Eigenvalues of Schrödinger operators and Dirichlet-to-Neumann maps

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Eigenvalues and eigenspaces of selfadjoint Schrödinger operators on \( \mathbb{R}^n \) are expressed in terms of Dirichlet-to-Neumann maps corresponding to Schrödinger operators on the upper and lower half space.

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1 Introduction

It is known that the eigenvalues of a Schrödinger operator \( A_D \) with Dirichlet boundary condition on a bounded domain \( \Omega \subset \mathbb{R}^n \) with a bounded, real-valued potential \( V \) coincide with the poles of the meromorphic operator function \( \mu \mapsto M^{\Omega}(\mu) \), where \( M^{\Omega}(\mu) \) is the Dirichlet-to-Neumann map of \(-\Delta + V - \mu\), see, e.g., [1, 2]. Moreover, for each eigenvalue \( \lambda \) the map

\[
\tau : \ker(A_D - \lambda) \to \text{ran} \, \text{Res}_\lambda M^\Omega, \quad u \mapsto \partial_\nu u|_{\partial \Omega}
\]

(\( \partial_\nu u|_{\partial \Omega} \) denotes the trace of the normal derivative of \( u \) at the boundary \( \partial \Omega \)) is an isomorphism between the eigenspace and the range of the residue of \( M^\Omega \) at \( \lambda \); cf. [2]. Such a result is also desirable for a selfadjoint Schrödinger operator \( A = -\Delta + V \) in \( L^2(\mathbb{R}^n) \), \( n \geq 2 \). In order to define an operator function which plays the role of \( M^\Omega \) we introduce the artificial “boundary” \( \Sigma := \mathbb{R}^{n-1} \times \{0\} \), which separates \( \mathbb{R}^n \) into \( \mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, \infty) \) and \( \mathbb{R}^n_- := \mathbb{R}^{n-1} \times (-\infty, 0) \), and consider the Dirichlet-to-Neumann maps \( M^{\pm}(\mu) \) in \( L^2(\Sigma) \) corresponding to the Schrödinger operators \(-\Delta + V - \mu \) on \( \mathbb{R}^n_\pm \), respectively. A natural candidate for the description of the eigenvalues of \( A \) is \( M(\mu) := (M^+(\mu) + M^-(\mu))^{-1} \); cf. [3] for a similar function defined in the case that \( \Sigma \) is a sphere. In Theorem 2.1 of this note we show that each pole of \( M \) is an eigenvalue of \( A \) but in general the analog of the map \( \tau \) is not bijective. We indicate in Theorem 2.2 that this drawback can be avoided by considering a certain \( 2 \times 2 \) block operator matrix function with entries formed by \( M^\pm \) and \( M \).

2 Characterization of eigenvalues and eigenspaces with Dirichlet-to-Neumann maps

Let \( n \geq 2 \) and denote by \( H^s(\mathbb{R}^n) \) and \( H^s(\Sigma) \) the Sobolev spaces of order \( s > 0 \) on \( \mathbb{R}^n \) and \( \Sigma \), respectively. Moreover, let \( V \in L^\infty(\mathbb{R}^n) \) be a real-valued potential. We consider the selfadjoint Schrödinger operator

\[
A u = -\Delta u + Vu, \quad \text{dom } A = H^2(\mathbb{R}^n),
\]

in \( L^2(\mathbb{R}^n) \). For \( \mu \) in the resolvent set \( \rho(A) \) of \( A \) we define

\[
\mathcal{N}^\pm_\mu := \{u^\pm_\mu \in H^2(\mathbb{R}^n_\pm) : (-\Delta + V - \mu)u^\pm_\mu = 0\},
\]

\[
\mathcal{N}_\mu := \{v_\mu^+ \oplus v_\mu^- \in \mathcal{N}^+_\mu \oplus \mathcal{N}^-_\mu : v_\mu^+|_\Sigma = v_\mu^-|_\Sigma\},
\]

where \( v|_\Sigma \) denotes the trace of a Sobolev function \( v \) at \( \Sigma \). Let \( \partial_\nu v := \frac{\partial v}{\partial \nu} \). One can show, that for every \( g \in H^{\frac{1}{2}}(\Sigma) \) there exists a unique element \( u_\mu \in \mathcal{N}_\mu \) with \( \partial_\nu u_\mu^+|_\Sigma = \partial_\nu u_\mu^-|_\Sigma = g \). Hence the operator-valued function \( M \) defined via

\[
\rho(A) \ni \mu \mapsto M(\mu), \quad M(\mu)(\partial_\nu u_\mu^+|_\Sigma - \partial_\nu u_\mu^-|_\Sigma) := u_\mu|_\Sigma
\]

is well-defined. \( M(\mu) \) is a bounded operator in \( L^2(\Sigma) \) with domain \( H^{\frac{1}{2}}(\Sigma) \) and range in \( H^{\frac{1}{2}}(\Sigma) \) for every \( \mu \in \rho(A) \). Moreover, for every \( g \in H^{\frac{1}{2}}(\Sigma) \) the function \( \mu \mapsto M(\mu)g \) is holomorphic and has poles of at most order one; cf. [2]. Note that for \( \mu \in \mathbb{C} \setminus \rho(A) \) the operator \( M(\mu) \) coincides with \((M^+(\mu) + M^-(\mu))^{-1}\), where \( M^\pm(\mu) \) denotes the Dirichlet-to-Neumann map with respect to \(-\Delta + V - \mu \) on \( \mathbb{R}^n_\pm \), i.e. \( M^\pm(\mu)u_\mu^+|_\Sigma = \mp \partial_\nu u_\mu^-|_\Sigma \) for \( u_\mu^\pm \in \mathcal{N}^\pm_\mu \), respectively.

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Theorem 2.1 If $\lambda \in \mathbb{R}$ is a pole of $M$ then $\lambda$ is an eigenvalue of $A$, but in general $\dim \text{ran } \text{Res}_\lambda M \leq \dim \ker(A - \lambda)$.

Proof. Let $\lambda \in \mathbb{R}$ be a pole of $M$. We show $\dim \ker(A - \lambda) \geq \dim \text{ran } \text{Res}_\lambda M$, from which, in particular, the first assertion follows. Let $\mu, \nu, \zeta \in \mathbb{C} \setminus \mathbb{R}$ be distinct and let $g \in H_1^2(\Sigma)$. For $j, k \in \{\mu, \nu, \zeta\}$ denote by $u_j$ the unique element in $N_j$ with $\partial_n u_j |_{\Sigma} - \partial_n u_j^+ |_{\Sigma} = g$ and choose $u_k$ analogously. Due to $u_j - u_k \in \text{dom } A$ and

$$ (A - j)(u_j - u_k) = (-\Delta + V - j)(u_j^+ - u_k) \oplus (-\Delta + V - j)(u_j^+ - u_k) = (j - k)u_k $$

we obtain $(A - j)^{-1}u_k \equiv \frac{u_j - u_k}{j - k}$ if $j \neq k$. Hence we get

$$ \left. \frac{1}{z - \nu} \left( (A - \mu)^{-1}u_\nu \right) \right|_{\Sigma} = \left. \frac{1}{z - \nu} \left( (A - \mu)^{-1}(u_\nu - u_\nu) \right) \right|_{\Sigma} = \left. \frac{1}{z - \nu} \left( \left[ \frac{u_\mu - u_\nu}{\mu - \nu} \right] \right) \right|_{\Sigma}, $$

and hence

$$ \left. \frac{1}{z - \nu} \left[ M(\mu) g - M(\nu) g \right] \right|_{\Sigma} = \left. \frac{1}{z - \nu} \left[ M(\mu) g - M(\nu) g \right] \right|_{\Sigma}. $$

By the spectral theorem one gets $i P u_\nu \equiv \lim_{\eta \to 0} \eta (A - (\lambda + i \eta))^{-1} u_\nu$, where $P$ denotes the orthogonal projection in $L^2(\mathbb{R}^n)$ onto $\ker(A - \lambda)$. As the map $v \mapsto [(A - \mu)^{-1}v] |_{\Sigma}$ is continuous from $L^2(\mathbb{R}^n)$ to $L^2(\Sigma)$ we get for $z = \lambda + i \eta$

$$ \left. \frac{1}{z - \nu} \left[ M(\mu) g - M(\nu) g \right] \right|_{\Sigma} = \lim_{\eta \to 0} \frac{i \eta}{(z - \mu)(z - \nu)} M(z) g = \text{Res}_\lambda M g. $$

We have shown $\{u|_{\Sigma} : u \in P N_{\nu}\} = \text{ran } \text{Res}_\lambda M$, hence $\dim \ker(A - \lambda) \geq \dim \text{ran } \text{Res}_\lambda M$. In general equality does not hold. For example for a potential $V$ reflection symmetric with respect to $\Sigma$ (i.e., $V(x', x_n) = V(x', -x_n)$) eigenfunctions with vanishing traces on $\Sigma$ may exist.

In order to characterize all eigenvalues and eigenspaces of $A$ we define the block operator matrix function $\mathcal{M}$ via

$$ \mu \mapsto \mathcal{M}(\mu) := \begin{pmatrix} M(\mu) & -M(\mu) M(\mu) \\ -M(\mu) M(\mu) & M(\mu) M(\mu) \end{pmatrix}, \quad \mu \in \mathbb{C} \setminus \mathbb{R}. $$

$\mathcal{M}(\mu)$ is an operator in $L^2(\Sigma) \times L^2(\Sigma)$ with domain $H_1^2(\Sigma) \times H_1^2(\Sigma)$ and range in $H_1^2(\Sigma) \times H_1^2(\Sigma)$. The function $\mathcal{M}$ is holomorphic in the strong sense and can be extended to a strongly holomorphic function (also denoted by $\mathcal{M}$) defined on $\rho(A)$. Similar functions were already considered in, e.g., [5] for the ODE case and in [6, 7] in an abstract setting.

Theorem 2.2 $\lambda \in \mathbb{R}$ is a pole of $\mathcal{M}$ and $\text{ran } \text{Res}_\lambda \mathcal{M}$ is finite-dimensional if and only if $\lambda$ is an isolated eigenvalue of $A$ with finite multiplicity. In this case the map

$$ \mathcal{T} : \ker(A - \lambda) \to \text{ran } \text{Res}_\lambda \mathcal{M}, \quad u \mapsto \left[ u|_{\Sigma}, -\partial_n u|_{\Sigma} \right]^T. $$

is bijective.

We omit the proof of Theorem 2.2, which uses methods similar to the proof of Theorem 2.1 and a unique continuation argument; cf. [4] for a similar reasoning.

Remark 2.3 With the help of the function $\mathcal{M}$ one can even characterize all (embedded and isolated) eigenvalues and the corresponding eigenspaces of $A$; cf. [4] for the case of a Schrödinger operator on an exterior domain.

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References