On the number of negative eigenvalues of the Laplacian on a metric graph

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Abstract. The number of negative eigenvalues of self adjoint Laplacians on metric graphs is calculated in terms of the boundary conditions and the underlying geometric structure. This extends and complements earlier results by Kostrykin and Schrader from [15].

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1. Introduction

Differential operators on metric graphs – also called quantum graphs – have attracted a lot of attention in the recent past, see, e.g., [18] for a brief survey, various applications and further references on quantum graphs. In many cases one is particularly interested in the differential expression $-\Delta$ and the spectral properties of its self adjoint realizations on the graph. The aim of this note is to derive a formula for the number of negative eigenvalues in terms of the boundary conditions.

Recall that one way to parameterize all self adjoint realizations T of $-\Delta$ in the L^2 space on the metric graph is as follows: The operator acts as $T\psi = -\Delta\psi$ on functions satisfying

$$A\underline{\psi} + B\underline{\psi'} = 0, \tag{1}$$

where $\underline{\psi}$ and $\underline{\psi'}$ denote the vertex values of the functions ψ and ψ' in the maximal domain of the Laplacian and $\{A, B\}$ are certain pairs of matrices (see Proposition 1 for the details). In [15] the number $n_-(T)$ of negative eigenvalues of the self adjoint Laplacian T in (1) was estimated from above by $n_+(AB^*)$, the number of positive eigenvalues of AB^*, and equality was obtained in the case of a star graph. In the present paper we use a different approach to calculate the exact number for every connected finite graph as

$$n_-(T) = n_+(AB^* + BM_0B^*),$$

where the correction term M_0 is a matrix that reflects geometric properties of the underlying metric graph. An immediate consequence of this result is a simple characterization of non-negative Laplacians, which is important in connection with the corresponding heat semigroups; cf. [15].

The article is organized as follows. In Section 2 we recall the necessary notations and we formulate the main result; cf. Theorem 1. The rest of this note is mainly devoted to the proof and a short discussion of the result. In Section 3 two different parameterizations of the self adjoint realizations from [14] and [17] are compared. A Titchmarsh-Weyl function M associated to the Dirichlet Laplacian is introduced in Section 4. Together with the pair of matrices $\{A, B\}$ the negative eigenvalues of the self adjoint Laplacian in (1) can be characterized in a convenient way; cf. Proposition 3. This connection is well known from abstract extension theory, but in order to be as self-contained as possible we reprove here some simple statements in our context. The reader is referred to, e.g., [4, §3], [5, §4] and also [22] for a general description of the spectral properties of the self adjoint extensions of a nonnegative operator in terms of abstract boundary conditions with the help of so-called boundary triplets and their Weyl functions, see also [3, 10, 23, 24, 25] for applications to quantum graphs. The actual proof of Theorem 1 is then given in Section 5. By applying a variational principle from [8] counting of the eigenvalues of a holomorphic matrix function is reduced to counting eigenvalues of a matrix. Finally, in Section 6 we make some remarks in order to illustrate the reflection of the graph in the obtained formula.

We want to point out that the proof of the main result can alternatively be deduced from [4, Theorem 5] (see also [9]). However, the present approach seems to be more direct and does not use the machinery of boundary triplets and relations (see also Section 6 for further comments on this connection) and is adopted to the notions used with quantum graphs.

2. Notations and the main result

A *quantum graph* is a differential operator acting on functions defined on a metric graph. Hence it can be seen as a triple consisting of

- a metric graph $\mathcal{G}$,
- a differential expression on the edges, and
- matching (boundary) conditions at the vertices.

Note that these parts are not independent, as, e.g., the matching conditions have to agree with the geometric structure.

Roughly speaking a *metric graph* $\mathcal{G}$ is a geometric object consisting of intervals (called *edges*) which are glued together in a certain way. More precisely, we have two sets of intervals, the internal and external edges, respectively: For each $i \in \mathcal{I}$ there is associated a finite interval $[0, a_i]$ with $a_i > 0$, and to each $e \in \mathcal{E}$ there corresponds a copy of the half line $\mathbb{R}^+ = [0, \infty)$. A metric graph $\mathcal{G}$ is then obtained by connecting the endpoints of the edges. Formally this can be done for instance by introducing an equivalence relation on the set of endpoints and identifying the equivalence classes as *vertices*. We will not go into more technical details here.

Within this note we are considering finite graphs only, i.e., the number of edges $|\mathcal{I}| + |\mathcal{E}|$ is finite, and we furthermore assume that $\mathcal{G}$ is connected. In what follows we are mainly using the notation from [14]. For a finite connected graph $\mathcal{G}$ define the space $L^2(\mathcal{G})$ as

$$L^2(\mathcal{G}) := \bigoplus_{e\in\mathcal{E}} L^2(\mathbb{R}^+) \;\oplus\; \bigoplus_{i\in\mathcal{I}} L^2(0, a_i) \tag{2}$$

and equip $L^2(\mathcal{G})$ with the usual scalar product induced by the scalar products in $L^2(\mathbb{R}^+)$ and $L^2(0, a_i)$, $i \in \mathcal{I}$, so that $L^2(\mathcal{G})$ becomes a Hilbert space. The elements $\psi \in L^2(\mathcal{G})$ will often be written with respect to the decomposition (2) in the form

$$\psi = \bigoplus_{e\in\mathcal{E}} \psi_e \;\oplus\; \bigoplus_{i\in\mathcal{I}} \psi_i, \qquad \psi_e \in L^2(\mathbb{R}_+),\ \psi_i \in L^2(0, a_i).$$

As this note deals with the Laplace operator $-\Delta$ here the differential expression is $-\frac{d^2}{dx^2}$ on each edge, more precisely,

$$-\Delta\psi = \bigoplus_{e\in\mathcal{E}} (-\psi_e'') \oplus \bigoplus_{i\in\mathcal{I}} (-\psi_i''), \qquad \psi \in \mathcal{D}_{\max} := \bigoplus_{e\in\mathcal{E}} H^2(\mathbb{R}^+) \oplus \bigoplus_{i\in\mathcal{I}} H^2(0, a_i),$$

where $H^2(0,a_i)$ and $H^2(\mathbb{R}^+)$ denote the usual second order Sobolev spaces. The operator defined on the maximal domain $\mathcal{D}_{\max}$ is denoted by $T_{\max}$.

In order to describe the matching conditions at the vertices we need the following notation. Consider the $n := |\mathcal{E}| + 2|\mathcal{I}|$ dimensional space

$$\mathcal{K} = \mathbb{C}^{|\mathcal{E}|} \oplus \mathbb{C}^{2|\mathcal{I}|}$$

and define the boundary or vertex values $\underline{\psi}, \underline{\psi'} \in \mathcal{K}$ of ψ and ψ' for $\psi \in \mathcal{D}_{\max}$ by

$$\underline{\psi} := \begin{pmatrix} \{\psi_e(0)\}_{e\in\mathcal{E}} \\ \left\{ \begin{array}{c} \psi_i(0) \\ \psi_i(a_i) \end{array} \right\}_{i\in\mathcal{I}} \end{pmatrix} \quad \text{and} \quad \underline{\psi'} := \begin{pmatrix} \{\psi'_e(0)\}_{e\in\mathcal{E}} \\ \left\{ \begin{array}{c} \psi'_i(0) \\ -\psi'_i(a_i) \end{array} \right\}_{i\in\mathcal{I}} \end{pmatrix}, \tag{3}$$

respectively. Boundary conditions are then given in the form

$$A\underline{\psi} + B\underline{\psi'} = 0, \qquad \psi \in \mathcal{D}_{\max},$$

where $\{A, B\}$ is a pair of $n \times n$ matrices. More precisely, these conditions are matching conditions of the vertex values of the functions, but for simplicity they will be refered to as *boundary conditions*. In order to avoid misunderstandings we point out that this notation is not related to the boundary of the underlying graph but rather originates from abstract extension theory. We will make use of the notion of Nevanlinna pairs given in the next definition, see, e.g., [6].

Definition 1 *A pair $\{A, B\}$ of $n \times n$ matrices is said to be a* Nevanlinna pair *if $A\,B^* = B\,A^*$ and the $n \times 2n$ matrix $[A\,, B]$ has maximal rank n.*

Alternatively, one can also deal with the corresponding subspaces in $\mathcal{K} \times \mathcal{K}$, which then are called Lagrangian. The next proposition, which gives a parametrization of all self adjoint realizations of the Laplacian, is a reformulation from [14].

Proposition 1 *For every Nevanlinna pair $\{A, B\}$ in $\mathcal{K}$ the operator $T = -\Delta$ defined for all $\psi \in \mathcal{D}_{\max}$ that satisfy*

$$A\underline{\psi} + B\underline{\psi'} = 0 \tag{4}$$

is a self adjoint realization of $-\Delta$ in $L^2(\mathcal{G})$ and, conversely, for every self adjoint realization T of $-\Delta$ in $L^2(\mathcal{G})$ there exists a (non-unique) Nevanlinna pair $\{A, B\}$ such that (4) *holds for all $\psi \in \operatorname{dom} T$.*

Note that this is a purely operator theoretic result, that does not take into account the underlying geometric structure. In view of quantum graphs only those $\{A, B\}$ are of interest which respect the structure of the metric graph, see, e.g., [16, 19, 20, 21].

In order to formulate our main result we will make use of the following notation. The symmetric 2×2 matrices $m_{i,0}$ are defined by

$$m_{i,0} = \frac{1}{a_i} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad i \in \mathcal{I},$$

and the symmetric $n \times n$ matrix M_0 in $\mathcal{K}$ is defined by

$$M_0 = \begin{pmatrix} 0_{|\mathcal{E}|} & 0 \\ 0 & M_{\mathcal{I},0} \end{pmatrix}, \quad \text{where} \quad M_{\mathcal{I},0} = \begin{pmatrix} m_{1,0} & & 0 \\ & \ddots & \\ 0 & & m_{|\mathcal{I}|,0} \end{pmatrix}. \tag{5}$$

Here the upper left block $0_{|\mathcal{E}|}$ of M_0 is the zero matrix in $\mathbb{C}^{|\mathcal{E}|}$ and the lower right block $M_{\mathcal{I},0}$ is the $2|\mathcal{I}| \times 2|\mathcal{I}|$ matrix with the 2×2 matrices $m_{i,0}$, $i = 1, \dots, |\mathcal{I}|$, as diagonal entries. Observe that M_0 is nonpositive.

For a self adjoint operator S such that $\sigma(S) \cap (-\infty, 0)$ $(\sigma(S) \cap (0, \infty))$ consists of finitely many eigenvalues with finite multiplicities we denote by $n_-(S)$ $(n_+(S)$, respectively) their total number, counted with multiplicities. The next statement is the main theorem of this note.

Theorem 1 *Let $\mathcal{G}$ be a connected finite graph, and let T be a self adjoint realization of the Laplacian in $L^2(\mathcal{G})$, that is,*

$$T = -\Delta, \qquad \operatorname{dom} T = \{\psi \in \mathcal{D}_{\max} : A\underline{\psi} + B\underline{\psi}' = 0\}, \tag{6}$$

where $\{A, B\}$ is a Nevanlinna pair, and let M_0 be the symmetric $n \times n$ matrix in (5). *Then the number of negative eigenvalues of T is given by*

$$n_-(T) = n_+(AB^* + BM_0B^*). \tag{7}$$

In particular, T is nonnegative if and only if the matrix $AB^* + BM_0B^*$ is nonpositive.

The proof of Theorem 1 will be given in Section 5. An important ingredient is a Titchmarsh-Weyl function M for the Dirichlet Laplacian in $L^2(\mathcal{G})$ which is defined and studied in Section 4. It will turn out, in particular, that the matrix M_0 in (5) coincides with the limit $M(0^-) = \lim_{\lambda \to 0^-} M(\lambda)$ of this matrix function; cf. Lemma 3. The proof of Theorem 1 also provides an alternative way of showing [15, Theorem 3.7]. The estimate $n_-(T) \le n_+(AB^*)$ in [15] (which, in general, is not sharp; cf. [13, Example 3.8]) follows directly from the fact that $BM_0B^* \le 0$. Furthermore, in the case of a star graph (that is, $\mathcal{I} = \emptyset$) we have $M_0 = 0$ (see (5)) and hence $n_-(T) = n_+(AB^*)$ holds.

3. Self adjoint realizations, revisited

There are various possibilities to parameterize the self adjoint realizations of $-\Delta$ in $L^2(\mathcal{G})$, i.e., the operators T that satisfy $T = T^* \subset T_{\max}$. In Proposition 1 we recalled a parametrization in terms of Nevanlinna pairs due to Kostrykin and Schrader [14]. In what follows we are going to recall another (equivalent) parametrization due to Kuchment [17] and make the connection between them explicit. Let us start with a simple but useful technical observation.

Lemma 1 *A pair $\{A, B\}$ of $n \times n$ matrices is a Nevanlinna pair if and only if*

$$\operatorname{ran} \begin{bmatrix} B^* \\ -A^* \end{bmatrix} = \ker [A\,, B] \tag{8}$$

holds.

Proof. The condition $A\,B^* = B\,A^*$ is equivalent to the inclusion $\subset$ in (8) and the maximality condition $\dim\ker[A\,,B] = n$ together with

$$\operatorname{ran}\begin{bmatrix} B^* \\ -A^* \end{bmatrix} = (\ker[B\,,-A]\,)^\perp \quad \text{and} \quad \dim(\ker[B\,,-A]\,)^\perp = n$$

shows (8). $\square$

The next proposition provides a slightly different description of the domains of the self adjoint realization of $-\Delta$ in $L^2(\mathcal{G})$; cf. [17, Theorem 6]. In contrast to Proposition 1, where different Nevanlinna pairs may lead to the same realization, here the correspondence is one-to-one. For a subspace $\mathcal{M} \subset \mathcal{K}$ we denote by $P_\mathcal{M}$ the orthogonal projection in $\mathcal{K}$ onto $\mathcal{M}$.

Proposition 2 *For every subspace $\mathcal{M} \subset \mathcal{K}$ and every symmetric matrix L in $\mathcal{M}$ the operator $T = -\Delta$ defined for all $\psi \in \mathcal{D}_{\max}$ that satisfy*

$$LP_\mathcal{M}\underline{\psi} + P_\mathcal{M}\underline{\psi}' = 0 \quad \text{and} \quad (I - P_\mathcal{M})\underline{\psi} = 0 \tag{9}$$

is a self adjoint realization of $-\Delta$ in $L^2(\mathcal{G})$ and, conversely, for every self adjoint realization T of $-\Delta$ in $L^2(\mathcal{G})$ there exists a unique subspace $\mathcal{M} \subset \mathcal{K}$ and a symmetric matrix L in $\mathcal{M}$ such that (9) *holds for all $\psi \in \operatorname{dom} T$.*

In the proof of our main result in Section 5 we are going to make use of both descriptions of one and the same operator. Therefore in the next lemma this connection is elaborated, see also [17, Corollary 5].

Lemma 2 *Let $\{A, B\}$ be a Nevanlinna pair in $\mathcal{K}$. Then the boundary condition*

$$A\underline{\psi} + B\underline{\psi}' = 0, \qquad \underline{\psi}, \underline{\psi}' \in \mathcal{K}, \tag{10}$$

can be written in the form

$$LP_\mathcal{M}\underline{\psi} + P_\mathcal{M}\underline{\psi}' = 0 \qquad \text{and} \qquad (I - P_\mathcal{M})\underline{\psi} = 0, \qquad \underline{\psi}, \underline{\psi}' \in \mathcal{K}, \tag{11}$$

where $\mathcal{M} := \operatorname{ran} B^*$ and L is defined by $L(B^*w) := P_\mathcal{M}A^*w$ for $w \in \mathcal{K}$. Conversely, let $\mathcal{M} \subset \mathcal{K}$ be a subspace and let L be a symmetric matrix in $\mathcal{M}$. Then $\{A, B\}$ defined by

$$A := \iota_\mathcal{M} L P_\mathcal{M} + (I - P_\mathcal{M}) \quad \text{and} \quad B := P_\mathcal{M}$$

is a Nevanlinna pair such that (11) *can be written in the form* (10)*. Here $\iota_\mathcal{M} : \mathcal{M} \to \mathcal{K}$ denotes the embedding.*

Proof. Note first that L is well-defined since $B^*w = 0$ together with $BA^*w = AB^*w = 0$ implies $A^*w \in \ker B = (\operatorname{ran} B^*)^\perp = \mathcal{M}^\perp$, that is, $P_\mathcal{M}A^*w = 0$. A straightforward calculation shows

$$\langle LB^*w, B^*v\rangle = \langle B^*w, LB^*v\rangle \quad \text{for all} \quad w, v \in \mathcal{M},$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product in $\mathcal{M}$, and hence L is symmetric. Assume now $A\underline{\psi} + B\underline{\psi}' = 0$ or, with Lemma 1 equivalently,

$$\begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix} \in \operatorname{ran} \begin{bmatrix} B^* \\ -A^* \end{bmatrix} = \ker [A \, , B].$$

Hence there exists a $u \in \mathcal{K}$ such that $\underline{\psi} = B^* u$ and $\underline{\psi}' = -A^* u$. In particular, this implies $\underline{\psi} \in \mathcal{M}$ and $(I - P_{\mathcal{M}})\underline{\psi} = 0$. Furthermore,

$$LP_{\mathcal{M}}\underline{\psi} + P_{\mathcal{M}}\underline{\psi}' = LB^* u + P_{\mathcal{M}}(-A^* u) = 0.$$

The proof of the converse statement is straight forward and left to the reader. $\square$

4. A Titchmarsh-Weyl function for the Dirichlet Laplacian

In this section we collect some results which follow from abstract extension theory. However, in order to keep the presentation self-contained we provide short proofs adapted for our context. We are going to introduce a matrix function M which contains the complete information on the spectrum of the self adjoint realizations of $-\Delta$ in $L^2(\mathcal{G})$. First we fix a special self adjoint realization T_D, namely the one with Dirichlet boundary conditions, i.e.,

$$T_D = -\Delta, \qquad \operatorname{dom} T_D = \{\psi \in \mathcal{D}_{\max} : \underline{\psi} = 0\}. \tag{12}$$

This self adjoint operator corresponds to the Nevanlinna pair $A = I$ and $B = 0$ in Proposition 1. The Dirichlet operator is of great technical importance for us as it serves as a reference operator. We remark that Dirichlet boundary conditions at the vertices do not respect the geometry of the underlying graph, and hence T_D does not correspond to $\mathcal{G}$ but rather to the set of disconnected intervals. More precisely, the operator T_D in (12) coincides with the orthogonal sum of the self adjoint differential operators $T_{D,j}$, $j \in \mathcal{I} \cup \mathcal{E}$, in $L^2(0, a_j)$ and $L^2(\mathbb{R}^+)$ with domains $H^2(0, a_j)$ and $H^2(\mathbb{R}^+)$, and Dirichlet boundary conditions at 0 and a_j if $j \in \mathcal{I}$ and at 0 if $j \in \mathcal{E}$, respectively. For $j \in \mathcal{I}$ the operator $T_{D,j}$ is positive and $\sigma(T_{D,j})$ consists of the simple eigenvalues $k^2\pi^2/a_j^2$, $k = 1, 2, \ldots$; for $j \in \mathcal{E}$ the operator $T_{D,j}$ is nonnegative and its spectrum $\sigma(T_{D,j}) = [0, \infty)$ is purely continuous. The next remark is an immediate consequence of these facts.

Remark 1 *If $\mathcal{E} = \emptyset$, then $\sigma(T_D)$ consists only of positive eigenvalues with multiplicity $\leq |\mathcal{I}|$ which accumulate to $+\infty$. If $\mathcal{I} = \emptyset$, then $\sigma(T_D) = [0, \infty)$ is purely continuous with multiplicity $|\mathcal{E}|$. If $\mathcal{E} \neq \emptyset$ and $\mathcal{I} \neq \emptyset$, then $\sigma(T_D) = [0, \infty)$ and there exists a sequence of embedded eigenvalues with multiplicity $\leq |\mathcal{I}|$ which accumulates to $+\infty$.*

Denote by $\sqrt{\cdot}$ the branch of the square root with a cut along $(-\infty, 0]$ fixed by $\sqrt{\lambda} \geq 0$ for $\lambda \in [0, \infty)$ and define the functions m_e, $e \in \mathcal{E}$, and m_i, $i \in \mathcal{I}$, by

$$m_e(\lambda) := -\sqrt{-\lambda} \quad \text{and} \quad m_i(\lambda) := \frac{\sqrt{\lambda}}{\sin \sqrt{\lambda} a_i} \begin{pmatrix} -\cos \sqrt{\lambda} a_i & 1 \\ 1 & -\cos \sqrt{\lambda} a_i \end{pmatrix}, \tag{13}$$

respectively. The functions m_e and m_i are defined for all λ in their maximal domains of holomorphy which are $\mathbb{C}\setminus[0,\infty)$ for m_e and $\mathbb{C}\setminus\sigma_p(T_{D,i})$ for m_i. Furthermore, we define the $n\times n$ matrix function M by

$$M(\lambda) := \begin{pmatrix} m_e(\lambda)I_{|\mathcal{E}|} & 0 \\ 0 & M_{\mathcal{I}}(\lambda) \end{pmatrix}, \text{ where } M_{\mathcal{I}}(\lambda) := \begin{pmatrix} m_1(\lambda) & & 0 \\ & \ddots & \\ 0 & & m_{|\mathcal{I}|}(\lambda) \end{pmatrix} \qquad (14)$$

and $I_{|\mathcal{E}|}$ is the identity matrix in $\mathbb{C}^{|\mathcal{E}|}$. Here the $2|\mathcal{I}|\times 2|\mathcal{I}|$ matrix function $M_{\mathcal{I}}$ is defined for all λ in the complement of $\bigcup_{i\in\mathcal{I}}\sigma_p(T_{D,i})$ in $\mathbb{C}$. It can be verified that the imaginary part of $M(\lambda)$ is nonnegative for $\lambda\in\mathbb{C}^+$ and $M(\lambda)^* = M(\bar\lambda)$ holds for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$, and hence M is a so-called Nevanlinna or Riesz-Herglotz function; cf. [12]. This together with the special form of M implies the assertions in the next lemma which can also be shown by direct computation.

Lemma 3 *The $n\times n$ matrix function M in (14) is holomorphic on $\mathbb{C}\setminus[0,\infty)$, for $\lambda\in(-\infty,0)$ the values $M(\lambda)$ are symmetric, and*

$$(M'(\lambda)x,x) > 0 \qquad \textit{and} \qquad \lim_{\lambda\to-\infty}(M(\lambda)x,x) = -\infty$$

hold for all $x\in\mathcal{K}$, $x\neq 0$. Furthermore, the limit $M(0^-) = \lim_{\lambda\to 0^-}M(\lambda)$ coincides with the symmetric $n\times n$ matrix M_0 in (5).

According to the next lemma the values $M(\lambda)$ of the matrix function M act as Dirichlet-to-Neumann operators.

Lemma 4 *For all $\psi_\lambda\in\ker(T_{\max}-\lambda)$ and $\lambda\notin\sigma(T_D)$ the function M has the property*

$$M(\lambda)\underline{\psi_\lambda} = \underline{\psi'_\lambda}.$$

Proof. Observe first that for the maximal operators $T_{\max,j}$ on the edges,

$$\psi_{j,\lambda}\in\ker(T_{\max,j}-\lambda) = \begin{cases} \operatorname{span}\{\sin\sqrt{\lambda}x, \cos\sqrt{\lambda}x\}, & j\in\mathcal{I}, \\ \operatorname{span}\{\exp(-\sqrt{-\lambda}x)\}, & j\in\mathcal{E}, \end{cases}$$

and $\lambda\notin\sigma(T_D)$ the functions m_i and m_e in (13) satisfy

$$m_e(\lambda)\psi_{e,\lambda}(0) = \psi'_{e,\lambda}(0) \quad \text{and} \quad m_i(\lambda)\begin{pmatrix}\psi_{i,\lambda}(0)\\ \psi_{i,\lambda}(a_i)\end{pmatrix} = \begin{pmatrix}\psi'_{i,\lambda}(0)\\ -\psi'_{i,\lambda}(a_i)\end{pmatrix}.$$

Now the assertion follows from (3) and the definition of the function M in (14). $\square$

From the construction of M it is clear that its singularities coincide with the spectral points of the Dirichlet operator. In the next proposition it will be shown how the function M is related to eigenvalues of self adjoint realizations of $-\Delta$.

Proposition 3 *Let T be the self adjoint realization of $-\Delta$ in $L^2(\mathcal{G})$ with the boundary conditions $LP_{\mathcal{M}}\underline{\psi} + P_{\mathcal{M}}\underline{\psi'} = 0$ and $(I - P_{\mathcal{M}})\underline{\psi} = 0$; cf. Proposition 2. Then for each $\lambda \notin \sigma(T_D)$ it holds*

$$\lambda \in \sigma_p(T) \quad \textit{if and only if} \quad \ker\,(L + P_{\mathcal{M}}M(\lambda)P_{\mathcal{M}}) \neq \{0\}$$

and

$$\dim\ker\,(T-\lambda) = \dim\ker\,(L + P_{\mathcal{M}}M(\lambda)P_{\mathcal{M}}).$$

Furthermore,

$$\{u \in \mathcal{K}:\ u = \underline{\psi_\lambda} \ \textit{for some}\ \ \psi_\lambda \in \ker\,(T-\lambda)\,\} = \ker\,(L + P_{\mathcal{M}}M(\lambda)P_{\mathcal{M}}).$$

Proof. Let us first mention that the mapping $\psi \mapsto \underline{\psi}$ from $\ker\,(T_{\max} - \lambda)$ to $\mathcal{K}$ is a bijection if $\lambda \notin \sigma(T_D)$. Assume now $\lambda \in \sigma_p(T)$ and let ψ_λ be a corresponding eigenfunction. Then $\underline{\psi_\lambda} \neq 0$ satisfies the boundary conditions in the assumptions and by Lemma 4 it holds $M(\lambda)\underline{\psi_\lambda} = \underline{\psi'_\lambda}$. As $P_{\mathcal{M}}\underline{\psi_\lambda} = \underline{\psi_\lambda}$ this implies

$$(L + P_{\mathcal{M}}M(\lambda)P_{\mathcal{M}})\underline{\psi_\lambda} = LP_{\mathcal{M}}\underline{\psi_\lambda} + P_{\mathcal{M}}\underline{\psi'_\lambda} = 0,$$

which shows $\underline{\psi_\lambda} \in \ker\,(L + P_{\mathcal{M}}M(\lambda)P_{\mathcal{M}})$.

Conversely, let $u \in \ker\,(L + P_{\mathcal{M}}M(\lambda)P_{\mathcal{M}})$. Then there exists a unique function $\psi_\lambda \in \ker\,(T_{\max} - \lambda)$ such that $\underline{\psi_\lambda} = u$. By assumption $u \in \mathcal{M}$ and hence $(I - P_{\mathcal{M}})\underline{\psi_\lambda} = 0$. Again employing Lemma 4 gives $LP_{\mathcal{M}}\underline{\psi_\lambda} + P_{\mathcal{M}}\underline{\psi'_\lambda} = 0$ and hence ψ_λ is an eigenfunction of T at the eigenvalue λ. □

For completeness we mention that the relation

$$\ker\,(L + P_{\mathcal{M}}M(\lambda)P_{\mathcal{M}}) = \ker\,(A + BM(\lambda)).$$

holds for all $\lambda \notin \sigma(T_D)$ if $\mathcal{M} \subset \mathcal{K}$ and L are connected with the Nevanlinna pair $\{A, B\}$ as in Lemma 2; cf. [1].

5. Proof of the main result

The essential ingredient in the proof of Theorem 1 is a variational principle for self adjoint matrix (and operator) functions which can be found in, e.g., [8] and will be briefly recalled. Let τ be a function defined on some interval $\mathcal{D} \subset \mathbb{R}$ whose values $\tau(\lambda)$ are symmetric $k \times k$ matrices. We say that $\lambda \in \mathcal{D}$ is an eigenvalue of τ if $\ker \tau(\lambda) \neq \{0\}$. The set of eigenvalues is denoted by $\sigma_p(\tau)$. The following proposition is a special case of [8, Theorem 2.1].

Proposition 4 *Let $\tau : (-\infty, 0) \to \mathbb{C}^{k\times k}$ be a symmetric, real analytic matrix function such that $\tau(0^-) = \lim_{\lambda\to 0^-} \tau(\lambda)$ exists and $\langle \tau'(\lambda)x, x\rangle < 0$ for all $\lambda \in (-\infty, 0)$, $x \in \mathbb{C}^k$, $x \neq 0$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{C}^k$. Then there exists $\lambda_0 < 0$ such that $(-\infty, \lambda_0) \cap \sigma_p(\tau) = \emptyset$ and $\sigma_p(\tau) \cap (-\infty, 0)$ consists of finitely many eigenvalues with total multiplicity*

$$n_-(\tau(0^-)) - n_-(\tau(\lambda_0)).$$

If $\tau(\lambda)$ is of the form $L-\lambda$ with some symmetric matrix L then this statement is evident, for nonlinear functions τ it follows from variational principles.

Proof of Theorem 1. Let now $\{A, B\}$ be a Nevanlinna pair and let T be the corresponding self adjoint realization of the Laplacian in $L^2(\mathcal{G})$; cf. (6). Define $\mathcal{M} := \operatorname{ran} B^* \subset \mathcal{K}$ and $L(B^*u) := P_{\mathcal{M}} A^* u$, $u \in \mathcal{K}$, as in Lemma 2. According to Remark 1 and Proposition 3 a point $\lambda \in (-\infty, 0)$ is an eigenvalue of T with multiplicity n_λ if and only if 0 is an eigenvalue of the matrix

$$L + P_{\mathcal{M}} M(\lambda) P_{\mathcal{M}}, \quad \text{(or, equivalently, of} \quad -L - P_{\mathcal{M}} M(\lambda) P_{\mathcal{M}}\,)$$

in $\mathcal{M}$ with multiplicity n_λ. Therefore the total number $n_-(T)$ (counting multiplicities) of negative eigenvalues of T coincides with

$$\sum_{\lambda \in (-\infty, 0)} \dim \ker \left(-L - P_{\mathcal{M}} M(\lambda) P_{\mathcal{M}}\right).$$

Observe that the sum is finite since $n_-(T) < \infty$, which follows from Remark 1, [2, §9, Theorem 3] and the fact that both T and T_D are self adjoint extensions of a symmetric operator with finite defect. Thus we have to count the negative eigenvalues of the symmetric matrix function

$$\tau(\lambda) := -L - P_{\mathcal{M}} M(\lambda) P_{\mathcal{M}}, \qquad \lambda \in (-\infty, 0).$$

The scalar product in $\mathcal{M}$ will again be denoted by $\langle \cdot, \cdot \rangle$ in order to distinguish it from the scalar product $(\cdot, \cdot)$ in $\mathcal{K}$. It follows from Lemma 3 that $\langle \tau'(\lambda) x, x \rangle < 0$ holds for all $x \in \mathcal{M}$, $x \neq 0$, and that $(M(\lambda)x, x)$ tends to $-\infty$ for $\lambda \to -\infty$, $x \neq 0$. Hence for λ_0 sufficiently small $\tau(\lambda_0)$ is positive definite and this implies $n_-(\tau(\lambda_0)) = 0$. Therefore Proposition 4 can be applied and yields

$$n_-(T) = n_-(\tau(0^-)) = n_-(-L - P_{\mathcal{M}} M(0^-) P_{\mathcal{M}}) = n_+(L + P_{\mathcal{M}} M_0 P_{\mathcal{M}});$$

cf. Lemma 3. We are now rewriting the quadratic form related to the matrix $L + P_{\mathcal{M}} M_0 P_{\mathcal{M}}$. Let $x \in \mathcal{M} = \operatorname{ran} B^*$ and $u \in \mathbb{C}^n$ such that $x = B^* u$. Then

$$\langle (L + P_{\mathcal{M}} M_0 P_{\mathcal{M}}) x, x \rangle = \langle L B^* u + P_{\mathcal{M}} M_0 B^* u, B^* u \rangle$$

and since $LB^*u = P_{\mathcal{M}} A^* u$ the expression above coincides with

$$\begin{aligned} \langle P_{\mathcal{M}}(A^*u + M_0 B^* u), B^* u \rangle &= (A^* u + M_0 B^* u, B^* u) \\ &= ((BA^* + BM_0 B^*) u, u). \end{aligned}$$

Therefore the number $n_-(T) = n_+(L + P_{\mathcal{M}} M_0 P_{\mathcal{M}})$ coincides with the number of positive eigenvalues of $BA^* + BM_0 B^*$, i.e., the relation (7) holds. This completes the proof of Theorem 1. $\square$

6. Concluding remarks

Theorem 1 shows how the number of negative eigenvalues depends on the boundary conditions and some aspects of the geometry of the underlying graph. We are going to add some observations in order to illustrate this dependence a little further.

Monotonicity. Assume we have a family of metric graphs $\mathcal{G}_t$ of the same structure but different size, more precisely, the internal edges are of length $t \cdot a_i$, where $t \in \mathbb{R}^+$, and fix some self adjoint boundary conditions for the Laplacian. Then the number of negative eigenvalues is decreasing if the parameter t is decreasing, due to the $1/t$-dependence of the non-positive matrix M_0. This is in accordance with the well known fact, that for a single interval the spectrum is pushed up if the interval shrinks.

Adding a vertex. The following observation is not directly connected to our question, but illustrates one important aspect. Assume that there is given a graph $\mathcal{G}$ and fix some self adjoint boundary conditions for the Laplacian. Then one can create a new quantum graph $\mathcal{G}_{\mathrm{new}}$ by adding one vertex in an interior point of one edge and by posing natural boundary conditions (in this case this means that ψ and ψ' are continous at the new vertex) and leaving the boundary conditions at the old vertices unchanged. Obviously, the eigenvalues of the corresponding self adjoint realizations $T(\mathcal{G})$ and $T(\mathcal{G}_{\mathrm{new}})$ coincide. According to Theorem 1 the number of positive eigenvalues of the corresponding matrices

$$AB^* + BM_0B^* \qquad \text{and} \qquad A_{\mathrm{new}}B^*_{\mathrm{new}} + B_{\mathrm{new}}M_{0\,\mathrm{new}}B^*_{\mathrm{new}}$$

coincide. However, in general, the individual eigenvalues of these matrices will differ. The reason for this at first sight surprising fact lies in the application of Proposition 4. It compares the number of eigenvalues in an interval, but not their position. And as a matter of fact the eigenvalues move differently for $\mathcal{G}$ and $\mathcal{G}_{\mathrm{new}}$.

Other parameterizations. Since Theorem 1 generalizes a result in [15] our formulation uses the same parametrization. In order to complete the picture we want to mention two other possible parameterizations. First the boundary conditions in (6) can also be written as

$$\mathrm{i}(S - I)\underline{\psi} - (S + I)\underline{\psi'} = 0, \tag{15}$$

where S is a unique unitary matrix, which allows an interpretation as a vertex scattering matrix, see [11] and [19]. Then formula (7) takes the form

$$n_-(T) = n_-(2\,\mathrm{Im}S + (S + I)M_0(S^* + I)).$$

Using the concept of linear relations in $\mathcal{K}$ (these are subspaces in $\mathcal{K}\times\mathcal{K}$, see, e.g., [7] for general references on these "multivalued operators"), one can parameterize the self adjoint boundary conditions also by self adjoint linear relations Θ in $\mathcal{K}$ via $\{\underline{\psi}, \underline{\psi'}\} \in \Theta$. The connection to the other parameterizations is then given by

$$\begin{aligned}\Theta = &-B^{-1}A = \{\{\underline{\psi}, \underline{\psi'}\} : A\underline{\psi} + B\underline{\psi'} = 0\} = \{\{B^*w, -A^*w\} : w \in \mathcal{K}\} \\ = &-L \oplus \{\{0, w_\infty\} : w_\infty \in \mathcal{M}^\perp\} \\ = &\ \mathrm{i}(S + I)^{-1}(S - I),\end{aligned}$$

where the matrices are identified with the graphs of the corresponding linear maps and the products and inverses are understood in the sense of linear relations. We mention that Θ can also be identified with $\ker[A, B]$ which is denoted by $\mathcal{M}(A, B)$ in [14]. Then $\lambda \in \sigma_p(T)$ if and only if $\ker(M(\lambda) - \Theta) \neq \{0\}$ and the crucial Proposition 3 follows directly from considering only the operator parts.

Schrödinger operators. Finally, we want to mention that from the proof it is clear that our approach works also for those Schrödinger operators $-\Delta + V$ for which the Titchmarsh-Weyl function of the corresponding Dirichlet operator satisfies Lemma 3. More general Schrödinger operators will be studied in an forthcoming publication.

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