On the adjoint of a symmetric operator

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Abstract

In general it is a non-trivial task to determine the adjoint S^* of an unbounded symmetric operator S in a Hilbert or Krein space. We propose a method to specify S^* explicitly which makes use of two boundary mappings that satisfy an abstract Green's identity, a surjectivity condition and give rise to a self-adjoint extension of S. We show for various concrete examples how convenient and easily applicable this technique is.

AMS Subject Classification (2000). Primary 47B25;
Secondary 47A06, 47B50, 47E05, 47F05.

1 Introduction

Let S be a densely defined unbounded linear operator in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and suppose that S is symmetric, i.e. $\langle Sf, g\rangle = \langle f, Sg\rangle$ for all $f, g \in \operatorname{dom} S$, or, equivalently, the adjoint operator S^* is an extension of S. For many investigations, e.g. for the description of the self-adjoint extensions of S and their spectral properties, it is necessary to determine the adjoint operator S^*. Within the classical extension theory due to J. von Neumann [27] this is possible in an implicit way in terms of deficiency elements, but apart from just using the definition there exists no general technique to compute the domain $\operatorname{dom} S^*$ of the adjoint operator explicitly and to specify the action of S^* on elements $f \in \operatorname{dom} S^*$ which do not belong to the domain of the symmetric operator S.

In this paper we propose an abstract method to verify that a given operator is the adjoint of a symmetric operator, and in various examples we show how convenient this technique is. We illustrate the approach for the well-known case of a Sturm–Liouville operator on a finite interval. Suppose that $q \in L^1(a, b)$ is a real-valued function, let $\ell(f) := -f'' + qf$ and denote by $\mathcal{D}_{\max}$ the linear space of absolutely continuous functions $f \in L^2(a, b)$ with absolutely continuous derivatives f' such that $\ell(f)$ also belongs to $L^2(a, b)$. The maximal operator $S_{\max} f = \ell(f)$ is defined for all $f \in \mathcal{D}_{\max}$ and the minimal operator is defined as

$$S_{\min} f = \ell(f), \quad \operatorname{dom} S_{\min} = \big\{ f \in \mathcal{D}_{\max} : f(a) = f'(a) = f(b) = f'(b) = 0 \big\}.$$

Integration by parts shows that $S_{\min}$ is a symmetric operator in the Hilbert space $(L^2(a, b), \langle\cdot,\cdot\rangle)$ and that the maximal operator $S_{\max}$ is a restriction of $S^*_{\min}$. The fact that $S_{\max}$ coincides with $S^*_{\min}$ is much more difficult to verify and requires a deeper analysis. According to our main result Theorem 2.3 and Corollary 2.5 the following suffices to prove $S_{\max} = S^*_{\min}$: find two boundary mappings Γ_0 and Γ_1 mapping $\mathcal{D}_{\max}$ into some suitable boundary Hilbert space

$(\mathcal{G}, (\cdot,\cdot))$ such that

$$\langle S_{\max} f, g\rangle - \langle f, S_{\max} g\rangle = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) \tag{1.1}$$

holds for all $f, g \in \mathcal{D}_{\max}$, the map $(\Gamma_0, \Gamma_1)^\top : \mathcal{D}_{\max} \to \mathcal{G} \oplus \mathcal{G}$ is onto and the restriction of $S_{\max}$ to $\ker \Gamma_0$ contains (and then automatically coincides with) a self-adjoint operator. By computing the left-hand side in (1.1) it is easy to read off that

$$\mathcal{G} = \mathbb{C}^2, \qquad \Gamma_0 f = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f'(a) \\ -f'(b) \end{pmatrix}$$

is a possible choice. Indeed, the identity (1.1) and the surjectivity condition are obviously satisfied and so it remains to check that the boundary conditions $f(a) = f(b) = 0$ determine a self-adjoint differential operator. This can be seen elementary by checking that for all $g \in L^2(a,b)$ and some $\lambda_\pm \in \mathbb{C}^\pm$ the differential equation $\ell(f) - \lambda_\pm f = g$ has a solution $f \in \mathcal{D}_{\max}$ which satisfies $f(a) = f(b) = 0$.

Sometimes one does not start with a symmetric operator but with some maximal operator T and poses the question whether T is the adjoint of a symmetric operator. Theorem 2.3 can be used to answer this question affirmatively and to find this symmetric operator using boundary mappings. We emphasize that T is not assumed to be closed and that the boundary mappings are not assumed to be bounded with respect to the graph norm of T, but both properties follow from the statement. The method proposed in Theorem 2.3 is inspired by the theory of isometric and unitary operators between indefinite inner product spaces (see, e.g. [5, 13, 31]) and the concept of boundary triples used in extension theory of symmetric operators, cf. [12, 14, 17]. Very roughly speaking, we trace back the problem to determine the adjoint to the much easier problem to check self-adjointness. There are many abstract and concrete results about self-adjointness in the literature but hardly any that show that an operator is the adjoint of a symmetric operator.

The paper is organized as follows. Section 2 contains the main result on the adjoint of a symmetric operator (Theorem 2.3). Since we also want to cover the case of non-densely defined symmetric operators, we formulate the results in the more general language of linear relations (the operator case is formulated in Corollary 2.5). Moreover, we allow the linear relation to act in a Krein space rather than a Hilbert space; a Krein space is a space with an indefinite inner product which is the direct and orthogonal sum of a Hilbert and an anti-Hilbert space. In a couple of remarks at the end of Section 2 various alternative sufficient conditions for the applicability of Theorem 2.3 and Corollary 2.5 are given. In Section 3 we apply this technique to various problems. First we consider as a simple well known example a Sturm–Liouville differential expression which is regular or in the limit circle case at both end-points. As a trickier problem we investigate a block operator matrix with first and second order differential expressions as entries in Section 3.2. Such type of block operator matrices have been considered from a different point of view in many papers, cf. [1, 21, 22]. The case of a uniformly elliptic second order differential expression (with an indefinite weight function) on a bounded domain is treated in Section 3.3 in a similar way as in [9, 18, 28]. Our last example on multiplication operators in

L^2-spaces is more abstract and is connected with functional models for operator-valued Nevanlinna or Riesz–Herglotz functions, see [6, 15, 26]. We note that here the symmetric operator is in general non-densely defined.

2 Main theorem

Let $\mathcal{H}$ be a Hilbert or Krein space with inner product $\langle\cdot,\cdot\rangle$ and equip the space $\mathcal{H}^2 \equiv \mathcal{H}\oplus\mathcal{H}$ with the usual inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}^2}$. In the case that $(\mathcal{H},\langle\cdot,\cdot\rangle)$ is a Krein space, all topological notions in $\mathcal{H}$ and $\mathcal{H}^2$ are understood with respect to some Hilbert space norm $\|\cdot\|$ such that $\langle\cdot,\cdot\rangle$ is $\|\cdot\|$-continuous. By $\mathcal{L}(\mathcal{H})$ we denote the set of bounded linear operators defined on $\mathcal{H}$.. In the following we will frequently make use of an indefinite inner product $[\![\cdot,\cdot]\!]_{\mathcal{H}^2}$ on $\mathcal{H}^2$ defined by

$$[\![\hat f,\hat g]\!]_{\mathcal{H}^2} := \langle \mathcal{J}_{\mathcal{H}}\hat f,\hat g\rangle_{\mathcal{H}^2}, \quad \mathcal{J}_{\mathcal{H}} := \begin{pmatrix} 0 & -iI_{\mathcal{H}} \\ iI_{\mathcal{H}} & 0\end{pmatrix}, \quad \hat f,\hat g\in\mathcal{H}^2; \tag{2.1}$$

or explicitly:

$$\left[\!\!\left[\begin{pmatrix} f_1\\ f_2\end{pmatrix},\begin{pmatrix} g_1\\ g_2\end{pmatrix}\right]\!\!\right]_{\mathcal{H}^2} = i\langle f_1,g_2\rangle - i\langle f_2,g_1\rangle, \qquad \begin{pmatrix} f_1\\ f_2\end{pmatrix},\begin{pmatrix} g_1\\ g_2\end{pmatrix}\in\mathcal{H}^2.$$

Observe that $[\![\cdot,\cdot]\!]_{\mathcal{H}^2}$ is an indefinite inner product also in the case when $(\mathcal{H},\langle\cdot,\cdot\rangle)$ is a Hilbert space.

In this note we study linear operators and, more generally, linear relations in the space $\mathcal{H}$, see, e.g. [4, 16]. Recall that a linear relation in $\mathcal{H}$ is a linear subspace of $\mathcal{H}^2$ and that a linear operator can always be identified with a linear relation via its graph. The elements of a linear relation T are pairs denoted by $\hat f = \{f;f'\}\in T$. A linear relation T is said to be *closed* if T is closed as a subspace of $\mathcal{H}^2$. The *domain*, *kernel*, *range* and *multivalued part* of a linear relation T in $\mathcal{H}$ are defined as

$$\begin{aligned} \operatorname{dom}T &= \{f\in\mathcal{H}\colon \exists\, f' \text{ s.t. } \{f;f'\}\in T\}, & \ker T &= \{f\in\mathcal{H}\colon \{f;0\}\in T\},\\ \operatorname{ran}T &= \{f'\in\mathcal{H}\colon \exists\, f \text{ s.t. } \{f;f'\}\in T\}, & \operatorname{mul}T &= \{f'\in\mathcal{H}\colon \{0;f'\}\in T\}, \end{aligned}$$

respectively. Obviously, a linear relation T is (the graph of) an operator if and only if $\operatorname{mul}T=\{0\}$. The *inverse* of a linear relation T is defined as

$$T^{-1} = \big\{\{f';f\}\colon \{f;f'\}\in T\big\};$$

note that $\operatorname{ran}T = \operatorname{dom}T^{-1}$ and $\operatorname{mul}T = \ker T^{-1}$ hold.

Let T be a closed linear relation in $\mathcal{H}$. A point $\lambda\in\mathbb{C}$ belongs to the *resolvent set* $\rho(T)$ if

$$(T-\lambda)^{-1} = \big\{\{f'-\lambda f;f\}\colon \{f;f'\}\in T\big\}$$

is an everywhere defined bounded operator. The *spectrum* $\sigma(T)$ is the complement of $\rho(T)$ in $\mathbb{C}$. It is not difficult to check that for $\lambda\in\mathbb{C}$ such that $\ker(T-\lambda)=\{0\}$ the identity

$$T = \big\{\{(T-\lambda)^{-1}h;(I_{\mathcal{H}}+\lambda(T-\lambda)^{-1})h\}\colon h\in\operatorname{ran}(T-\lambda)\big\} \tag{2.2}$$

holds. The *adjoint* T^* of a linear relation T in $\mathcal{H}$ is defined as the orthogonal companion of T with respect to $[\![\cdot,\cdot]\!]_{\mathcal{H}^2}$, i.e.

$$\begin{aligned} T^* := T^{[\![\perp]\!]} &= \big\{\hat f \in \mathcal{H}^2 \colon [\![\hat f, \hat g]\!]_{\mathcal{H}^2} = 0 \text{ for all } \hat g \in T\big\} \\ &= \big\{\{f;f'\} \in \mathcal{H}^2 \colon \langle f', g\rangle = \langle f, g'\rangle \text{ for all } \{g;g'\} \in T\big\}. \end{aligned} \tag{2.3}$$

Here the symbol * is also used for the adjoint in the case when $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Krein space. It is easy to see that (2.3) generalizes the usual definition of the adjoint of a densely defined operator. Observe that the adjoint T^* is a closed linear relation in $\mathcal{H}$. A linear relation T is called *symmetric* (*self-adjoint*) if $T \subset T^*$ ($T = T^*$, respectively). We note that for a self-adjoint relation T the spectrum $\sigma(T)$ is symmetric with respect to the real line. If $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space, then $\sigma(T)$ is real.

Next the notion of boundary triples will be recalled; see, e.g. [11, 12, 14, 15, 17, 20, 25]. This concept is nowadays very popular in extension and spectral theory of symmetric and self-adjoint operators since a boundary triple can be used to describe all closed extensions of a symmetric operator which are restrictions of the adjoint, in particular, all self-adjoint extensions. Besides the inner product $[\![\cdot,\cdot]\!]_{\mathcal{H}^2}$ we make use of a second indefinite inner product $[\![\cdot,\cdot]\!]_{\mathcal{G}^2}$ on $\mathcal{G}^2$ defined as in (2.1), where $\mathcal{G}$ is a Hilbert space.

Definition 2.1. *Let S be a closed symmetric relation in $\mathcal{H}$. We say that $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is a* boundary triple for S^* *if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and $\Gamma_0, \Gamma_1 \colon S^* \to \mathcal{G}$ are linear mappings such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} \colon S^* \to \mathcal{G} \oplus \mathcal{G}$ is surjective, and the relation*

$$[\![\hat f, \hat g]\!]_{\mathcal{H}^2} = [\![\Gamma \hat f, \Gamma \hat g]\!]_{\mathcal{G}^2} \tag{2.4}$$

holds for all $\hat f, \hat g \in S^*$.

If S is a closed symmetric relation in $\mathcal{H}$, then a boundary triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ for S^* exists if and only if S admits self-adjoint extensions in $\mathcal{H}$. We note that a boundary triple for S^* is not unique. If $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is a boundary triple for S^*, then $\Gamma = (\Gamma_0, \Gamma_1)^\top \colon S^* \to \mathcal{G}^2$ is continuous with respect to the graph norm of S^* and the mapping

$$\Theta \mapsto A_\Theta := \ker(\Gamma_1 - \Theta\Gamma_0) = \big\{\hat f \in S^* : \Gamma \hat f = \{\Gamma_0 \hat f; \Gamma_1 \hat f\} \in \Theta\big\} = \Gamma^{-1}(\Theta)$$

establishes a bijective correspondence between the closed linear relations Θ in $\mathcal{G}$ and the closed extensions $A_\Theta \subset S^*$ of S; see, e.g. [14, 15, 17]. If $\operatorname{mul}\Theta \neq \{0\}$, then the expression $\Gamma_1 - \Theta\Gamma_0$ has to be interpreted in the sense of linear relations, i.e.

$$\Gamma_1 - \Theta\Gamma_0 = \big\{\{\hat f; \Gamma_1 \hat f - y\} : \hat f \in S^*,\ \{\Gamma_0 \hat f; y\} \in \Theta\big\}.$$

It is important to note that the identity $A_\Theta^* = A_{\Theta^*}$ holds. This implies that A_Θ is a closed symmetric (self-adjoint) extension of S if and only if Θ is a closed symmetric (self-adjoint, respectively) relation in $\mathcal{G}$.

Remark 2.2. In many applications the closed symmetric relation S is a densely defined symmetric operator. Then S^* is also an operator and it is more natural to define the boundary mappings on $\operatorname{dom} S^*$ instead of (the graph of) S^*. More precisely, if $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is a boundary triple for S^*, then for $f \in \operatorname{dom} S^*$ and $\hat{f} = \{f; S^* f\}$ we write $\Gamma_i f$ instead of $\Gamma_i \hat{f}$, $i = 0, 1$. Then (2.4) turns into

$$\langle S^* f, g \rangle - \langle f, S^* g \rangle = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \qquad f, g \in \operatorname{dom} S^*. \tag{2.5}$$

Relation (2.5) is sometimes called *abstract Green's identity* or *abstract Lagrange identity*. Later this terminology will become clearer.

The following theorem is the main result of this note. It provides a method to determine the adjoint of a symmetric operator or relation. The idea is based on the theory of isometric and unitary operators in Krein spaces and the concept of boundary triplets, cf. [5, 12, 13, 31].

Theorem 2.3. *Let T be a linear relation in the Hilbert or Krein space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and let $(\mathcal{G}, (\cdot, \cdot))$ be a Hilbert space. Assume that $\Gamma = \binom{\Gamma_0}{\Gamma_1} : T \to \mathcal{G} \oplus \mathcal{G}$ is a linear mapping such that the following conditions are satisfied:*

(i) *there exists a symmetric relation Θ in $\mathcal{G}$ such that*

$$\ker(\Gamma_1 - \Theta\Gamma_0) = \big\{ \hat{f} \in T : \Gamma \hat{f} = \{\Gamma_0 \hat{f}; \Gamma_1 \hat{f}\} \in \Theta \big\}$$

contains a self-adjoint relation A in $(\mathcal{H}, \langle \cdot, \cdot \rangle)$,

(ii) $\operatorname{ran} \Gamma = \mathcal{G} \oplus \mathcal{G}$,

(iii) $[\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} = [\![\Gamma \hat{f}, \Gamma \hat{g}]\!]_{\mathcal{G}^2}$ *for all $\hat{f}, \hat{g} \in T$.*

Then $S := \ker \Gamma$ is a closed symmetric relation in $\mathcal{H}$ such that $S^* = T$ and $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is a boundary triple for S^*. Furthermore, Θ is a self-adjoint relation in $\mathcal{G}$ and $A = \ker(\Gamma_1 - \Theta\Gamma_0) = A_\Theta$ holds.

Remark 2.4. We point out that in the assumptions of Theorem 2.3, T is not assumed to be closed and that the boundary mappings are not assumed to be continuous with respect to the graph norm of T. It is part of the conclusion that T is closed and Γ is bounded. For the applicability of the method it is essential that closedness of T and boundedness of Γ do not have to be checked, see the examples in Section 3.

If T is a linear operator and the mappings Γ_0, Γ_1 are defined on $\operatorname{dom} T$ instead of T (cf. Remark 2.2), then Theorem 2.3 reduces to the following corollary, which in the Hilbert space case, under the additional assumption that T is closed, coincides with [12, Theorem 1.13].

Corollary 2.5. *Let T be a linear operator in the Hilbert or Krein space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and let $(\mathcal{G}, (\cdot, \cdot))$ be a Hilbert space. Assume that $\Gamma = \binom{\Gamma_0}{\Gamma_1} : \operatorname{dom} T \to \mathcal{G} \oplus \mathcal{G}$ is a linear mapping such that the following conditions are satisfied:*

(i) *there exists a symmetric relation Θ in $\mathcal{G}$ such that*

$$T \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0) = T \upharpoonright \big\{ f \in \operatorname{dom} T : \Gamma f = \{\Gamma_0 f; \Gamma_1 f\} \in \Theta \big\}$$

has a self-adjoint restriction A in $(\mathcal{H}, \langle \cdot, \cdot \rangle)$,

(ii) $\operatorname{ran} \Gamma = \mathcal{G} \oplus \mathcal{G}$,

(iii) $\langle Tf, g\rangle - \langle f, Tg\rangle = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)$ *for all* $f, g \in \operatorname{dom} T$.

Then $S := T \upharpoonright \ker \Gamma$ *is a densely defined closed symmetric operator in* $\mathcal{H}$ *such that* $S^* = T$ *and* $(\mathcal{G}, \Gamma_0, \Gamma_1)$ *is a boundary triple for* S^*. *Furthermore,* Θ *is a self-adjoint relation in* $\mathcal{G}$ *and* $A = T \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0) = A_\Theta$ *holds.*

Proof of Theorem 2.3. The symmetric relation Θ can be extended to a maximal symmetric relation $\widetilde{\Theta}$ in $\mathcal{G}$, i.e. $\widetilde{\Theta}$ is symmetric and $i \in \rho(\widetilde{\Theta})$ or $-i \in \rho(\widetilde{\Theta})$. Without loss of generality assume the former and define an operator $W = (W_{ij})_{i,j=1}^2 \in \mathcal{L}(\mathcal{G} \oplus \mathcal{G})$ by

$$W := \begin{pmatrix} I_\mathcal{G} + i(\widetilde{\Theta} - i)^{-1} & -(\widetilde{\Theta} - i)^{-1} \\ (\widetilde{\Theta} - i)^{-1} & I_\mathcal{G} + i(\widetilde{\Theta} - i)^{-1} \end{pmatrix}. \tag{2.6}$$

Let $\mathcal{J}_\mathcal{G}$ be as in (2.1) such that $[\![\cdot,\cdot]\!]_{\mathcal{G}^2} = (\mathcal{J}_\mathcal{G}\cdot,\cdot)_{\mathcal{G}^2}$ holds. Then $(\widetilde{\Theta}^* + i)^{-1} \in \mathcal{L}(\mathcal{G})$ and

$$(\widetilde{\Theta} - i)^{-1} - (\widetilde{\Theta}^* + i)^{-1} = 2i(\widetilde{\Theta}^* + i)^{-1}(\widetilde{\Theta} - i)^{-1};$$

this and a straightforward calculation show that $W^*W = I_\mathcal{G}$ and $W^*\mathcal{J}_\mathcal{G}W = \mathcal{J}_\mathcal{G}$. Since condition (iii) holds for $\Gamma\colon T \to \mathcal{G} \oplus \mathcal{G}$, we conclude that the mapping

$$\Gamma^W = \begin{pmatrix} \Gamma_0^W \\ \Gamma_1^W \end{pmatrix} := \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = W\Gamma \,:\, T \to \mathcal{G} \oplus \mathcal{G}$$

satisfies a corresponding condition

$$\begin{aligned} [\![\Gamma^W \hat{f}, \Gamma^W \hat{g}]\!]_{\mathcal{G}^2} &= [\![W\Gamma\hat{f}, W\Gamma\hat{g}]\!]_{\mathcal{G}^2} = \big(\mathcal{J}_\mathcal{G} W\Gamma\hat{f}, W\Gamma\hat{g}\big)_{\mathcal{G}^2} \\ &= \big(W^*\mathcal{J}_\mathcal{G} W\Gamma\hat{f}, \Gamma\hat{g}\big)_{\mathcal{G}^2} = \big(\mathcal{J}_\mathcal{G}\Gamma\hat{f}, \Gamma\hat{g}\big)_{\mathcal{G}^2} = [\![\Gamma\hat{f}, \Gamma\hat{g}]\!]_{\mathcal{G}^2} = [\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} \end{aligned} \tag{2.7}$$

for all $\hat{f}, \hat{g} \in T$, and since W is isometric, $\ker \Gamma = \ker \Gamma^W$ holds. Observe also that for $\hat{f} \in T$ the element $\{\Gamma_0\hat{f}; \Gamma_1\hat{f}\}$ belongs to the symmetric relation $\widetilde{\Theta}$ if and only if $\{\Gamma_0^W\hat{f}; \Gamma_1^W\hat{f}\}$ belongs to the symmetric relation

$$\big\{\{W_{11}u + W_{12}v; W_{21}u + W_{22}v\} \colon \{u; v\} \in \widetilde{\Theta}\big\}.$$

Making use of $\widetilde{\Theta} = \big\{\{(\widetilde{\Theta} - i)^{-1}x; (I_\mathcal{G} + i(\widetilde{\Theta} - i)^{-1})x\} : x \in \operatorname{ran}(\widetilde{\Theta} - i)\big\}$, cf. (2.2), and inserting the entries W_{ij} from (2.6) we conclude that $\{\Gamma_0\hat{f}; \Gamma_1\hat{f}\} \in \widetilde{\Theta}$ implies $\Gamma_0^W\hat{f} = 0$, i.e.

$$\ker(\Gamma_1 - \widetilde{\Theta}\Gamma_0) \subset \ker \Gamma_0^W$$

and hence by condition (i)

$$A \subset \ker(\Gamma_1 - \Theta\Gamma_0) \subset \ker(\Gamma_1 - \widetilde{\Theta}\Gamma_0) \subset \ker \Gamma_0^W. \tag{2.8}$$

Suppose now that $\hat{f}, \hat{g} \in \ker \Gamma_0^W$. From (2.7) we obtain

$$[\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} = [\![\Gamma^W\hat{f}, \Gamma^W\hat{g}]\!]_{\mathcal{G}^2} = 0$$

and hence $\ker \Gamma_0^W$ is a symmetric relation in $\mathcal{H}$. Therefore (2.8) and the self-adjointness of A imply

$$\ker \Gamma_0^W \subset \left(\ker \Gamma_0^W\right)^* \subset A^* = A \subset \ker \Gamma_0^W,$$

which implies equality everywhere, i.e. $\ker(\Gamma_1 - \Theta\Gamma_0) = \Gamma^{-1}(\Theta) = A = \ker \Gamma_0^W$ is self-adjoint in $\mathcal{H}$. Moreover,

$$S := \ker \Gamma = \ker \Gamma^W \subset \ker \Gamma_0^W$$

is a symmetric relation in $\mathcal{H}$.

Assume that Θ is not self-adjoint. Then $\Theta \subsetneq \Theta^*$, and since $\operatorname{ran} \Gamma = \mathcal{G} \oplus \mathcal{G}$, we have $\Gamma^{-1}(\Theta) \subsetneq \Gamma^{-1}(\Theta^*)$. Assumption (iii) implies $\Gamma^{-1}(\Theta^*) \subset (\Gamma^{-1}(\Theta))^*$, which is a contradiction to the self-adjointness of $\Gamma^{-1}(\Theta)$. Hence Θ is self-adjoint. It follows now that also $WW^* = I_{\mathcal{G}}$ and hence $\operatorname{ran} \Gamma^W = \mathcal{G} \oplus \mathcal{G}$ by assumption (ii).

Let us verify that $S = T^*$ holds. For $\hat{f} \in S = \ker \Gamma^W$ and $\hat{g} \in T$ we have $[\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} = [\![\Gamma^W \hat{f}, \Gamma^W \hat{g}]\!]_{\mathcal{G}^2} = 0$ by (2.7); hence $\hat{f} \in T^*$. On the other hand, since $A \subset T$ is self-adjoint, each element $\hat{f} \in T^*$ necessarily belongs to $A = \ker \Gamma_0^W$. For arbitrary $\hat{g} \in T$ this implies

$$0 = [\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} = [\![\Gamma^W \hat{f}, \Gamma^W \hat{g}]\!]_{\mathcal{G}^2} = -i(\Gamma_1^W \hat{f}, \Gamma_0^W \hat{g}).$$

It follows from condition (ii) and $\operatorname{ran} \Gamma = \operatorname{ran} \Gamma^W$ that $\operatorname{ran} \Gamma_0^W = \mathcal{G}$ holds and this gives $\Gamma_1^W \hat{f} = 0$. Hence we have $\hat{f} \in \ker \Gamma_0^W \cap \ker \Gamma_1^W = S$. Therefore $S = T^*$ and, in particular, S is a closed linear relation in $\mathcal{H}$.

Since $S^* = T^{**} = \overline{T}$, it remains to show that T is closed. Let $(\hat{f}_n) \in T$ be a sequence converging to $\hat{f}$. For $\hat{z} \in \mathcal{G} \oplus \mathcal{G}$ we choose $\hat{g} \in T$ such that $\Gamma \hat{g} = \mathcal{J}_{\mathcal{G}} \hat{z}$ holds. From

$$\lim_{n\to\infty} (\Gamma \hat{f}_n, \hat{z})_{\mathcal{G}^2} = \lim_{n\to\infty} [\![\Gamma \hat{f}_n, \Gamma \hat{g}]\!]_{\mathcal{G}^2} = \lim_{n\to\infty} [\![\hat{f}_n, \hat{g}]\!]_{\mathcal{H}^2} = [\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} \tag{2.9}$$

we conclude that $\Gamma \hat{f}_n$ converges weakly to some $\hat{x} \in \mathcal{G} \oplus \mathcal{G}$. Let $\hat{h} \in T$ be such that $\Gamma \hat{h} = \hat{x}$. Then (2.9) implies

$$\begin{aligned} [\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} &= \lim_{n\to\infty} (\Gamma \hat{f}_n, \hat{z})_{\mathcal{G}^2} = (\hat{x}, \hat{z})_{\mathcal{G}^2} = (\Gamma \hat{h}, \mathcal{J}_{\mathcal{G}}^{-1} \Gamma \hat{g})_{\mathcal{G}^2} \\ &= [\![\Gamma \hat{h}, \Gamma \hat{g}]\!]_{\mathcal{G}^2} = [\![\hat{h}, \hat{g}]\!]_{\mathcal{H}^2} \end{aligned}$$

and therefore $[\![\hat{f} - \hat{h}, \hat{g}]\!]_{\mathcal{H}^2} = 0$. Since $\hat{g} \in T$, we conclude $\hat{f} - \hat{h} \in T^* = S \subset T$. Now $\hat{h} \in T$ implies $\hat{f} \in T$. We have shown $S^* = T$. By conditions (ii) and (iii) it follows that $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is a boundary triple for S^*. □

Remark 2.6. In the proof of Theorem 2.3 we have also shown that $(\mathcal{G}, \Gamma_0^W, \Gamma_1^W)$ is a boundary triple for $S^* = T$ such that $A_\Theta = \ker \Gamma_0^W$ holds.

Remark 2.7. In applications it is often convenient to choose the symmetric relation Θ in condition (i) as one of the self-adjoint relations $\Theta_0 = \{\{0; g\} : g \in \mathcal{G}\}$ or $\Theta_1 = \Theta_0^{-1} = 0$. In this case one has to verify that $\ker \Gamma_0 = \ker(\Gamma_1 - \Theta_0 \Gamma_0)$ or $\ker \Gamma_1$, respectively, contains a self-adjoint relation in $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. For Corollary 2.5, (i) reduces to the statement:

$$T \restriction \{f \in \operatorname{dom} T \colon \Gamma_0 f = 0\} \qquad (\text{or } T \restriction \{f \in \operatorname{dom} T \colon \Gamma_1 f = 0\}, \text{respectively})$$

has a self-adjoint restriction.

Remark 2.8. If $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space, then condition (i) can be replaced by one of the following conditions

(i)$'$ there exist a symmetric relation Θ in $\mathcal{G}$ and $\lambda_\pm \in \mathbb{C}^\pm$ such that for every $h \in \mathcal{H}$ there are $\hat{f}_\pm = \{f_\pm; f'_\pm\} \in T$ with

$$f'_\pm - \lambda_\pm f_\pm = h \quad \text{and} \quad \{\Gamma_0 \hat{f}_\pm; \Gamma_1 \hat{f}_\pm\} \in \Theta;$$

(i)$''$ $\ker\Gamma$ is closed, $\operatorname{codim}(\operatorname{ran}(\ker\Gamma - \lambda_\pm)) = n < \infty$ for some $\lambda_\pm \in \mathbb{C}^\pm$ and there exists a symmetric relation Θ in $\mathcal{G}$ such that

$$\dim\big(\ker(\Gamma_1 - \Theta\Gamma_0)/\ker\Gamma\big) = n.$$

Indeed, (i) can be replaced by (i)$'$ since condition (iii) in Theorem 2.3 implies that the linear relation $A := \ker(\Gamma_1 - \Theta\Gamma_0)$ is symmetric. Then by (i)$'$ we have $\operatorname{ran}(A - \lambda_\pm) = \mathcal{H}$ for some $\lambda_\pm \in \mathbb{C}^\pm$. Therefore A is self-adjoint and hence (i) holds. The fact that (i) can be replaced by (i)$''$ is a consequence of the symmetry of the n-dimensional extension $A := \ker(\Gamma_1 - \Theta\Gamma_0)$ of the closed symmetric relation $S = \ker\Gamma$.

Remark 2.9. If $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert or Krein space, condition (i) can be replaced by

(i)$'''$ there exist a symmetric relation Θ in $\mathcal{G}$ and $\lambda \in \mathbb{R}$ such that for every $h \in \mathcal{H}$ there is an $\hat{f} = \{f; f'\} \in T$ with

$$f' - \lambda f = h \quad \text{and} \quad \{\Gamma_0 \hat{f}; \Gamma_1 \hat{f}\} \in \Theta.$$

Remark 2.10. Condition (iii) in Theorem 2.3 can be replaced by one of the following conditions

(iii)$'$ the sesquilinear form D defined on T by $D[\hat{f}, \hat{g}] := \langle f', g\rangle - (\Gamma_1\hat{f}, \Gamma_0\hat{g})$, $\hat{f} = \{f; f'\}$, $\hat{g} = \{g; g'\} \in T$ is symmetric;

(iii)$''$ $D[\hat{f}, \hat{f}] = \langle f', f\rangle - (\Gamma_1\hat{f}, \Gamma_0\hat{f})$ is real for all $\hat{f} = \{f; f'\} \in T$.

In the case that T is an operator, one defines $D[f, g] := \langle Tf, g\rangle - (\Gamma_1 f, \Gamma_0 g)$ for $f, g \in \operatorname{dom} T$. For Sturm–Liouville operators the form D is nothing else than the Dirichlet form.

3 Applications

In this section we apply the general method to determine the adjoint of a symmetric operator for various examples. As a simple problem we first discuss the well-known case of a Sturm–Liouville operator in Section 3.1. Afterwards we investigate a block operator matrix with first and second order differential operators as entries. In Section 3.3 a second order elliptic differential expression with an indefinite weight function on a bounded domain is considered and finally, in Section 3.4, we deal with multiplication operators in L^2-spaces which are connected with functional models for operator-valued Nevanlinna or Riesz–Herglotz functions.

3.1 Sturm–Liouville operators in limit circle case

We consider a Sturm–Liouville operator on an interval (a,b) which is regular or in limit circle case at both end-points in order to illustrate the general method from the previous section. The form of the adjoint of the minimal operator is of course well known and can be derived by other means, which usually require quite lengthy calculations; see, e.g. [32, 33, 34].

Let p,q,w be real-valued functions on the interval (a,b) such that $1/p$, $q,w \in L^1_{\mathrm{loc}}(a,b)$ and $w(x)>0$ almost everywhere. Let $L^2_w(a,b)$ denote the space of (equivalence classes) of complex-valued measurable functions on (a,b) such that $|f|^2 w \in L^1(a,b)$ and equip $L^2_w(a,b)$ with the inner product

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}w(x)dx.$$

In the Hilbert space $(L^2_w(a,b), \langle\cdot,\cdot\rangle)$ we consider the operator

$$Tf = \frac{1}{w}\big(-(pf')' + qf\big)$$

defined on

$$\operatorname{dom} T = \Big\{ f \in L^2_w(a,b)\colon\ f,\, pf' \text{ absolutely continuous on } (a,b),\\ \frac{1}{w}\big(-(pf')'+qf\big) \in L^2_w(a,b)\Big\}.$$

With W_p denoting the modified Wronskian

$$W_p(f,g)(x) := p(x)\big(f(x)g'(x) - f'(x)g(x)\big)$$

we have the following Lagrange identity for $a \le c < d \le b$ and $f,g \in \operatorname{dom} T$,

$$\int_c^d (Tf)(x)\overline{g(x)}w(x)dx - \int_c^d f(x)\overline{(Tg)(x)}w(x)dx = W_p(f,\overline{g})(d) - W_p(f,\overline{g})(c).$$

It follows from this identity that for $f,g\in\operatorname{dom} T$ the limits $\lim_{c\to a} W_p(f,\overline{g})(c)$, $\lim_{d\to b} W_p(f,\overline{g})(d)$ exist because $f,g,Tf,Tg \in L^2_w(a,b)$. Since the equation is regular or in limit circle case at both end-points, for every $\lambda\in\mathbb{C}$, all solutions of

$$\frac{1}{w}\big(-(pf')'+qf\big) = \lambda f \tag{3.1}$$

are in $L^2_w(a,b)$. Fix a $\lambda\in\mathbb{R}$ and let θ_a, ϕ_a be two linearly independent, real-valued solutions of (3.1) such that $W_p(\theta_a,\phi_a)\equiv 1$. Note that $W_p(f,g)$ is constant if f and g are both solutions of (3.1). The functions θ_a, ϕ_a will be used for boundary mappings connected with the left end-point. Similarly, let θ_b, ϕ_b also be two linearly independent, real-valued solutions of (3.1) such that $W_p(\theta_b,\phi_b)\equiv 1$ (one could also use a different λ for b).

Define boundary mappings for $f\in\operatorname{dom} T$ by

$$\Gamma_0 f := \begin{pmatrix} \lim\limits_{x\to a} W_p(f,\phi_a)(x) \\ \lim\limits_{x\to b} W_p(f,\phi_b)(x)\end{pmatrix}, \qquad \Gamma_1 f := \begin{pmatrix} -\lim\limits_{x\to a} W_p(f,\theta_a)(x) \\ \lim\limits_{x\to b} W_p(f,\theta_b)(x)\end{pmatrix}.$$

All limits exist since the functions f, ϕ_a, ϕ_b, θ_a, θ_b are in $\operatorname{dom} T$. It is well known that the self-adjoint restrictions of T can be described with the help of these limits, cf. [32, 33, 34]. The boundary triple which is obtained in the following statement as a consequence of Corollary 2.5 was also used in [2].

Corollary 3.1. *The operator T is the adjoint of the densely defined closed symmetric operator*

$$\begin{aligned} Sf &= \frac{1}{w}\bigl(-(pf')' + qf\bigr), \\ \operatorname{dom} S &= \left\{ f \in \operatorname{dom} T : \begin{array}{c} \lim_{x\to a} W_p(f,\phi_a)(x) = \lim_{x\to a} W_p(f,\theta_a)(x) = 0 \\ \lim_{x\to b} W_p(f,\phi_b)(x) = \lim_{x\to b} W_p(f,\theta_b)(x) = 0 \end{array} \right\}, \end{aligned}$$

in the Hilbert space $(L^2_w(a,b), \langle\cdot,\cdot\rangle)$, and $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$ is a boundary triple for T.

Remark 3.2. If T is regular at, e.g. the left end-point a (i.e. a is finite and $1/p, q, w$ are integrable at a), then one can choose θ_a, ϕ_a such that they satisfy the initial conditions

$$\begin{aligned} \theta_a(a) &= 1, & \phi_a(a) &= 0, \\ (p\theta_a')(a) &= 0, & (p\phi_a')(a) &= 1. \end{aligned}$$

For the boundary mappings at a one then gets

$$(\Gamma_0 f)_1 = f(a), \qquad (\Gamma_1 f)_1 = (pf')(a),$$

where $(\ )_1$ denotes the first component of a two-vector. Similarly, if the right end-point b is regular, then the second components of the boundary mappings can be chosen as

$$(\Gamma_0 f)_2 = f(b), \qquad (\Gamma_1 f)_2 = -(pf')(b).$$

Proof of Corollary 3.1. We show that all conditions of Corollary 2.5 are satisfied. To see that $\operatorname{ran}\binom{\Gamma_0}{\Gamma_1} = \mathcal{G}^2$, consider a function $f \in \operatorname{dom} T$ that is equal to θ_a in a neighbourhood of a and vanishes identically in a neighbourhood of b, which gives $\Gamma_0 f = \binom{1}{0}$, $\Gamma_1 f = \binom{0}{0}$. Using three similar functions we obtain the surjectivity of $\binom{\Gamma_0}{\Gamma_1}$, i.e. (ii).

Next we show that the abstract Green's identity is satisfied. For $f, g \in \operatorname{dom} T$ we have

$$\begin{aligned} & W_p(f,\theta_a)\overline{W_p(g,\phi_a)} - W_p(f,\phi_a)\overline{W_p(g,\theta_a)} \\ &= p^2(f\theta_a' - f'\theta_a)(\overline{g}\phi_a' - \overline{g}'\phi_a) - p^2(f\phi_a' - f'\phi_a)(\overline{g}\theta_a' - \overline{g}'\theta_a) \\ &= p^2\Bigl(f\overline{g}'(\theta_a\phi_a' - \theta_a'\phi_a) - f'\overline{g}(\theta_a\phi_a' - \theta_a'\phi_a)\Bigr) \\ &= W_p(\theta_a,\phi_a)W_p(f,\overline{g}) = W_p(f,\overline{g}) \end{aligned}$$

and a similar relation for θ_b, ϕ_b. Hence

$$
\begin{aligned}
&\langle Tf, g\rangle - \langle f, Tg\rangle \\
&= \lim_{d\to b} W_p(f,\overline{g})(d) - \lim_{c\to a} W_p(f,\overline{g})(c) \\
&= \lim_{d\to b}\Big(W_p(f,\theta_b)(d)\overline{W_p(g,\phi_b)(d)} - W_p(f,\phi_b)(d)\overline{W_p(g,\theta_b)(d)}\Big) \\
&\quad - \lim_{c\to a}\Big(W_p(f,\theta_a)(c)\overline{W_p(g,\phi_a)(c)} - W_p(f,\phi_a)(c)\overline{W_p(g,\theta_a)(c)}\Big) \\
&= (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g),
\end{aligned}
$$

which shows condition (iii). Finally, we verify condition (i)$'$ in Remark 2.8 with the self-adjoint relation $\Theta = \{\{0; g\} : g \in \mathbb{C}^2\}$. For $\lambda \in \mathbb{C}\setminus\mathbb{R}$, let ψ_a, ψ_b be non-trivial solutions of (3.1) such that

$$
\lim_{x\to a} W_p(\psi_a, \phi_a) = 0, \qquad \lim_{x\to b} W_p(\psi_b, \phi_b) = 0,
$$

respectively. It is possible to find such solutions since the space of solutions of (3.1) is two-dimensional. Now it is easy to show that for $f \in L^2_w(a,b)$, the function

$$
y(x) = \frac{1}{W_p(\psi_b,\psi_a)}\left(\psi_b(x)\int_a^x \psi_a(t)f(t)w(t)dt + \psi_a(x)\int_x^b \psi_b(t)f(t)w(t)dt\right)
$$

is a solution of $(T-\lambda)y = f$ and $\Gamma_0 y = 0$. All integrals exist since $f, \psi_a, \psi_b \in L^2_w(a,b)$, and $W_p(\psi_b,\psi_a) \neq 0$ because otherwise, the function ψ_a would be an eigenfunction of the symmetric operator $T \restriction \ker\Gamma_0$ corresponding to a non-real eigenvalue. This can be done for λ in the upper and lower half planes, and hence condition (i)$'$ in Remark 2.8 is satisfied. $\square$

Remark 3.3. In a similar way one can prove Corollary 3.1 also for an indefinite weight w that satisfies $w \neq 0$ almost everywhere and $w \in L^1_{\mathrm{loc}}(a,b)$. In this case $L^2_w(a,b)$ is a Krein space rather than a Hilbert space. Instead of (i)$'$ in Remark 2.8 we use (i)$'''$ in Remark 2.9 with a real λ. If ψ_a, ψ_b (constructed as above) are linearly independent, then we can find a solution of $(T-\lambda)y = f$, $\Gamma_0 y = 0$ for every $f \in L^2_w(a,b)$ as in the case $w > 0$. Otherwise, let χ_a be a solution of (3.1) such that $\lim_{x\to a} W_p(\chi_a,\theta_a) = 0$, which in this case must be linearly independent of ψ_b. Hence $W_p\big(\chi_a,\psi_b\big) \neq 0$ and for $f \in L^2_w(a,b)$, the function

$$
y(x) = \frac{1}{W_p(\psi_b,\chi_a)}\left(\psi_b(x)\int_a^x \chi_a(t)f(t)w(t)dt + \chi_a(x)\int_x^b \psi_b(t)f(t)w(t)dt\right)
$$

is a solution of $(T-\lambda)y = f$, $(\Gamma_0 y)_2 = 0$, $(\Gamma_1 y)_1 = 0$. This shows that (i)$'''$ is satisfied with

$$
\Theta = \left\{\left\{\begin{pmatrix}\alpha\\0\end{pmatrix}; \begin{pmatrix}0\\\beta\end{pmatrix}\right\} :\ \alpha,\beta\in\mathbb{C}\right\},
$$

which clearly is a symmetric (and even a self-adjoint) relation.

3.2 A block operator matrix

In this subsection we consider a block operator matrix of the form

$$\begin{pmatrix} -\dfrac{d}{dx}p\dfrac{d}{dx}+q & -\dfrac{d}{dx}\bar{b}+\bar{c} \\ b\dfrac{d}{dx}+c & d \end{pmatrix} \tag{3.2}$$

in the Hilbert space $\mathcal{H} = L^2(0,1)\oplus L^2(0,1)$ with inner product $\langle\cdot,\cdot\rangle$. Operators of this type have been considered in many papers and books, see, e.g. [1, 10, 21, 22], but usually under relatively restrictive assumptions on the coefficients, and apart from [10] only self-adjoint realizations were studied and not the maximal operator or boundary triples. However, note that in contrast to [10] the off-diagonal entries in (3.2) are unbounded first order differential operators, which leads also to a different form of boundary mappings. Let us assume that the coefficients satisfy the following conditions

$$p>0; \quad q,d \text{ real-valued}; \quad \frac{1}{p},b,c,d\in L^\infty(0,1); \quad q\in L^1(0,1). \tag{3.3}$$

The expression in (3.2) is only formal; we define a maximal operator T more carefully by

$$T\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -\big(pf'+\bar{b}g\big)'+qf+\bar{c}g \\ bf'+cf+dg \end{pmatrix} \tag{3.4}$$

with domain

$$\operatorname{dom} T = \left\{\begin{pmatrix} f \\ g \end{pmatrix}\in\mathcal{H}\colon\ f,\,pf'+\bar{b}g \text{ absolutely continuous on } [0,1], \right. \\ \left. -\big(pf'+\bar{b}g\big)'+qf,\ bf'\in L^2(0,1)\right\}. \tag{3.5}$$

Note that in general the domain is not diagonal, i.e. it is not of the form $\mathcal{D}_1\oplus\mathcal{D}_2$.

For $\binom{f}{g}\in\operatorname{dom} T$ we define the boundary mappings

$$\Gamma_0\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} f(0) \\ -f(1) \end{pmatrix}, \qquad \Gamma_1\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} (pf'+\bar{b}g)(0) \\ (pf'+\bar{b}g)(1) \end{pmatrix}. \tag{3.6}$$

Using Corollary 2.5 we show the following statement.

Corollary 3.4. *The operator T defined in (3.4)–(3.5) is the adjoint of the densely defined closed symmetric operator $S = T\restriction\operatorname{dom} S$, where*

$$\operatorname{dom} S = \left\{\begin{pmatrix} f \\ g \end{pmatrix}\in\operatorname{dom} T:\ \begin{matrix} f(0)=(pf'+\bar{b}g)(0)=0 \\ f(1)=(pf'+\bar{b}g)(1)=0 \end{matrix}\right\},$$

and $(\mathbb{C}^2,\Gamma_0,\Gamma_1)$ is a boundary triple for T.

Proof. In order to show (iii) in Corollary 2.5 we verify condition (iii)$''$ in Remark 2.10. Let $\binom{f}{g} \in \operatorname{dom} T$ and calculate

$$\begin{aligned}
D\left[\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right] &= \left\langle T\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle - \left(\Gamma_1\begin{pmatrix} f \\ g \end{pmatrix}, \Gamma_0\begin{pmatrix} f \\ g \end{pmatrix}\right) \\
&= \left\langle -(pf' + \bar{b}g)' + qf + \bar{c}g, f\right\rangle + \left\langle bf' + cf + dg, g\right\rangle \\
&\quad - (pf' + \bar{b}g)(0)\overline{f(0)} + (pf' + \bar{b}g)(1)\overline{f(1)} \\
&= -\int_0^1 (pf' + \bar{b}g)'\bar{f} + \int_0^1 q|f|^2 + \langle \bar{c}g, f\rangle + \langle bf', g\rangle + \langle cf, g\rangle + \langle dg, g\rangle \\
&\quad - (pf' + \bar{b}g)(0)\overline{f(0)} + (pf' + \bar{b}g)(1)\overline{f(1)} \\
&= \int_0^1 (pf' + \bar{b}g)\overline{f'} + \int_0^1 q|f|^2 + \langle \bar{c}g, f\rangle + \langle bf', g\rangle + \langle cf, g\rangle + \langle dg, g\rangle \\
&= \int_0^1 \left(p|f'|^2 + 2\operatorname{Re}(bf'\bar{g}) + q|f|^2\right) + 2\operatorname{Re}\langle cf, g\rangle + \langle dg, g\rangle.
\end{aligned}$$

Since the latter expression is real, assumption (iii)$''$ in Remark 2.10 is satisfied.

To show the surjectivity of Γ, it is sufficient to consider $g = 0$ and a function f that satisfies $f \equiv 1$ near 0, $f \equiv 0$ near 1, a function for which $f(x) = \int_0^x \frac{1}{p(t)}dt$ near 0 and $f \equiv 0$ near 1 and similar functions for the right end-point.

We show condition (i)$'''$ in Remark 2.9 with $\Theta = \{\{0; g\} : g \in \mathcal{G}\}$, i.e. we show that for some $\lambda \in \mathbb{R}$ and every $\binom{u}{v} \in \mathcal{H}$ there exists $\binom{f}{g} \in \operatorname{dom} A$ such that $(A - \lambda)\binom{f}{g} = \binom{u}{v}$, where $A = T \restriction \ker \Gamma_0$. First we see that the form D is bounded from below: due to the assumptions (3.3), the form D can be estimated as follows

$$\begin{aligned}
&D\left[\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right] \\
&\geq (\operatorname{ess\,inf} p)\|f'\|^2 - 2\|b\|_\infty\|f'\|\,\|g\| - \|q\|_1\|f\|_\infty^2 - 2\|c\|_\infty\|f\|\,\|g\| - \|d\|_\infty\|g\|^2 \\
&\geq (\operatorname{ess\,inf} p)\|f'\|^2 - 2\|b\|_\infty\left(\varepsilon\|f'\|^2 + \frac{1}{4\varepsilon}\|g\|^2\right) - \|q\|_1\left(\varepsilon\|f'\|^2 + C_\varepsilon\|f\|^2\right) \\
&\quad - \|c\|_\infty\left(\|f\|^2 + \|g\|^2\right) - \|d\|_\infty\|g\|^2,
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary and C_ε is some positive constant depending only on ε; for the estimate of $\|f'\|\,\|g\|$ the geometric–quadratic inequality was used, for the estimate of $\|f\|_\infty$ see, e.g. [24, IV-(1.19)]. If ε is chosen sufficiently small, then

$$D\left[\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right] \geq \gamma\left\|\begin{pmatrix} f \\ g \end{pmatrix}\right\|^2$$

for some $\gamma \in \mathbb{R}$. Now choose $\lambda < \min\{\gamma, \operatorname{ess\,inf} d\}$ such that

$$\operatorname{ess\,inf}\left(p - \frac{|b|^2}{d - \lambda}\right) > 0. \tag{3.7}$$

It follows that λ cannot be an eigenvalue of A since every eigenvalue of A must be contained in the numerical range of D. Let $u, v \in L^2(0,1)$. The equation

$(A-\lambda)\binom{f}{g} = \binom{u}{v}$, which we have to solve, is explicitly given by

$$-\big(pf' + \overline{b}g\big)' + (q-\lambda)f + \overline{c}g = u, \tag{3.8}$$

$$bf' + cf + (d-\lambda)g = v. \tag{3.9}$$

Solving (3.9) for g yields

$$g = -\frac{b}{d-\lambda}f' - \frac{c}{d-\lambda}f + \frac{v}{d-\lambda}\,; \tag{3.10}$$

plugging this into (3.8) we obtain

$$\left(-\left(p - \frac{|b|^2}{d-\lambda}\right)f' + \frac{\overline{b}c}{d-\lambda}f - \frac{\overline{b}v}{d-\lambda}\right)' \\ - \frac{b\overline{c}}{d-\lambda}f' + \left(q - \lambda - \frac{|c|^2}{d-\lambda}\right)f = u - \frac{\overline{c}v}{d-\lambda}\,,$$

which can be written as

$$(-Pf' + Rf + V)' - \overline{R}f' + Qf = U \tag{3.11}$$

with

$$P = p - \frac{|b|^2}{d-\lambda}\,, \quad R = \frac{\overline{b}c}{d-\lambda}\,, \quad Q = q - \lambda - \frac{|c|^2}{d-\lambda}\,,$$
$$U = u - \frac{\overline{c}v}{d-\lambda}\,, \quad V = -\frac{\overline{b}v}{d-\lambda}\,.$$

Due to the assumptions (3.3) and relation (3.7), these functions satisfy

$$\frac{1}{P}, R \in L^\infty(0,1); \quad Q \in L^1(0,1); \quad U, V \in L^2(0,1).$$

Introducing new variables

$$F := f, \quad G := -Pf' + Rf + V$$

we can write (3.11) as a canonical (or Dirac) system,

$$J\begin{pmatrix} F \\ G \end{pmatrix}' + \begin{pmatrix} \dfrac{|R|^2}{P} - Q & -\dfrac{\overline{R}}{P} \\ -\dfrac{R}{P} & \dfrac{1}{P} \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} -U - \dfrac{\overline{R}}{P}V \\ \dfrac{1}{P}V \end{pmatrix}, \tag{3.12}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The potential is a Hermitian matrix with entries in $L^1(0,1)$. The boundary conditions $\Gamma_0\binom{f}{g} = 0$ transform into

$$F(0) = 0, \quad F(1) = 0. \tag{3.13}$$

If we denote the left-hand side of (3.12) by $\tau_{\text{can}}\binom{F}{G}$, then the operator $A_{\text{can}}\binom{F}{G} := \tau_{\text{can}}\binom{F}{G}$ with domain

$$\operatorname{dom} A_{\text{can}} = \left\{\begin{pmatrix} F \\ G \end{pmatrix} \in \mathcal{H} : \ F, G \text{ absolutely continuous on } [0,1], \tau_{\text{can}}\begin{pmatrix} F \\ G \end{pmatrix} \in \mathcal{H}\right\}$$

is self-adjoint in $\mathcal{H}$ and the spectrum of $A_{\rm can}$ consists only of eigenvalues; see, e.g. [33, Satz 15.11]. Since λ is not an eigenvalue of A, 0 is not an eigenvalue of $A_{\rm can}$. Hence $A_{\rm can}$ is boundedly invertible and (3.12), (3.13) has an absolutely continuous solution $\binom{F}{G}$ for every right-hand side in $L^2(0,1) \oplus L^2(0,1)$. Transforming this solution back, i.e. setting $f := F$ and defining g by (3.10), we see that (3.8), (3.9) has a solution in $\operatorname{dom} A$ for every right-hand side $\binom{u}{v} \in \mathcal{H}$; the conditions about absolute continuity are satisfied since $f = F$ and $pf' + \overline{b}g = -G$. This shows that (i)$'''$ in Remark 2.9 is satisfied. Hence we have proved the corollary. $\square$

3.3 Second order elliptic operators with indefinite weights

Boundary triple methods for elliptic differential operators on bounded domains were recently investigated in various papers; see, e.g. [3, 7, 29, 30]. The following example is similar to and heavily inspired by [9, 10, 28] and the fundamental article [18] by G. Grubb, where it appears in a slightly different form.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n > 1$, with C^∞-boundary $\partial\Omega$ and let

$$(\mathcal{L}f)(x) = -\sum_{j,k=1}^{n} \big(\partial_j\, a_{jk}\, \partial_k f\big)(x) \,+\, a(x)f(x)$$

be a second order differential expression on Ω with smooth coefficients $a_{jk}, a \in C^\infty(\overline{\Omega})$ such that $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x \in \overline{\Omega}$ and a is real. Moreover, we assume that there exists $C > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq C\sum_{k=1}^{n} \xi_k^2$$

holds for all $x \in \overline{\Omega}$ and all $\xi = \{\xi_1;\ldots;\xi_n\} \in \mathbb{R}^n$, i.e. $\mathcal{L}$ is a uniformly elliptic differential expression.

The Sobolev space of kth order on Ω is denoted by $H^k(\Omega)$ and the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$ is denoted by $H_0^k(\Omega)$. Sobolev spaces on the boundary of Ω are denoted by $H^s(\partial\Omega)$, $s \in \mathbb{R}$. Let $n(x) = \{n_1(x);\ldots;n_n(x)\}$ be the outward normal vector on $\partial\Omega$ and denote by $f|_{\partial\Omega}$ and $\frac{\partial f}{\partial\nu}|_{\partial\Omega} = \sum a_{jk}\, n_j\, \partial_k f|_{\partial\Omega}$ the traces of a function $f \in C^\infty(\overline{\Omega})$ and its normal derivative. According to [23] the trace map can be extended to a linear mapping defined on

$$\mathcal{D}_{\max} = \big\{ f \in L^2(\Omega) \colon \mathcal{L}f \in L^2(\Omega) \big\}$$

with values in $H^{-1/2}(\partial\Omega)$. Let $(\cdot,\cdot)_{1/2,-1/2}$ be the extension of the $L^2(\partial\Omega)$ inner product $(\cdot,\cdot)$ to $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ and let $\iota_\pm : H^{\pm 1/2}(\partial\Omega) \to L^2(\partial\Omega)$ be isomorphisms such that $(x,y)_{1/2,-1/2} = (\iota_+ x, \iota_- y)$ holds for all $x \in H^{1/2}(\partial\Omega)$ and $y \in H^{-1/2}(\partial\Omega)$.

In the following we assume that the Dirichlet problem for $\mathcal{L}$ is uniquely solvable, i.e.

(D) for every $g \in L^2(\Omega)$ there exists a unique function $f \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\mathcal{L}f = g$ holds.

Let w be a real-valued function on Ω such that $w(x) \neq 0$ for a.e. $x \in \Omega$ and $w, \frac{1}{w} \in L^\infty(\Omega)$, and equip the space $L^2(\Omega)$ with the (in general indefinite) inner product

$$\langle f, g \rangle = \int_\Omega f(x)\overline{g(x)}w(x)\,dx.$$

Observe that $f \in L^2(\Omega)$ if and only if $wf \in L^2(\Omega)$ and also if and only if $\frac{1}{w}f \in L^2(\Omega)$. We consider the operator

$$Tf = \frac{1}{w}\,\mathcal{L}f, \qquad \operatorname{dom} T = \mathcal{D}_{\max} = \big\{ f \in L^2(\Omega) \colon \mathcal{L}f \in L^2(\Omega) \big\}, \tag{3.14}$$

in $L^2(\Omega)$ defined in the sense of distributions. The assumption (D) implies that the domain of T can be decomposed into a direct sum:

$$\operatorname{dom} T = \big(H^2(\Omega) \cap H^1_0(\Omega) \big) \dotplus \ker T.$$

The functions $f \in \operatorname{dom} T$ will be decomposed accordingly, we write $f = f_D + f_0$, where $f_D \in H^2(\Omega) \cap H^1_0(\Omega)$ and $f_0 \in \ker T$. Let $\mathcal{G} = L^2(\partial\Omega)$ and define the boundary mappings by

$$\Gamma_0 f := \iota_- f_0|_{\partial\Omega} \quad \text{and} \quad \Gamma_1 f := -\iota_+ \frac{\partial f_D}{\partial \nu}\Big|_{\partial\Omega}, \quad f = f_D + f_0 \in \operatorname{dom} T.$$

The boundary mappings Γ_0 and Γ_1 are well defined since $f_0|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ and $\frac{\partial f_D}{\partial \nu}|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$, cf. [23]. Corollary 2.5 implies now the following statement, cf. [9, Proposition 3.1].

Corollary 3.5. *The operator T defined in* (3.14) *is the adjoint of the densely defined closed symmetric operator*

$$Sf = \frac{1}{w}\,\mathcal{L}f, \qquad \operatorname{dom} S = \left\{ f \in H^2(\Omega) \cap H^1_0(\Omega) : \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\},$$

in the Krein or Hilbert space $(L^2(\Omega), \langle\cdot,\cdot\rangle)$ *and* $(L^2(\partial\Omega), \Gamma_0, \Gamma_1)$ *is a boundary triple for T.*

Proof. By the classical trace theorem on $H^2(\Omega)$ the mapping

$$H^2(\Omega) \cap H^1_0(\Omega) \ni f_D \mapsto \frac{\partial f_D}{\partial \nu}\Big|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$$

is surjective and according to [19, Theorem 2.1] the same holds for the map $\ker T \ni f_0 \mapsto f_0|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$.. This implies that condition (ii) in Corollary 2.5 holds. In order to verify condition (iii) in Corollary 2.5 we show that

$$D[f,f] = \langle Tf, f \rangle - (\Gamma_1 f, \Gamma_0 f)$$

is real for $f \in \operatorname{dom} T$, cf. condition (iii)$''$ in Remark 2.10. In fact, since $Tf_0 = 0$ we find

$$\begin{aligned} D[f,f] &= \langle Tf_D, f_D \rangle + \langle Tf_D, f_0 \rangle - \langle f_D, Tf_0 \rangle - (\Gamma_1 f, \Gamma_0 f) \\ &= \langle Tf_D, f_D \rangle + \int_\Omega \Big((\mathcal{L}f_D)(x)\overline{f_0(x)} - f_D(x)\overline{\mathcal{L}f_0(x)} \Big) dx - (\Gamma_1 f, \Gamma_0 f) \end{aligned}$$

and from Green's identity (which is applicable since $f_D \in H^2(\Omega)$ and $f_0 \in \mathcal{D}_{\max}$, cf. [23]) we conclude that the above expression takes the form

$$\langle Tf_D, f_D\rangle + \int_{\partial\Omega} \left(f_D|_{\partial\Omega} \overline{\frac{\partial f_0}{\partial \nu}\Big|_{\partial\Omega}} - \frac{\partial f_D}{\partial \nu}\Big|_{\partial\Omega} \overline{f_0|_{\partial\Omega}} \right) d\sigma + \left(\frac{\partial f_D}{\partial \nu}\Big|_{\partial\Omega}, f_0|_{\partial\Omega} \right)_{\frac{1}{2},-\frac{1}{2}},$$

where we have also used the definition of Γ_0 and Γ_1. From $f_D|_{\partial\Omega} = 0$ we obtain

$$D[f,f] = \langle Tf_D, f_D\rangle = \int_\Omega (\mathcal{L}f_D)(x)\overline{f_D(x)}\, dx,$$

which is real by Green's identity. Finally, it follows immediately from assumption (D) that condition (i)$'''$ in Remark 2.9 holds for $\lambda = 0$ and the self-adjoint relation $\Theta = \{\{0; g\} : g \in L^2(\partial\Omega)\}$. □

3.4 Multiplication operators in L^2

Let $(\mathcal{G}, (\cdot,\cdot)_\mathcal{G})$ be a Hilbert space and Q a Nevanlinna or Riesz–Herglotz function whose values are bounded operators in $\mathcal{G}$, i.e. Q is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, $Q(\overline{\lambda}) = Q(\lambda)^*$ and $\operatorname{Im} Q(\lambda) := \frac{1}{2i}\big(Q(\lambda) - Q(\lambda)^*\big) \geq 0$ if $\operatorname{Im}\lambda > 0$. Then Q has an integral representation of the form

$$Q(\lambda) = \lambda B + C + \int_\mathbb{R} \Big(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \Big) d\Sigma(t), \tag{3.15}$$

where B and C are bounded self-adjoint operators with $B \geq 0$ and Σ is an operator-valued measure, i.e. a mapping $\Sigma\colon \mathcal{B}_b(\mathbb{R}) \to \mathcal{L}(\mathcal{G})$, where $\mathcal{B}_b(\mathbb{R})$ denotes the set of bounded Borel subsets of $\mathbb{R}$ and $\mathcal{L}(\mathcal{G})$ denotes the set of bounded operators, with the properties

$$\Sigma(\varnothing) = 0, \quad \Sigma(\Delta) \geq 0 \text{ for } \Delta \in \mathcal{B}_b(\mathbb{R}), \quad \Sigma \text{ is strongly countably additive.}$$

Moreover, Σ satisfies the following property,

$$\int_\mathbb{R} \frac{1}{1+t^2}\, d\Sigma(t) \;\text{ is a bounded operator in } \mathcal{G}.$$

As in [8] and [26] we define the space $L^2(\mathbb{R}, \mathcal{G}, \Sigma)$ as follows. Let $C_{00}(\mathbb{R}, \mathcal{G})$ be the set of strongly continuous functions on $\mathbb{R}$ with compact support such that the values are in a finite dimensional subspace of $\mathcal{G}$ (this subspace depends on the function). For $f, g \in C_{00}(\mathbb{R}, \mathcal{G})$ the following semi-inner product exists,

$$\langle f, g\rangle_\Sigma = \int_\mathbb{R} \big(d\Sigma(t) f(t), g(t)\big)_\mathcal{G} = \lim_{d(\pi_n)\to 0} \sum_{k=1}^n \big(\Sigma(\Delta_k) f(\xi_k), g(\xi)\big)_\mathcal{G},$$

where $\pi_n = (t_k)_{k=0}^n$, $t_0 < \cdots < t_n$, is a partition such that $\operatorname{supp} f, \operatorname{supp} g \subset [t_0, t_n]$, $d(\pi_n)$ its diameter, $\Delta_k := (t_{k-1}, t_k]$ and $\xi_k \in [t_{k-1}, t_k]$. The space $L^2(\mathbb{R}, \mathcal{G}, \Sigma)$ is defined as the completion of $C_{00}(\mathbb{R}, \mathcal{G})$ and then factorization with respect to the set of f for which $\langle f, f\rangle_\Sigma = 0$. It can be easily shown that if $f \in L^2(\mathbb{R}, \mathcal{G}, \Sigma)$ and φ is a scalar, bounded, continuous function defined on $\mathbb{R}$, then $\varphi f \in L^2(\mathbb{R}, \mathcal{G}, \Sigma)$ with $\|\varphi f\| \leq \|\varphi\|_\infty \|f\|$. This shows also that the multiplication operator by the independent variable in $L^2(\mathbb{R},\mathcal{G},\Sigma)$ with maximal domain is self-adjoint since it is symmetric and for $\lambda\in\mathbb{C}\setminus\mathbb{R}$, the resolvent operator $f(t)\mapsto\frac{1}{t-\lambda}f(t)$ is a bounded operator.

For $x\in\mathcal{G}$, the function $\frac{1}{\sqrt{1+t^2}}x$ is in $L^2(\mathbb{R},\mathcal{G},\Sigma)$ since it can be approximated by functions of the form $\varphi_n(t)x$, where φ_n has compact support. Moreover, one can show that

$$\left\|\frac{1}{\sqrt{1+t^2}}x\right\|_\Sigma^2\le\left\|\int_{\mathbb{R}}\frac{1}{1+t^2}\,d\Sigma(t)\right\|\,\|x\|_{\mathcal{G}}^2\,. \tag{3.16}$$

For a function f for which $\sqrt{1+t^2}f(t)\in L^2(\mathbb{R},\mathcal{G},\Sigma)$ one can define $\int_{\mathbb{R}}d\Sigma(t)f(t)$ using the Riesz representation theorem and (3.16) such that

$$\left(\int_{\mathbb{R}}d\Sigma(t)f(t),x\right)_{\mathcal{G}}=\left\langle\sqrt{1+t^2}\,f(t),\,\frac{1}{\sqrt{1+t^2}}x\right\rangle_\Sigma\qquad\text{for every } x\in\mathcal{G}.$$

Let

$$\mathcal{G}_0:=\ker B,\quad \mathcal{G}_1:=\mathcal{G}_0^\perp,\quad B_1:=B\restriction\mathcal{G}_1,$$

$$\mathcal{H}_B:=\operatorname{ran}B_1^{1/2}\quad\text{with inner product}\quad\langle x,y\rangle_{\mathcal{H}_B}:=\big(B_1^{-1/2}x,B_1^{-1/2}y\big)_{\mathcal{G}}$$

$$\mathcal{H}:=L^2(\mathbb{R},\mathcal{G},\Sigma)\oplus\mathcal{H}_B\quad\text{with inner product }\langle\cdot,\cdot\rangle.$$

In the Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle)$ define the linear relation

$$T:=\left\{\left\{\begin{pmatrix}f(t)\\x\end{pmatrix};\begin{pmatrix}g(t)\\y\end{pmatrix}\right\}\in\mathcal{H}\oplus\mathcal{H}\colon\ \begin{matrix}g(t)=tf(t)-c\text{ for some }c\in\mathcal{G}\\ \text{and a.e. }t\in\mathbb{R},\ x=Bc\end{matrix}\right\}$$

and the boundary mappings $\Gamma_0,\Gamma_1\colon T\to\mathcal{G}$ by

$$\Gamma_0\left\{\begin{pmatrix}f(t)\\x\end{pmatrix};\begin{pmatrix}g(t)\\y\end{pmatrix}\right\}:=c,$$

$$\Gamma_1\left\{\begin{pmatrix}f(t)\\x\end{pmatrix};\begin{pmatrix}g(t)\\y\end{pmatrix}\right\}:=y+Cc+\int_{\mathbb{R}}d\Sigma(t)\frac{1}{1+t^2}\Big(f(t)+tg(t)\Big),$$

where c is as above, i.e. $c=tf(t)-g(t)$. The integral in the definition of Γ_1 is well defined since

$$\sqrt{1+t^2}\frac{1}{1+t^2}\Big(f(t)+tg(t)\Big)=\frac{1}{\sqrt{1+t^2}}f(t)+\frac{t}{\sqrt{1+t^2}}g(t)\in L^2(\mathbb{R},\mathcal{G},\Sigma).$$

Remark 3.6. Note that if $B=0$, the space $\mathcal{H}$ can be identified with $L^2(\mathbb{R},\mathcal{G},\Sigma)$. Moreover, the relation T is an operator if and only if $B=0$ and the space $L^2(\mathbb{R},\mathcal{G},\Sigma)$ does not contain non-zero constants.

As a consequence of Theorem 2.3 we obtain the following corollary which is a generalization of [15, Proposition 5.3] and [26, Proposition 7.9].

Corollary 3.7. *Assume that the function Q in* (3.15) *satisfies* $0\in\rho(\operatorname{Im}Q(i))$. *Then the relation T is the adjoint of the closed symmetric operator*

$$S\begin{pmatrix}f(t)\\0\end{pmatrix}=\begin{pmatrix}tf(t)\\-\displaystyle\int_{\mathbb{R}}d\Sigma(t)f(t)\end{pmatrix},$$

$$\operatorname{dom}S=\left\{\begin{pmatrix}f(t)\\0\end{pmatrix}\in\mathcal{H}\colon tf(t)\in L^2(\mathbb{R},\mathcal{G},\Sigma),\ \int_{\mathbb{R}}d\Sigma(t)f(t)\in\mathcal{H}_B\right\},$$

in the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *and* $(\mathcal{G}, \Gamma_0, \Gamma_1)$ *is a boundary triple for* T.

Remark 3.8. It can be shown in the same way as in [15, 26] that the so-called Weyl function associated with the boundary triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ in Corollary 3.7 coincides with the given Nevanlinna function Q in (3.15).

Proof of Corollary 3.7. We show that the assumptions of Theorem 2.3 are satisfied. The relation $\ker \Gamma_0$ is the orthogonal sum of the maximal multiplication operator with the independent variable in $L^2(\mathbb{R}, \mathcal{G}, d\Sigma)$ and the completely multi-valued relation $\{0\} \oplus \mathcal{H}_B$ and hence self-adjoint, i.e. (i) is satisfied with $\Theta = \{\{0; g\} : g \in L^2(\partial\Omega)\}$.

To show (iii), we employ Remark 2.10. For

$$\left\{ \begin{pmatrix} f(t) \\ x \end{pmatrix}; \begin{pmatrix} g(t) \\ y \end{pmatrix} \right\} \in T$$

we calculate

$$\begin{aligned}
&D\left[\left\{ \begin{pmatrix} f(t) \\ x \end{pmatrix}; \begin{pmatrix} g(t) \\ y \end{pmatrix} \right\}, \left\{ \begin{pmatrix} f(t) \\ x \end{pmatrix}; \begin{pmatrix} g(t) \\ y \end{pmatrix} \right\}\right] \\
&= \left\langle \begin{pmatrix} g(t) \\ y \end{pmatrix}, \begin{pmatrix} f(t) \\ x \end{pmatrix} \right\rangle - \left(y + Cc + \int_{\mathbb{R}} d\Sigma(t) \frac{1}{1+t^2} \Big(f(t) + tg(t) \Big), \, c \right)_{\mathcal{G}} \\
&= \langle g, f\rangle_\Sigma + \big(B_1^{-1/2} y, B_1^{-1/2} Bc\big)_{\mathcal{G}} - (y, c)_{\mathcal{G}} - (Cc, c)_{\mathcal{G}} \\
&\quad - \left\langle \frac{1}{\sqrt{1+t^2}} \Big(f(t) + tg(t) \Big), \frac{1}{\sqrt{1+t^2}} c \right\rangle_\Sigma \\
&= \langle g, f\rangle_\Sigma - (Cc, c)_{\mathcal{G}} \\
&\quad - \left\langle \frac{1}{\sqrt{1+t^2}} f(t) + \frac{t}{\sqrt{1+t^2}} g(t), \, \frac{t}{\sqrt{1+t^2}} f(t) - \frac{1}{\sqrt{1+t^2}} g(t) \right\rangle_\Sigma \\
&= -(Cc, c)_{\mathcal{G}} - \left\langle \frac{t}{1+t^2} f, f \right\rangle_\Sigma + 2 \operatorname{Re} \left\langle \frac{1}{1+t^2} f, g \right\rangle_\Sigma + \left\langle \frac{t}{1+t^2} g, g \right\rangle_\Sigma,
\end{aligned}$$

which is real and implies Green's identity (iii) by condition (iii)$''$.

The surjectivity of Γ, i.e. the relation (ii) is shown as follows. The element

$$\hat{f} = \left\{ \begin{pmatrix} \frac{1}{1+t^2} a \\ 0 \end{pmatrix}; \begin{pmatrix} \frac{t}{1+t^2} a \\ Ba \end{pmatrix} \right\}$$

has boundary values $\Gamma_0 \hat{f} = 0$, $\Gamma_1 \hat{f} = \operatorname{Im} Q(i) a.$. Note that the functions $\frac{1}{1+t^2} a$ and $\frac{t}{1+t^2} a$ are in $L^2(\mathbb{R}, \mathcal{G}, \Sigma)$ since they are products of scalar functions in $L^2(\mathbb{R}, \frac{1}{1+t^2} dt)$ and a fixed element in $\mathcal{G}$. The deficiency element

$$\hat{g} = \left\{ \begin{pmatrix} \frac{1}{t-i} b \\ Bb \end{pmatrix}; \begin{pmatrix} \frac{i}{t-i} b \\ iBb \end{pmatrix} \right\}$$

has boundary values $\Gamma_0 \hat{g} = b$, $\Gamma_1 \hat{g} = Q(i) b$. Since $\operatorname{Im} Q(i)$ is surjective, this shows that $\Gamma = \binom{\Gamma_0}{\Gamma_1}$ is surjective. □

Acknowledgement

The second author (M.L.) gratefully acknowledges the support of EPSRC, grant no. EP/E037844/1.

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