvent of the Dirichlet Laplacian times a projection matrix. In this sense the nonrelativistic limit connects the Dirac operators (1.1) with their nonrelativistic counterparts. More precisely, if $S^{\Omega_\pm}$ denotes the self-adjoint Dirichlet Laplacian in $L^2(\Omega_\pm)$, $z < 0$ is fixed, and $P_+ = \mathrm{diag}(1,0)$ is the projection onto the upper spinor, then for $c > 0$ sufficiently large we conclude in Corollary 3.7 that

$$\big\| \big(H_\kappa^{\Omega_\pm,c} - (z + c^2/2)\big)^{-1} - (S^{\Omega_\pm} - z)^{-1} P_+ \big\|_{L^2(\Omega_\pm;\mathbb{C}^2) \to L^2(\Omega_\pm;\mathbb{C}^2)} \leq C c^{-1/2},$$

where $C > 0$ is some constant (depending on the choice of $z < 0$). Although our proof of the operator norm convergence of the resolvents follows a similar strategy as in [12], where the three-dimensional situation is treated, we are more general here in the sense that even Lipschitz domains (in contrast to C^2-domains) are allowed. Furthermore, the technical estimates in Lemma 3.4 are obtained in a more direct and efficient way than in [12].

The nonrelativistic limit is of interest by itself, as it gives a physical interpretation of the operators $H_\kappa^{\Omega_\pm,c}$, and it can also be used to transfer spectral inequalities and spectral geometry results for sufficiently large $c > 0$ from Laplacians to Dirac operators. We illustrate this aspect briefly in the end of Section 3, where a version of the classical Faber-Krahn inequality is obtained for the Dirac operators $H_\kappa^{\Omega_+,c}$ with sufficiently large $c > 0$ (see Corollary 3.8). The interested reader is referred to the monographs [34, 40, 50] for an introduction to general spectral geometry and to [3, 7, 12, 20, 23, 31, 41, 56] for some recent related spectral inequality and geometry results for Dirac operators. Note that, although also $H_\kappa^{\Omega_-,c}$ converges in the nonrelativistic limit to the Dirichlet Laplacian in Ω_-, spectral implications are only of interest for the bounded domain Ω_+, as the spectrum of $H_\kappa^{\Omega_-,c}$ is always purely essential and given by $(-\infty, -c^2/2] \cup [c^2/2, \infty)$, see Proposition 2.2. Let us also mention here that the nonrelativistic limit of Dirac operators has been studied in many different settings in the mathematical literature. The classical case of three-dimensional Dirac operators with regular potentials is treated in the monograph [55], where also further references can be found. For one-dimensional Dirac operators we refer the reader to [28] and the related papers [24, 25, 33, 36], and for the nonrelativistic limit of Dirac operators with singular potentials in two and three dimensions see [6, 9, 10, 11, 17].

Notations. The Pauli spin matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and they fulfil the relations

$$\sigma_k \sigma_j + \sigma_j \sigma_k = 2 I_2 \delta_{jk}, \qquad j,k \in \{1,2,3\}, \tag{1.2}$$

where I_2 denotes the 2×2 identity matrix and δ_{jk} is the Kronecker delta. Furthermore, we use the abbreviations $\sigma \cdot \nabla := \sigma_1 \partial_1 + \sigma_2 \partial_2$ and $\sigma \cdot v := \sigma_1 v_1 + \sigma_2 v_2$ for $v = (v_1, v_2)^T \in \mathbb{C}^2$.

Throughout this paper $\Omega_+ \subset \mathbb{R}^2$ is a bounded Lipschitz domain with boundary denoted by Σ and we set $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega_+}$. Moreover, ν is the unit outward normal vector associated to Ω_+. For an open set $U \subset \mathbb{R}^2$ and $s \in \mathbb{R}$ the Sobolev space of order s is denoted by $H^s(U)$ and $H^r(\Sigma)$ stands for the Sobolev space of order $r \in [-1,1]$ on Σ; cf. [44, Chapter 3] for definitions and more details. Sobolev spaces of vector-valued functions are defined component-wise and denoted by $H^s(U;\mathbb{C}^n)$ and $H^r(\Sigma;\mathbb{C}^n)$ for $n \in \mathbb{N}$.

We use the notation $\boldsymbol{t}_\Sigma^\pm$ for the Dirichlet trace operator and recall that

$$\boldsymbol{t}_\Sigma^\pm : H^s(\Omega_\pm) \to H^{s-1/2}(\Sigma) \tag{1.3}$$

is bounded for $s \in (1/2, 3/2)$ and $\boldsymbol{t}_\Sigma^\pm$ is also bounded as an operator from $H^s(\Omega_\pm)$ to $H^1(\Sigma)$ for $s > 3/2$, see [44, Theorem 3.38] and [13, Theorem 3.6]. The Dirichlet trace operator on $H^s(\mathbb{R}^2)$ is defined as $\boldsymbol{t}_\Sigma := \boldsymbol{t}_\Sigma^+ R_+$, where R_+ is the operator which restricts a function in $H^s(\mathbb{R}^2)$ to Ω_+, and has by definition analogous mapping properties as $\boldsymbol{t}_\Sigma^+$. Trace operators acting in vector valued Sobolev spaces are defined component-wise.

Finally, we fix the branch of the complex square root such that $\operatorname{Im} \sqrt{w} > 0$ for all $w \in \mathbb{C} \setminus [0,\infty)$ and mention that generic positive constants which may change in-between lines are denoted by C.

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2. Dirac operators, Laplacians, and associated integral operators

In this section we collect preparatory material on Dirac and Laplace operators, and potential and boundary integral operators that are used throughout this paper.

2.1. The free Dirac and Laplace operator.

The free Laplacian in $\mathbb{R}^2$ is defined by

$$S := -\Delta, \qquad \operatorname{dom} S := H^2(\mathbb{R}^2) \subset L^2(\mathbb{R}^2),$$

and the free Dirac operator is given by

$$H^c := -ic(\sigma \cdot \nabla) + \frac{c^2}{2}\sigma_3, \quad \operatorname{dom} H^c := H^1(\mathbb{R}^2;\mathbb{C}^2) \subset L^2(\mathbb{R}^2;\mathbb{C}^2),$$

where $c > 0$ is a constant which usually denotes the speed of light. Using the relations from (1.2) one sees that S and H^c are connected via the formula

$$(H^c)^2 = \Big(c^2 S + \frac{c^4}{4}\Big) I_2. \tag{2.1}$$

It is well-known that both operators, H^c and S, are self-adjoint in L^2 and that their spectra are given by

$$\sigma(S) = [0,\infty) \quad \text{and} \quad \sigma(H^c) = (-\infty, -c^2/2] \cup [c^2/2, \infty); \tag{2.2}$$

see, e.g., [16, Section 1.2] and [53, Example 8.1]. In particular, S is a nonnegative operator. For $w' \in \rho(S) = \mathbb{C} \setminus \sigma(S)$ the resolvent $R_{w'} := (S - w')^{-1}$ is an integral operator with kernel $G_{w'}(x) := (1/(2\pi))K_0\big(-i\sqrt{w'}|x|\big)$, $x \in \mathbb{R}^2 \setminus \{0\}$; i.e.

$$R_{w'}v(x) = \int_{\mathbb{R}^2} G_{w'}(x-y)v(y)\,dy, \qquad v \in L^2(\mathbb{R}^2),\ x \in \mathbb{R}^2, \tag{2.3}$$

see, e.g., [54, eq. (7.53)]. Here, K_0 denotes the zeroth modified Bessel function of second kind. Note that $R_{w'}$ is not only bounded as an operator in $L^2(\mathbb{R}^2)$, but also acts as a bounded operator from $L^2(\mathbb{R}^2)$ to $H^2(\mathbb{R}^2)$. Using the identity (2.1) one concludes for $w \in \rho(H^c)$ that

$$R^c_w := (H^c - wI_2)^{-1} = c^{-2}(H^c + wI_2)R_{w'}I_2, \qquad w' = w^2/c^2 - c^2/4. \tag{2.4}$$

The above representation of R^c_w shows that R^c_w is bounded as an operator from $L^2(\mathbb{R}^2;\mathbb{C}^2)$ to $H^1(\mathbb{R}^2;\mathbb{C}^2)$. Moreover, (2.3) implies

$$R^c_w u(x) = \int_{\mathbb{R}^2} G^c_w(x-y)u(y)\,dy, \qquad u \in L^2(\mathbb{R}^2;\mathbb{C}^2),\ x \in \mathbb{R}^2,$$

with

$$\begin{aligned} G^c_w(x) := \frac{\sqrt{w^2/c^2 - c^2/4}}{2\pi c} & K_1\big(-i\sqrt{w^2/c^2-c^2/4}|x|\big)\frac{\sigma\cdot x}{|x|} \\ & + \frac{1}{2\pi}K_0\big(-i\sqrt{w^2/c^2-c^2/4}|x|\big)\left(\frac{1}{2}\sigma_3 + \frac{w}{c^2}I_2\right), \end{aligned} \tag{2.5}$$

where K_1 is the first order modified Bessel function of second kind.

2.2. **Potential operators and boundary integral operators.** For $w \in \rho(H^c)$ and $w' \in \rho(S)$ the potential operators associated to the free Dirac and Laplace operator are defined by

$$\begin{aligned} \Phi^c_w &: L^2(\Sigma;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2), \\ \Phi^c_w\varphi(x) &:= \int_\Sigma G^c_w(x-y_\Sigma)\varphi(y_\Sigma)\,d\sigma(y_\Sigma), \\ SL_{w'} &: L^2(\Sigma) \to L^2(\mathbb{R}^2), \\ SL_{w'}\psi(x) &:= \int_\Sigma G_{w'}(x-y_\Sigma)\psi(y_\Sigma)\,d\sigma(y_\Sigma). \end{aligned} \tag{2.6}$$

By [16, eq. (4.4)] (where Φ_z from [16] coincides with $c\Phi^c_w$, $w = zc$, if one chooses in [16] $m = c/2$) and [44, Theorem 6.12] (see also [35, eq. (19) and Lemma 2.6] for the exterior domain) these operators are well-defined and bounded. Moreover, according to [16, eq. (4.8)] and [35, Proposition 3.1 (ii)] their adjoints fulfil the relations

$$(\Phi^c_w)^* = \boldsymbol{t}_\Sigma R^c_{\overline{w}} \quad \text{and} \quad (SL_{w'})^* = \boldsymbol{t}_\Sigma R_{\overline{w'}}. \tag{2.7}$$

In particular, the mapping properties of $\boldsymbol{t}_\Sigma$, see below (1.3), and R^c_w as well as $R_{w'}$ show that $(\Phi^c_w)^* : L^2(\mathbb{R}^2;\mathbb{C}^2) \to H^{1/2}(\Sigma;\mathbb{C}^2)$ and $(SL_{w'})^* : L^2(\mathbb{R}) \to H^1(\Sigma)$ are bounded. A duality argument implies that Φ^c_w and $SL_{w'}$ can be extended to bounded operators mapping from $H^{-1/2}(\Sigma;\mathbb{C}^2)$ to $L^2(\mathbb{R}^2;\mathbb{C}^2)$ and from $H^{-1}(\Sigma)$ to $L^2(\mathbb{R}^2)$, respectively.

Next, we introduce for $w \in \rho(H^c)$ and $w' \in \rho(S)$ the following boundary integral operators

$$\begin{aligned} \mathcal{C}_w^c &: L^2(\Sigma;\mathbb{C}^2) \to L^2(\Sigma;\mathbb{C}^2),\\ \mathcal{C}_w^c\varphi(x_\Sigma) &:= \lim_{\varepsilon\searrow 0}\int_{\Sigma\setminus B(x_\Sigma,\varepsilon)} G_w^c(x_\Sigma - y_\Sigma)\varphi(y_\Sigma)\,d\sigma(y_\Sigma),\\ \mathcal{S}_{w'} &: L^2(\Sigma) \to L^2(\Sigma),\\ \mathcal{S}_{w'}\psi(x_\Sigma) &:= \int_\Sigma G_{w'}(x_\Sigma - y_\Sigma)\psi(y_\Sigma)\,d\sigma(y_\Sigma),\\ \mathcal{W}_{w'} &: L^2(\Sigma) \to L^2(\Sigma),\\ \mathcal{W}_{w'}\psi(x_\Sigma) &:= \lim_{\varepsilon\searrow 0}\int_{\Sigma\setminus B(x_\Sigma,\varepsilon)} \frac{\sqrt{w'}}{2\pi}K_1\big(-i\sqrt{w'}|x_\Sigma - y_\Sigma|\big)\\ &\qquad\qquad \cdot \frac{\overline{\mathbf{x_\Sigma} - \mathbf{y_\Sigma}}}{|x_\Sigma - y_\Sigma|}\psi(y_\Sigma)\,d\sigma(y_\Sigma), \end{aligned} \tag{2.8}$$

where we used the convention $\mathbf{v} = v_1 + iv_2 \in \mathbb{C}$ for $v = (v_1, v_2)^T \in \mathbb{R}^2$. The operators introduced above are well-defined and bounded, see [16, Theorem 4.3] (where $\mathcal{C}_z$ from [16] coincides with $c\mathcal{C}_w^c$, $w = zc$, if one chooses in [16] $m = c/2$) and [35, eqs. (25), (26) and (28), and Lemma 2.9]. Moreover, $\mathcal{S}_{w'}$ can also be viewed as a bounded operator from $H^r(\Sigma)$ to $H^{r+1}(\Sigma)$ for $r \in [-1,0]$. Indeed, for $r = 0$ this follows from [35, eq. (25)]. Then, the cases $r \in [-1,0)$ can be proven by using $\mathcal{S}_{w'}^* = \mathcal{S}_{\overline{w'}}$, see [35, Propositions 2.13 (iii) and 3.1 (iii)], duality and interpolation. Furthermore, later we shall also use that $\mathcal{S}_{w'} : L^2(\Sigma) \to H^1(\Sigma)$ is bijective and according to [21, eq. (9)] $\mathcal{S}_{w'}$ is nonnegative in $L^2(\Sigma)$ if $w' < 0$. From (2.5) and $G_{w'}(x) = (1/(2\pi))K_0\big(-i\sqrt{w'}|x|\big)$ we obtain that the just introduced boundary integral operators are connected via

$$\mathcal{C}_w^c = \begin{pmatrix} \big(\frac{1}{2} + \frac{w}{c^2}\big)\mathcal{S}_{w'} & \frac{1}{c}\mathcal{W}_{w'} \\ \frac{1}{c}(\mathcal{W}_{\overline{w'}})^* & \big(-\frac{1}{2} + \frac{w}{c^2}\big)\mathcal{S}_{w'} \end{pmatrix}, \quad w' = w^2/c^2 - c^2/4. \tag{2.9}$$

We end this section by providing difference estimates for $\mathcal{W}_{w'}$ and $\mathcal{S}_{w'}$.

Proposition 2.1. *Let $K \subset \rho(S) = \mathbb{C}\setminus[0,\infty)$ be compact. Then, there exists a $C = C(K) > 0$ such that for all $w', v' \in K$*

$$\|\mathcal{W}_{w'} - \mathcal{W}_{v'}\|_{L^2(\Sigma)\to L^2(\Sigma)}, \|\mathcal{S}_{w'} - \mathcal{S}_{v'}\|_{L^2(\Sigma)\to L^2(\Sigma)} \le C(K)|w' - v'|.$$

Proof. Let $c > 0$ be fixed such that $0 < c^2/4 < -\sup_{w'\in K\cap\mathbb{R}} w'$ if $K \cap \mathbb{R} \neq \emptyset$; otherwise just assume $c > 0$. We consider the open set $O = \mathbb{C}\setminus[-c^2/4,\infty)$ and the function

$$O \ni w' \mapsto f(w') := c\sqrt{w' + c^2/4}.$$

Note, that by the choice of c we have $w' + c^2/4 < 0$ for all $w' \in O \cap \mathbb{R}$. Moreover, our choice of the branch of the square root implies that f is holomorphic in O and $\operatorname{Im} f(w') > 0$, in particular, $f(w') \in \rho(H^c)$ for $w' \in O$. With the various definitions from above one sees

$$\mathcal{C}_{f(w')}^c = \begin{pmatrix} \Big(\frac{1}{2} + \frac{f(w')}{c^2}\Big)\mathcal{S}_{w'} & \frac{1}{c}\mathcal{W}_{w'} \\ \frac{1}{c}(\mathcal{W}_{\overline{w'}})^* & \Big(-\frac{1}{2} + \frac{f(w')}{c^2}\Big)\mathcal{S}_{w'} \end{pmatrix}. \tag{2.10}$$

Combining [18, Proposition 2.6] and [16, Theorem 4.3] (where $\mathcal{C}_z$ from [16] coincides with $c\mathcal{C}_w^c$, $w = zc$, if one chooses in [16] $m = c/2$) the mapping $\rho(H^c) \ni w \mapsto \mathcal{C}_w^c \in$

$\mathcal{L}(L^2(\Sigma;\mathbb{C}^2))$ is holomorphic. Thus, $O \ni w' \mapsto \mathcal{C}^c_{f(w')} \in \mathcal{L}(L^2(\Sigma;\mathbb{C}^2))$ is also holomorphic and the same holds for the individual blocks in the representation of $\mathcal{C}^c_{f(w')}$ in (2.10), which we denote by $\mathcal{C}^c_{f(w')}[j,k]$, $j,k \in \{1,2\}$. Hence, $O \ni w' \mapsto \mathcal{W}_{w'} = c\,\mathcal{C}^c_{f(w')}[1,2] \in \mathcal{L}(L^2(\Sigma))$ is holomorphic. Moreover, as $1/2+f(w')/c^2 \neq 0$ for all $w' \in O$, the mapping $O \ni w' \mapsto \mathcal{S}_{w'} = (1/2+f(w')/c^2)^{-1}\mathcal{C}^c_{f(w')}[1,1] \in \mathcal{L}(L^2(\Sigma))$ is also holomorphic. Now, the assertion follows from the holomorphicity and the fact that K is a compact subset of O by the choice of $c > 0$. □

2.3. **Dirac operators on domains.** Recall that Ω_+ denotes a bounded Lipschitz domain, $\Sigma = \partial\Omega_+$, $\Omega_- = \mathbb{R}^2\setminus\overline{\Omega}_+$ and ν is the unit outward normal vector associated to Ω_+. In this section we consider Dirac operators on $\Omega_\pm$ with boundary conditions. For $c > 0$ and $\kappa \in \mathbb{R}$ we define

$$\begin{aligned} H_\kappa^{\Omega_\pm,c} &:= -ic(\sigma\cdot\nabla) + \frac{c^2}{2}\sigma_3, \\ \operatorname{dom} H_\kappa^{\Omega_\pm,c} &:= \big\{f_\pm \in H^1(\Omega_\pm;\mathbb{C}^2) : \\ &\qquad \boldsymbol{t}_\Sigma^\pm f_\pm = \pm i(\sinh(\kappa)I_2 - \cosh(\kappa)\sigma_3)(\sigma\cdot\nu)\boldsymbol{t}_\Sigma^\pm f_\pm\big\}; \end{aligned} \tag{2.11}$$

cf. (1.1). Dirac operators with such boundary conditions appear in the description of graphene quantum dots, see [19, 20] and the references therein, and are the two-dimensional counterpart of the generalized MIT bag models considered in [7, 12, 30]. Furthermore, these operators are also strongly connected to Dirac operators with δ-shell potentials, since $H_\kappa^{\Omega_+,c} \oplus H_\kappa^{\Omega_-,c}$ coincides with the operator (formally) given by $-ic(\sigma\cdot\nabla) + (c^2/2)\sigma_3 + 2c(\sinh(\kappa)I_2 + \cosh(\kappa)\sigma_3)\delta_\Sigma$, where δ_Σ denotes the δ-shell potential supported on Σ, see, e.g., [16, Section 5.2], [26, Section 5] or [52, Section 7].

We summarize some important properties of $H_\kappa^{\Omega_\pm,c}$ in the next proposition.

Proposition 2.2. *Let $\kappa \in \mathbb{R}$ and $H_\kappa^{\Omega_\pm,c}$ be defined by (2.11). Then, $H_\kappa^{\Omega_\pm,c}$ is self-adjoint in $L^2(\Omega_\pm;\mathbb{C}^2)$ and the following holds:*

(i) $\sigma_{\mathrm{disc}}(H_\kappa^{\Omega_+,c}) = \sigma(H_\kappa^{\Omega_+,c}) \subset (-\infty,-c^2/2] \cup [c^2/2,\infty)$.
(ii) $\sigma(H_\kappa^{\Omega_-,c}) = (-\infty,-c^2/2] \cup [c^2/2,\infty)$.
(iii) *Define $D = \operatorname{diag}(\sqrt{2}e^{\kappa/2},\sqrt{2}e^{-\kappa/2})$. Then, for $w \in \rho(H^c)$ the operator $c^{-1}\sigma_3 + D\mathcal{C}^c_w D$ is boundedly invertible in $L^2(\Sigma;\mathbb{C}^2)$ and the resolvent formula*

$$(H_\kappa^{\Omega_+,c} \oplus H_\kappa^{\Omega_-,c} - wI_2)^{-1} = R^c_w - \Phi^c_w D(c^{-1}\sigma_3 + D\mathcal{C}^c_w D)^{-1}D(\Phi^c_{\overline{w}})^*$$

holds.

Proof. Let A_0, $A^\pm_{\eta,\tau,\lambda}$, Φ_z, $\mathcal{C}_z$ and $P_{\eta,\tau,\lambda}$ be the operators from [16, eqs. (1.4), (5.17), (4.4), (4.5) and (5.1)]. If one chooses in [16] $m = c/2$, $\eta = 2\sinh(\kappa)$, $\tau = 2\cosh(\kappa)$, $\lambda = 0$ and $w = zc$, then $H^c = cA_0$, $H_\kappa^{\Omega_\pm,c} = cA^\pm_{\eta,\tau,\lambda}$, $\mathcal{C}^c_w = c^{-1}\mathcal{C}_z$, $\Phi^c_w = c^{-1}\Phi_z$, $R^c_w = c^{-1}(A_0 - z)^{-1}$ and $2\sinh(\kappa)I_2 + 2\cosh(\kappa)\sigma_3 = \sigma_3 D^2 = P_{\eta,\tau,\lambda}$. Using these relations, we start by proving that $c^{-1}\sigma_3 + D\mathcal{C}^c_w D$ is boundedly invertible in $L^2(\Sigma;\mathbb{C}^2)$ for all $w \in \rho(H^c)$. It follows from [16, Proposition 5.5 (i)] that $I + c\sigma_3 D^2\mathcal{C}^c_w$ is boundedly invertible for $w \in \mathbb{C}\setminus\mathbb{R}$. Thus, the same is true for

$$c^{-1}\sigma_3 + D\mathcal{C}^c_w D = c^{-1}\sigma_3 D^{-1}(I_2 + c\sigma_3 D^2\mathcal{C}^c_w)D. \tag{2.12}$$

It remains to consider $w \in \mathbb{R} \cap \rho(H^c) = (-c^2/2, c^2/2)$. Since, $(\mathcal{C}_w^c)^* = \mathcal{C}_{\overline{w}}^c$, see [16, eq. (2.3) and Theorem 4.3 (iii)], $\mathcal{C}_w^c$ is self-adjoint in this case. Thus,

$$\begin{aligned}\|(c^{-1}\sigma_3 + D\mathcal{C}_w^c D)f\|^2_{L^2(\Sigma;\mathbb{C}^2)} &\\ = c^{-2}\|f\|^2_{L^2(\Sigma;\mathbb{C}^2)} &+ c^{-1}(D(\sigma_3\mathcal{C}_w^c + \mathcal{C}_w^c\sigma_3)Df, f)_{L^2(\Sigma;\mathbb{C}^2)} \\ &\qquad + \|D\mathcal{C}_w^c Df\|^2_{L^2(\Sigma;\mathbb{C}^2)} \\ \geq c^{-2}\|f\|^2_{L^2(\Sigma;\mathbb{C}^2)} &+ c^{-1}(D(\sigma_3\mathcal{C}_w^c + \mathcal{C}_w^c\sigma_3)Df, f)_{L^2(\Sigma;\mathbb{C}^2)}.\end{aligned} \tag{2.13}$$

Moreover, using (2.9) shows

$$\sigma_3\mathcal{C}_w^c + \mathcal{C}_w^c\sigma_3 = \begin{pmatrix} 1 + \frac{2w}{c^2} & 0 \\ 0 & 1 - \frac{2w}{c^2} \end{pmatrix} \mathcal{S}_{w^2/c^2 - c^2/4}, \tag{2.14}$$

which is a nonnegative operator since $|w/c^2| < 1/2$, and also $\mathcal{S}_{w^2/c^2-c^2/4} \geq 0$ as $w^2/c^2 - c^2/4 < 0$; cf. below (2.8). Thus,

$$\|(c^{-1}\sigma_3 + D\mathcal{C}_w^c D)f\|^2_{L^2(\Sigma;\mathbb{C}^2)} \geq c^{-2}\|f\|^2_{L^2(\Sigma;\mathbb{C}^N)},$$

and hence $c^{-1}\sigma_3 + D\mathcal{C}_w^c D$ is boundedly invertible in $L^2(\Sigma;\mathbb{C}^2)$. By (2.12) the same is true for $I + c\sigma_3 D^2\mathcal{C}_w^c$.

Now, combining [16, Theorem 5.6, Lemma 5.11 (ii) and Proposition 5.13] yields the main assertion, (i), (ii) and the resolvent formula

$$(H_\kappa^{\Omega_+,c} \oplus H_\kappa^{\Omega_-,c} - wI_2)^{-1} = R_w^c - \Phi_w^c(I_2 + c\sigma_3 D^2\mathcal{C}_w^c)^{-1} c\sigma_3 D^2(\Phi_{\overline{w}}^c)^*$$

for $w \in \rho(H^c)$. Applying (2.12) shows that this formula coincides with the formula from item (iii). □

2.4. **The Dirichlet Laplacian.** The Dirichlet Laplacian on $\Omega_\pm$ is defined by

$$\begin{aligned} S^{\Omega_\pm} &:= -\Delta, \\ \operatorname{dom} S^{\Omega_\pm} &:= \{f \in H^{3/2}(\Omega_\pm) : \Delta f \in L^2(\Omega_\pm), \boldsymbol{t}_\Sigma^\pm f = 0\}. \end{aligned} \tag{2.15}$$

It is well known that $S^{\Omega_\pm}$ is a nonnegative self-adjoint operator in $L^2(\Omega_\pm)$ associated to the closed nonnegative form

$$\mathfrak{s}^{\Omega_\pm}(g_\pm, h_\pm) = (\nabla g_\pm, \nabla h_\pm)_{L^2(\Omega_\pm)}, \quad g_\pm, h_\pm \in \operatorname{dom} \mathfrak{s}^{\Omega_\pm} = H_0^1(\Omega_\pm);$$

cf. [13, Theorem 6.9] and [37] (and for the unbounded Lipschitz domain Ω_- this can be found in, e.g, [4, Section 3]).

The next proposition contains a useful resolvent formula for the orthogonal sum $S^{\Omega_+} \oplus S^{\Omega_-}$ of the Dirichlet Laplacians in $L^2(\mathbb{R}^2) = L^2(\Omega_+) \oplus L^2(\Omega_-)$. In the terminology of boundary triplets, Weyl functions (or Q-functions and Dirichlet-to-Neumann maps) similar results can be found in, e.g., [2, Theorem 4.4], [8, Theorem 3.2], [12, Lemma 2.6] or [14, Theorem 8.6.3]. However, for the convenience of the reader we provide a short direct proof.

Proposition 2.3. *For $w' \in \mathbb{C} \setminus [0, \infty)$ let $SL_{w'}$ and $\mathcal{S}_{w'}$ be the integral operators defined in* (2.6) *and* (2.8). *Then, the resolvent formula*

$$(S^{\Omega_+} \oplus S^{\Omega_-} - w')^{-1} = R_{w'} - SL_{w'}\mathcal{S}_{w'}^{-1}(SL_{\overline{w'}})^*$$

holds.

Proof. Recall first that $\mathcal{S}_{w'}$, $w' \in \mathbb{C} \setminus [0,\infty)$, is bijective as an operator from $L^2(\Sigma)$ to $H^1(\Sigma)$ and that $(SL_{\overline{w'}})^* : L^2(\mathbb{R}^2) \to H^1(\Sigma)$; cf. (2.7). Now fix $w' \in \mathbb{C} \setminus [0,\infty)$, let $h \in L^2(\mathbb{R}^2)$ and consider

$$f := R_{w'} h - SL_{w'} \mathcal{S}_{w'}^{-1} (SL_{\overline{w'}})^* h. \tag{2.16}$$

Then it follows from the mapping properties of $R_{w'}$ and $SL_{w'}$ discussed below (2.3) and in [35, Lemma 2.6 (i)], respectively, that $f \in H^{3/2}(\mathbb{R}^2 \setminus \Sigma) \cap H^1(\mathbb{R}^2)$ and $\Delta f \in L^2(\Sigma)$. Moreover, applying the trace operator yields

$$\boldsymbol{t}_\Sigma f = \boldsymbol{t}_\Sigma R_{w'} h - \mathcal{S}_{w'} \mathcal{S}_{w'}^{-1} (SL_{\overline{w'}})^* h = (SL_{\overline{w'}})^* h - (SL_{\overline{w'}})^* h = 0,$$

and hence $f \in \operatorname{dom}(S^{\Omega_+} \oplus S^{\Omega_-})$. Furthermore, since $(-\Delta - w')(SL_w f)|_{\Omega_\pm} = 0$ holds, see [35, Lemma 2.6 (ii)], we conclude

$$(S^{\Omega_+} \oplus S^{\Omega_-} - w') f = h,$$

which together with (2.16) implies the resolvent formula. □

3. The nonrelativistic limit

In this section we show that the nonrelativistic limit of $H_\kappa^{\Omega_\pm, c}$ is given by the Dirichlet Laplacian on $\Omega_\pm$; cf. Theorem 3.6 and Corollary 3.7. In order to simplify the presentation we usually fix $z < 0$ and formulate the estimates and statements for all $c > 0$ sufficiently large; in fact, here we always tacitly assume that $c > 0$ is chosen such that $z + c^2/2 \in (-c^2/2, c^2/2)$ and $z + z^2/c^2 < 0$, and hence the resolvents of H^c and S and all corresponding integral operators are well defined; cf. (2.2).

We start by considering the nonrelativistic limit of the free Dirac operator. The following lemma is the two-dimensional counterpart of [12, Proposition 3.1]. For the convenience of the reader we provide a short direct proof.

Lemma 3.1. *Fix $z < 0$, let $R^c_{z+c^2/2}$ and R_z be the resolvents of H^c and S in Section 2.1, let $P_+ = \operatorname{diag}(1,0)$, and let $c > 0$ be sufficiently large. Then,*

$$\| R^c_{z+c^2/2} - R_z P_+ \|_{L^2(\mathbb{R}^2;\mathbb{C}^2) \to H^1(\mathbb{R}^2;\mathbb{C}^2)} \leq C c^{-1}.$$

Proof. According to (2.4) we have

$$\begin{aligned} R^c_{z+c^2/2} &= c^{-2}(H^c + (z + c^2/2) I_2) R_{z+z^2/c^2} I_2 \\ &= (-i(\sigma \cdot \nabla)/c + (z/c^2) I_2 + P_+) R_{z+z^2/c^2} I_2 \end{aligned}$$

and hence

$$R^c_{z+c^2/2} - R_z P_+ = c^{-1}(-i(\sigma \cdot \nabla) + (z/c) I_2) R_{z+z^2/c^2} I_2 + P_+ R_{z+z^2/c^2} I_2 - R_z P_+$$

For the first term on the right hand side we estimate

$$\begin{aligned} &\| c^{-1}(-i(\sigma \cdot \nabla) + (z/c) I_2) R_{z+z^2/c^2} I_2 \|_{L^2(\mathbb{R}^2;\mathbb{C}^2) \to H^1(\mathbb{R}^2;\mathbb{C}^2)} \\ &\leq c^{-1} \| -i(\sigma \cdot \nabla) + (z/c) I_2 \|_{H^2(\mathbb{R}^2;\mathbb{C}^2) \to H^1(\mathbb{R}^2;\mathbb{C}^2)} \| R_{z+z^2/c^2} \|_{L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)} \\ &\leq C c^{-1} \| R_{z+z^2/c^2} \|_{L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)} \end{aligned} \tag{3.1}$$

and for the remaining part we obtain

$$\begin{aligned}\|P_+R_{z+z^2/c^2}I_2 - R_zP_+\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to H^1(\mathbb{R}^2;\mathbb{C}^2)} \\ &= \frac{z^2}{c^2}\|R_{z+z^2/c^2}R_z\|_{L^2(\mathbb{R}^2)\to H^1(\mathbb{R}^2)} \\ &\le Cc^{-2}\|R_{z+z^2/c^2}\|_{L^2(\mathbb{R}^2)\to H^1(\mathbb{R}^2)}\|R_z\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \\ &\le Cc^{-2}\|R_{z+z^2/c^2}\|_{L^2(\mathbb{R}^2)\to H^2(\mathbb{R}^2)}.\end{aligned} \tag{3.2}$$

Furthermore, we use that R_z is bounded as an operator from $L^2(\mathbb{R}^2)$ to $H^2(\mathbb{R}^2)$ (see the comments below (2.3)) and $S-(z+z^2/c^2) \ge -(z+z^2/c^2) \ge |z|/2$, where the last estimate holds for $c>0$ sufficiently large. We then obtain

$$\begin{aligned}\|R_{z+z^2/c^2}\|_{L^2(\mathbb{R}^2)\to H^2(\mathbb{R}^2)} &= \|R_z(I+(z^2/c^2)R_{z+z^2/c^2})\|_{L^2(\mathbb{R}^2)\to H^2(\mathbb{R}^2)} \\ &\le \|R_z\|_{L^2(\mathbb{R}^2)\to H^2(\mathbb{R}^2)}\Big(1+\frac{z^2}{c^2}\|R_{z+z^2/c^2}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}\Big) \\ &\le \|R_z\|_{L^2(\mathbb{R}^2)\to H^2(\mathbb{R}^2)}\Big(1+\frac{2|z|}{c^2}\Big) \\ &\le C.\end{aligned} \tag{3.3}$$

Now, combining (3.1), (3.2), and (3.3) yields the assertion. □

In the following corollary we use the obtained results to compute the limit of $\Phi^c_{z+c^2/2}$ and $(\Phi^c_{z+c^2/2})^*$, as $c\to\infty$.

Corollary 3.2. *Fix $z<0$, let $\Phi^c_{z+c^2/2}$ and SL_z be the potential operators from (2.6), let $P_+=\operatorname{diag}(1,0)$, $\mathcal{M}_c=\operatorname{diag}(1,\sqrt{c})$, and let $c>0$ be sufficiently large. Then,*

$$\begin{aligned}\|(\Phi^c_{z+c^2/2})^*-(SL_z)^*P_+\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to H^{1/2}(\Sigma;\mathbb{C}^2)} &\le Cc^{-1}, \\ \|\Phi^c_{z+c^2/2}-SL_zP_+\|_{H^{-1/2}(\Sigma;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)} &\le Cc^{-1},\end{aligned}$$

and $\mathcal{M}_c(\Phi^c_{z+c^2/2})^*$ as well as $\Phi^c_{z+c^2/2}\mathcal{M}_c$ are uniformly bounded operators from $L^2(\mathbb{R}^2;\mathbb{C}^2)$ to $H^{1/2}(\Sigma;\mathbb{C}^2)$ and from $H^{-1/2}(\Sigma;\mathbb{C}^2)$ to $L^2(\mathbb{R}^2;\mathbb{C}^2)$, respectively.

Proof. The first inequality is a consequence of the previous lemma, (2.7) and the mapping properties of the trace operator discussed below (1.3). The second inequality follows by duality. Moreover, since $P_+\mathcal{M}_c=P_+$, the assertions regarding the uniform boundedness are simple consequences of the two inequalities. □

Having suitable estimates for the resolvent of the free Dirac operator, the potential operator and its adjoint, it remains to consider the convergence of the term $(c^{-1}\sigma_3+D\mathcal{C}^c_{z+c^2/2}D)^{-1}$; cf. Proposition 2.2 (iii).

Lemma 3.3. *Fix $z<0$, let $\mathcal{C}^c_{z+c^2/2}$ and $\mathcal{S}_z$ be the boundary integral operators from (2.8), let $P_+=\operatorname{diag}(1,0)$, and let $c>0$ be sufficiently large. Then,*

$$\|\mathcal{C}^c_{z+c^2/2}-\mathcal{S}_zP_+\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)} \le Cc^{-1}.$$

Proof. In the following we consider $c>0$ sufficiently large such that $z+z^2/c^2$ is contained in the compact set $\overline{B(z,|z|/2)}\subset\rho(S)$. Hence, by using the representation of $\mathcal{C}^c_{z+c^2/2}$ from (2.9) (with $w = z + c^2/2$) and the convergence estimates from Proposition 2.1 we obtain

$$\begin{aligned}\|\mathcal{C}^c_{z+c^2/2} - \mathcal{S}_z P_+\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)} &= \left\| \begin{pmatrix} \left(1+\frac{z}{c^2}\right)\mathcal{S}_{z+z^2/c^2} - \mathcal{S}_z & \frac{1}{c}\mathcal{W}_{z+z^2/c^2} \\ \frac{1}{c}(\mathcal{W}_{z+z^2/c^2})^* & \frac{z}{c^2}\mathcal{S}_{z+z^2/c^2} \end{pmatrix} \right\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)} \\ &\leq C\Big(c^{-1}\|\mathcal{W}_{z+z^2/c^2}\|_{L^2(\Sigma)\to L^2(\Sigma)} + c^{-2}\|\mathcal{S}_{z+z^2/c^2}\|_{L^2(\Sigma)\to L^2(\Sigma)} \\ &\qquad + \|\mathcal{S}_{z+z^2/c^2} - \mathcal{S}_z\|_{L^2(\Sigma)\to L^2(\Sigma)}\Big) \\ &\leq Cc^{-1}. \end{aligned}$$

□

Lemma 3.4. *Fix $z < 0$, let $\mathcal{C}^c_{z+c^2/2}$ and $\mathcal{S}_z$ be the boundary integral operators from (2.8), let $D = \operatorname{diag}(\sqrt{2}e^{\kappa/2}, \sqrt{2}e^{-\kappa/2})$, $\mathcal{M}_c = \operatorname{diag}(1, \sqrt{c})$, and let $c > 0$ be sufficiently large. Then,*

$$\|\mathcal{M}_c^{-1}(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\|_{L^2(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)} \leq C\sqrt{c},$$
$$\|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\mathcal{M}_c^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)} \leq C\sqrt{c}.$$

Proof. In the following let $f \in L^2(\Sigma;\mathbb{C}^2)$. Using (2.13) (with $w = z + c^2/2$) gives

$$\begin{aligned}&\|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)f\|^2_{L^2(\Sigma;\mathbb{C}^2)} \\ &\qquad \geq c^{-2}\|f\|^2_{L^2(\Sigma;\mathbb{C}^2)} + c^{-1}(D(\sigma_3\mathcal{C}^c_{z+c^2/2} + \mathcal{C}^c_{z+c^2/2}\sigma_3)Df, f)_{L^2(\Sigma;\mathbb{C}^2)}\end{aligned}$$

and from (2.14) (with $w = z + c^2/2$) and $P_+ = \operatorname{diag}(1, 0)$ we obtain

$$\sigma_3\mathcal{C}^c_{z+c^2/2} + \mathcal{C}^c_{z+c^2/2}\sigma_3 = \begin{pmatrix} 2+\frac{2z}{c^2} & 0 \\ 0 & -\frac{2z}{c^2} \end{pmatrix}\mathcal{S}_{z+z^2/c^2} = 2(P_+ + (z/c^2)\sigma_3)\mathcal{S}_{z+z^2/c^2},$$

so that

$$\begin{aligned}&\|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)f\|^2_{L^2(\Sigma;\mathbb{C}^2)} \\ &\qquad \geq c^{-1}\big(\big(c^{-1} + 2D(P_+ + (z/c^2)\sigma_3)\mathcal{S}_{z+z^2/c^2}D\big)f, f\big)_{L^2(\Sigma;\mathbb{C}^2)}.\end{aligned}$$

Using the Cauchy-Schwarz inequality we see that

$$\begin{aligned}&\big(\big(c^{-1} + 2D(P_+ + (z/c^2)\sigma_3)\mathcal{S}_{z+z^2/c^2}D\big)f, f\big)_{L^2(\Sigma;\mathbb{C}^2)} \\ &\quad = \big((c^{-1} + 2DP_+\mathcal{S}_zD)f, f\big)_{L^2(\Sigma;\mathbb{C}^2)} \\ &\qquad - \big(\big(2DP_+\mathcal{S}_zD - 2D(P_+ + (z/c^2)\sigma_3)\mathcal{S}_{z+z^2/c^2}D\big)f, f\big)_{L^2(\Sigma;\mathbb{C}^2)} \\ &\quad \geq \big((c^{-1} + 2DP_+\mathcal{S}_zD)f, f\big)_{L^2(\Sigma;\mathbb{C}^2)} \\ &\qquad - 2\|D(P_+\mathcal{S}_z - (P_+ + (z/c^2)\sigma_3)\mathcal{S}_{z+z^2/c^2})D\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\|f\|^2_{L^2(\Sigma;\mathbb{C}^2)},\end{aligned}$$

and hence

$$\begin{aligned}&\|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)f\|^2_{L^2(\Sigma;\mathbb{C}^2)} \\ &\quad \geq c^{-1}\big(\big(c^{-1} + 2DP_+\mathcal{S}_zD\big)f, f\big)_{L^2(\Sigma;\mathbb{C}^2)} \\ &\qquad - 2c^{-1}\|D(P_+\mathcal{S}_z - (P_+ + (z/c^2)\sigma_3)\mathcal{S}_{z+z^2/c^2})D\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\|f\|^2_{L^2(\Sigma;\mathbb{C}^2)}.\end{aligned} \tag{3.4}$$

It follows from Proposition 2.1 that

$$\|P_+\mathcal{S}_z - (P_+ + (z/c^2)\sigma_3)\mathcal{S}_{z+z^2/c^2}\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)} \leq Cc^{-2}$$

and, in particular, for $c>0$ sufficiently large we conclude

$$\|D(P_+\mathcal{S}_z - (P_+ + (z/c^2)\sigma_3)\mathcal{S}_{z+z^2/c^2})D\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)} \leq \frac{c^{-1}}{4}.$$

Plugging this observation in (3.4) yields

$$\begin{aligned}
&\|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)f\|^2_{L^2(\Sigma;\mathbb{C}^2)}\\
&\quad\geq c^{-1}\big((c^{-1} + 2DP_+\mathcal{S}_zD)f,f\big)_{L^2(\Sigma;\mathbb{C}^2)} - \frac{c^{-2}}{2}\|f\|^2_{L^2(\Sigma;\mathbb{C}^2)}\\
&\quad\geq c^{-1}\big((c^{-1}/2 + 2DP_+\mathcal{S}_zD)f,f\big)_{L^2(\Sigma;\mathbb{C}^2)}.
\end{aligned}$$

We continue estimating and make use of [43, Lemma 3.2] in the penultimate inequality below

$$\begin{aligned}
&\|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)f\|^2_{L^2(\Sigma;\mathbb{C}^2)}\\
&\quad\geq c^{-1}\big((\mathcal{M}_c^{-2}P_-/2 + 2DP_+\mathcal{S}_zD)f,f\big)_{L^2(\Sigma;\mathbb{C}^2)}\\
&\quad= c^{-1}\big(\|\mathcal{M}_c^{-1}P_-f\|^2_{L^2(\Sigma;\mathbb{C}^2)}/2 + 2\big(\mathcal{S}_zP_+Df,P_+Df\big)_{L^2(\Sigma;\mathbb{C}^2)}\big)\\
&\quad= c^{-1}\big(\|\mathcal{M}_c^{-1}P_-f\|^2_{L^2(\Sigma;\mathbb{C}^2)}/2 + +2\big(\mathcal{S}_zP_+D\mathcal{M}_c^{-1}f,P_+D\mathcal{M}_c^{-1}f\big)_{L^2(\Sigma;\mathbb{C}^2)}\big)\\
&\quad\geq Cc^{-1}\big(\|\mathcal{M}_c^{-1}P_-f\|^2_{H^{-1/2}(\Sigma;\mathbb{C}^2)} + \|P_+D\mathcal{M}_c^{-1}f\|^2_{H^{-1/2}(\Sigma;\mathbb{C}^2)}\big)\\
&\quad\geq Cc^{-1}\|\mathcal{M}_c^{-1}f\|^2_{H^{-1/2}(\Sigma;\mathbb{C}^2)}.
\end{aligned}$$

Recall from Proposition 2.2 (iii) that the operator $c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D$ is boundedly invertible in $L^2(\Sigma;\mathbb{C}^2)$ since $z + c^2/2 \in \rho(H^c)$ for $c>0$ sufficiently large. Now, let $g\in L^2(\Sigma;\mathbb{C}^2)$ and $f = (c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}g \in L^2(\Sigma;\mathbb{C}^2)$. Then,

$$\begin{aligned}
\|\mathcal{M}_c^{-1}(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}g\|_{H^{-1/2}(\Sigma;\mathbb{C}^2)} &= \|\mathcal{M}_c^{-1}f\|_{H^{-1/2}(\Sigma;\mathbb{C}^2)}\\
&\leq C\sqrt{c}\|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)f\|_{L^2(\Sigma;\mathbb{C}^2)}\\
&= C\sqrt{c}\|g\|_{L^2(\Sigma;\mathbb{C}^2)},
\end{aligned}$$

which proves the first norm estimate. The second norm estimate follows by duality. □

Lemma 3.5. *Fix $z<0$, let $\mathcal{C}^c_{z+c^2/2}$ and $\mathcal{S}_z$ be the boundary integral operators from* (2.8)*, let $D = \operatorname{diag}(\sqrt{2}e^{\kappa/2}, \sqrt{2}e^{-\kappa/2})$, $P_+ = \operatorname{diag}(1,0)$, and let $c>0$ be sufficiently large. Then,*

$$\|P_+(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}P_+ - D^{-1}\mathcal{S}_z^{-1}P_+D^{-1}\|_{H^1(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)} \leq Cc^{-1/2}.$$

Proof. The identity

$$\begin{aligned}
&P_+(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}P_+ - D^{-1}\mathcal{S}_z^{-1}P_+D^{-1}\\
&= P_+(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}(D\mathcal{S}_zP_+D - c^{-1}\sigma_3 - D\mathcal{C}^c_{z+c^2/2}D)D^{-1}\mathcal{S}_z^{-1}P_+D^{-1}\\
&= P_+\mathcal{M}_c^{-1}(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}(D\mathcal{S}_zP_+D - c^{-1}\sigma_3 - D\mathcal{C}^c_{z+c^2/2}D)D^{-1}\mathcal{S}_z^{-1}P_+D^{-1}
\end{aligned}$$

together with the estimates from the two previous lemmas and the fact that $\mathcal{S}_z$ acts as an isomorphic operator from $L^2(\Sigma)$ to $H^1(\Sigma)$ (see Proposition 2.3) leads to the assertion. □

Theorem 3.6. *Let $H_\kappa^{\Omega_\pm,c}$ be defined by (2.11) and let $S^{\Omega_\pm}$ be the Dirichlet Laplacian on $\Omega_\pm$ from (2.15). Fix $z<0$, let $P_+=\operatorname{diag}(1,0)$, and let $c>0$ be sufficiently large. Then,*

$$\begin{aligned}\|(H_\kappa^{\Omega_+,c}\oplus H_\kappa^{\Omega_-,c}-(z+c^2/2))^{-1}&\\ -(S^{\Omega_+}\oplus S^{\Omega_-}-z)^{-1}P_+\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}&\leq Cc^{-1/2}.\end{aligned}$$

Proof. Applying the resolvent formulas from Proposition 2.2 (iii) and Proposition 2.3 gives us

$$\begin{aligned}(H_\kappa^{\Omega_+,c}\oplus H_\kappa^{\Omega_-,c}&-(z+c^2/2))^{-1}-(S^{\Omega_+}\oplus S^{\Omega_-}-z)^{-1}P_+\\ &=R^c_{z+c^2/2}-R_zP_+\\ &\quad-\Phi^c_{z+c^2/2}D(c^{-1}\sigma_3+D\mathcal{C}_{z+c^2/2}D)^{-1}D(\Phi^c_{z+c^2/2})^*\\ &\quad+SL_zP_+DD^{-1}\mathcal{S}_z^{-1}P_+D^{-1}D(SL_z)^*P_+,\end{aligned}$$

which can be rewritten as

$$(H_\kappa^{\Omega_+,c}\oplus H_\kappa^{\Omega_-,c}-(z+c^2/2))^{-1}-(S^{\Omega_+}\oplus S^{\Omega_-}-z)^{-1}P_+=\sum_{j=1}^4 D_j \tag{3.5}$$

with

$$\begin{aligned}D_1&:=R^c_{z+c^2/2}-R_zP_+,\\ D_2&:=(SL_zP_+-\Phi^c_{z+c^2/2})D(c^{-1}\sigma_3+D\mathcal{C}^c_{z+c^2/2}D)^{-1}D(\Phi^c_{z+c^2/2})^*,\\ D_3&:=SL_zP_+D(c^{-1}\sigma_3+D\mathcal{C}^c_{z+c^2/2}D)^{-1}D((SL_z)^*P_+-(\Phi^c_{z+c^2/2})^*),\\ D_4&:=SL_zP_+D\big(D^{-1}\mathcal{S}_z^{-1}P_+D^{-1}-(c^{-1}\sigma_3+D\mathcal{C}^c_{z+c^2/2}D)^{-1}\big)D(SL_z)^*P_+.\end{aligned}$$

Next, we estimate $D_1,\dots,D_4$ separately. It follows from Lemma 3.1 that D_1 can be estimated by

$$\|D_1\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\leq Cc^{-1}.$$

Furthermore, with Corollary 3.2 and Lemma 3.4 we obtain for D_2

$$\begin{aligned}
&\|D_2\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\quad= \|(SL_zP_+ - \Phi^c_{z+c^2/2})D(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}D(\Phi^c_{z+c^2/2})^*\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\quad= \|(SL_zP_+ - \Phi^c_{z+c^2/2})D\\
&\qquad\cdot (c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\mathcal{M}_c^{-1}D\mathcal{M}_c(\Phi^c_{z+c^2/2})^*\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\quad\le \|(SL_zP_+ - \Phi^c_{z+c^2/2})D\|_{L^2(\Sigma;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\qquad\cdot \|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\mathcal{M}_c^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\\
&\qquad\cdot \|D\mathcal{M}_c(\Phi^c_{z+c^2/2})^*\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to H^{1/2}(\Sigma;\mathbb{C}^2)}\\
&\quad\le \|(SL_zP_+ - \Phi^c_{z+c^2/2})D\|_{H^{-1/2}(\Sigma;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\qquad\cdot \|(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\mathcal{M}_c^{-1}\|_{H^{1/2}(\Sigma;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\\
&\qquad\cdot \|D\mathcal{M}_c(\Phi^c_{z+c^2/2})^*\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to H^{1/2}(\Sigma;\mathbb{C}^2)}\\
&\quad\le Cc^{-1}c^{1/2}\\
&\quad\le Cc^{-1/2}.
\end{aligned}$$

Next, let us consider D_3. Similar as above we get

$$\begin{aligned}
\|D_3\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)} &= \|SL_zP_+D(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\\
&\qquad\cdot D((SL_z)^*P_+ - (\Phi^c_{z+c^2/2})^*)\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&= \|SL_zP_+D\mathcal{M}_c^{-1}(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\\
&\qquad\cdot D((SL_z)^*P_+ - (\Phi^c_{z+c^2/2})^*)\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\le \|SL_zP_+D\|_{H^{-1/2}(\Sigma;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\qquad\cdot \|\mathcal{M}_c^{-1}(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\|_{L^2(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)}\\
&\qquad\cdot \|D((SL_z)^*P_+ - (\Phi^c_{z+c^2/2})^*)\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\Sigma;\mathbb{C}^2)}\\
&\le \|SL_zP_+D\|_{H^{-1/2}(\Sigma;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\qquad\cdot \|\mathcal{M}_c^{-1}(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\|_{L^2(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)}\\
&\qquad\cdot \|D((SL_z)^*P_+ - (\Phi^c_{z+c^2/2})^*)\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to H^{1/2}(\Sigma;\mathbb{C}^2)}\\
&\le Cc^{1/2}c^{-1}\\
&\le Cc^{-1/2}.
\end{aligned}$$

Finally, we estimate D_4 using Lemma 3.5 and the mapping properties of SL_z (see below (2.7)) as follows

$$\begin{aligned}
&\|D_4\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\quad= \|SL_zP_+D\big(D^{-1}\mathcal{S}_z^{-1}P_+D^{-1} - (c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}\big)\\
&\qquad\cdot D(SL_z)^*P_+\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\quad\le \|SL_zP_+D\|_{H^{-1/2}(\Sigma;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)}\\
&\qquad\cdot \big\|D^{-1}\mathcal{S}_z^{-1}P_+D^{-1} - P_+(c^{-1}\sigma_3 + D\mathcal{C}^c_{z+c^2/2}D)^{-1}P_+\big\|_{H^1(\Sigma;\mathbb{C}^2)\to H^{-1/2}(\Sigma;\mathbb{C}^2)}\\
&\qquad\cdot \|D(SL_z)^*P_+\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to H^1(\Sigma;\mathbb{C}^2)}\\
&\quad\le Cc^{-1/2}.
\end{aligned}$$

These four separate estimates together with (3.5) lead to the assertion. $\square$

As an immediate consequence we obtain that the Dirichlet Laplacian on $\Omega_\pm$ is the nonrelativistic limit of $H_\kappa^{\Omega_\pm,c}$ from (2.11).

Corollary 3.7. *Let $H_\kappa^{\Omega_\pm,c}$ be defined by (2.11) and let $S^{\Omega_\pm}$ be the Dirichlet Laplacian on $\Omega_\pm$ from (2.15). Fix $z<0$, let $P_+ = \operatorname{diag}(1,0)$, and let $c>0$ be sufficiently large. Then,*

$$\|(H_\kappa^{\Omega_\pm,c} - (z+c^2/2))^{-1} - (S^{\Omega_\pm} - z)^{-1} P_+\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)\to L^2(\mathbb{R}^2;\mathbb{C}^2)} \leq C c^{-1/2}.$$

As an interesting application of Corollary 3.7 we conclude below that the Faber-Krahn inequality is valid for $H_\kappa^{\Omega_+,c}$ if $c>0$ is sufficiently large. Here, the essential observation is that by Corollary 3.7 the positive eigenvalues of $H_\kappa^{\Omega_+,c} - c^2/2$ converge to the eigenvalues of S^{Ω_+}; this can be shown in the same way as in [12, Section 3.4]. More precisely, if we denote the positive discrete eigenvalues of $H_\kappa^{\Omega_+,c}$ by

$$\frac{c^2}{2} \leq \lambda_1(H_\kappa^{\Omega_+,c}) \leq \lambda_2(H_\kappa^{\Omega_+,c}) \leq \dots,$$

cf. Proposition 2.2 (i), and the positive discrete eigenvalues of S^{Ω_+}, which are the only elements in the spectrum of S^{Ω_+}, see, e.g., [53, below eq. (10.34)], by

$$0 < \mu_1(S^{\Omega_+}) \leq \mu_2(S^{\Omega_+}) \leq \dots,$$

then for $j \in \mathbb{N}$

$$\lambda_j(H_\kappa^{\Omega_+,c}) - \frac{c^2}{2} \to \mu_j(S^{\Omega_+}) \quad \text{as } c \to \infty. \tag{3.6}$$

Based on this observation one can transfer spectral geometry results and spectral inequalities from the Dirichlet Laplacian on Ω_+ to $H_\kappa^{\Omega_+,c}$ for $c>0$ sufficiently large. As an example, we present the Faber-Krahn inequality for Dirac operators for sufficiently large $c>0$; see [12, Corollary 1.3] for the three-dimensional analogue and further eigenvalue inequalities.

Corollary 3.8. *Let $H_\kappa^{\Omega_+,c}$ be defined by (2.11) and let $\lambda_1(H_\kappa^{\Omega_+,c})$ be the first positive eigenvalue $H_\kappa^{\Omega_+}$. Furthermore, let $H_\kappa^{\mathbb{D},c}$ and $\lambda_1(H_\kappa^{\mathbb{D},c})$ be the same quantities for the choice $\Omega_+ = \mathbb{D}$, where $\mathbb{D}$ denotes a disc in $\mathbb{R}^2$ with $|\mathbb{D}| = |\Omega_+|$, and $c>0$ be sufficiently large. Then, $\lambda_1(H_\kappa^{\mathbb{D},c}) \leq \lambda_1(H_\kappa^{\Omega_+,c})$ with equality for all c large enough if and only if $\Omega_+ = \mathbb{D}$.*

Proof. The statement follows from the classical Faber-Krahn inequality for the Dirichlet Laplacian in [32, 39], see also [34, Theorem 3.2.1], and (3.6). □

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