Finite Rank Perturbations in Pontryagin Spaces and a Sturm-Liouville Problem with λ-rational Boundary Conditions

Jussi Behrndt and Friedrich Philipp

Dedicated with great pleasure to Heinz Langer on the occasion of his eightieth birthday

Abstract. For selfadjoint operators A_1 and A_2 in a Pontryagin space Π_κ such that the resolvent difference of A_1 and A_2 is n-dimensional it is shown that the dimensions of the spectral subspaces corresponding to open intervals in gaps of the essential spectrum differ at most by $n + 2\kappa$. This is a natural extension of a classical result on finite rank perturbations of selfadjoint operators in Hilbert spaces to the indefinite setting. With the help of an explicit operator model for scalar rational functions it is shown that the estimate is sharp. Furthermore, the general perturbation result and the operator model are illustrated with an application to a singular Sturm-Liouville problem, where the boundary condition depends rationally on the eigenparameter.

1. Introduction

Spectral theory of selfadjoint operators in indefinite inner product spaces is a classical area of operator theory which is strongly influenced by the contributions of T.Ya. Azizov, I.S. Iohvidov, M.G. Krein, H. Langer, L.S. Pontryagin, and many others. The monographs [4, 15, 36] provide a comprehensive introduction and detailed overview on the main developments in this field until the end of the eighties, and we also refer the reader there for further references and historical information.

In this note we are mostly interested in a perturbation problem for selfadjoint operators in Pontryagin spaces, which is a natural generalization of a classical and very useful Hilbert space result. More precisely, let A_1 and A_2 be selfadjoint operators in a Pontryagin space Π_κ with κ negative squares and assume that the resolvent difference of A_1 and A_2 is a finite rank operator, that is, for some $n \in \mathbb{N}$ we have

$$\dim \operatorname{ran} \left((A_1 - \lambda_0)^{-1} - (A_2 - \lambda_0)^{-1} \right) = n \tag{1.1}$$

for some (and hence for all) $\lambda_0 \in \rho(A_1) \cap \rho(A_2)$. Assume, in addition, that $\Delta \subset \mathbb{R}$ is an open interval which is a gap of the essential spectrum of A_1 or, equivalenty, a gap of the essential spectrum of A_2, and denote by $\operatorname{eig}(A_1, \Delta)$ and $\operatorname{eig}(A_2, \Delta)$ the dimensions of the spectral subspaces of A_1 and A_2, respectively, corresponding to Δ; cf. [37, 39, 40] for the construction and properties of the spectral function. In our main result in Theorem 3.1 we prove the estimate

$$|\operatorname{eig}(A_1, \Delta) - \operatorname{eig}(A_2, \Delta)| \leq n + 2\kappa, \tag{1.2}$$

which also turns out to be optimal. We emphasize that in the special case $\kappa = 0$ our result reduces to a standard fact in spectral and perturbation theory of selfadjoint operators in Hilbert spaces: The dimensions of the spectral subspaces corresponding to an open interval in a gap of the essential spectra of two selfadjoint operators A_1 and A_2 in a Hilbert space such that (1.1) holds differ at most by n, see, e.g. [14, Chapter 9.3, Theorem 3].

In order to show that the estimate in (1.2) is sharp we first provide an explicit operator or matrix model for a special class of rational functions in Section 4.1 which has been used in a similar context, but in a more abstract form, also in the recent papers [8, 9]. We do not discuss the precise construction via boundary triples, intermediate extensions and associated Weyl functions here in the Introduction, but we wish to emphasize the following interesting conclusion of the model: For a given set of pairwise distinct real numbers $\mu_1, \ldots, \mu_m$ and $\nu_1, \ldots, \nu_m$ we explicitely construct a Pontryagin space $(\mathbb{C}^m, [\cdot, \cdot])$ and two matrices A_1 and A_2 such that $A_1 - A_2$ is a rank one matrix (and hence also (1.1) holds with $n = 1$) and

$$\sigma_{\mathrm{p}}(A_1) = \{\mu_1, \ldots, \mu_m\} \quad \text{and} \quad \sigma_{\mathrm{p}}(A_2) = \{\nu_1, \ldots, \nu_m\}.$$

Here, A_1 is a diagonal matrix and A_2 is of a certain structured form, see Section 4.1 for more details.

As an illustration of the general perturbation result Theorem 3.1 and an application of the operator model in Section 4 we consider a λ-dependent Sturm-Liouville boundary value problem of the type

$$-f'' + qf = \lambda f, \qquad s(\lambda) f(0) + t(\lambda) f'(0) = 0, \tag{1.3}$$

on the half line $[0, \infty)$, where s and t are rational functions such that $r = s/t$ belongs to the special class of rational functions in Section 4, the potential $q \in L^1_{\mathrm{loc}}((0, \infty))$ is real valued, and the differential expression is regular at 0 and in the limit point case at ∞. Such types of boundary value problems have been considered in various works and different linearization techniques were developed in the past; for references see Section 5. Here we use a coupling technique for symmetric operators and associated boundary triples from [20] to construct a linear operator $\widetilde{A}$ in a Pontryagin space $L^2((0, \infty)) \times \mathbb{C}^m$ such that the eigenvalues of $\widetilde{A}$ coincide with the eigenvalues of the boundary value problem (1.3). With the help of the operator model for the function $r = s/t$

from Section 4.1 we are able to specify $\widetilde{A}$ explicitely. As implicitly mentioned above, we are particularly interested in the eigenvalues of this linearization. In addition to proving the already known fact that their geometric multiplicity is one, we also obtain information on the signature of the complete root subspace of isolated eigenvalues of $\widetilde{A}$.

2. Preliminaries

A Pontryagin space with κ negative squares is an indefinite inner product space $(\Pi_\kappa, [\cdot\,,\cdot])$ which admits a decomposition

$$\Pi_\kappa = \Pi_+ \, [\dot{+}] \, \Pi_-,$$

where $[\dot{+}]$ denotes the direct $[\cdot\,,\cdot]$-orthogonal sum, $(\Pi_\pm, \pm[\cdot\,,\cdot])$ are Hilbert spaces and $\dim \Pi_- = \kappa$. For a detailed treatment of Pontryagin spaces and operators therein we refer to the monographs [4, 15] and [36].

For the rest of this section let $(\Pi_\kappa, [\cdot\,,\cdot])$ be a Pontryagin space with κ negative squares. A (*closed*) *linear relation* in Π_κ is a (closed) linear subspace of $\Pi_\kappa \times \Pi_\kappa$. Linear operators in Π_κ are viewed as linear relations via their graphs. We shall usually omit the term "linear" and just speak of relations and operators. For a relation A in Π_κ the adjoint A^+ is defined by

$$A^+ := \big\{\{h, k\} : [g, h] = [f, k] \text{ for all } \{f, g\} \in A\big\}.$$

Note that A^+ is always closed. A relation A in Π_κ is called *symmetric* if $A \subset A^+$ and *selfadjoint* if $A = A^+$.

For the algebraic notions and operations related to relations, such as kernel, range, domain, multivalued part, as well as sum, product, and inverse, we refer to [16, 34], and for a detailed study of symmetric and selfadjoint relations in Pontryagin and Krein spaces we refer to [28, 29] and the references therein. We only recall that the *resolvent set* $\rho(A)$ of a relation A in Π_κ is defined as the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda)^{-1} \in \mathcal{B}(\Pi_\kappa)$, where $\mathcal{B}(\Pi_\kappa)$ denotes the space of bounded and and everywhere defined operators in Π_κ. The *spectrum* of A is defined as the complement of $\rho(A)$, i.e. $\sigma(A) = \mathbb{C} \setminus \rho(A)$. The *point spectrum* $\sigma_{\mathrm{p}}(A)$ of A is the set of all $\lambda \in \mathbb{C}$ such that $\ker(A - \lambda) \neq \{0\}$. For $\lambda \in \sigma_{\mathrm{p}}(A)$ the *root subspace* of A corresponding to λ is defined by $\mathcal{L}_\lambda(A) := \bigcup_{n \in \mathbb{N}} \ker((A - \lambda)^n)$. The dimension of $\mathcal{L}_\lambda(A)$ is called the *algebraic multiplicity* of the eigenvalue λ.

If A is a selfadjoint relation in Π_κ and $\rho(A) \neq \varnothing$ then we shall define the *essential spectrum* $\sigma_{\mathrm{ess}}(A)$ as the complement of the isolated eigenvalues of A with finite algebraic multiplicities in $\sigma(A)$. We also mention that the spectrum of a selfadjoint relation in Π_κ is always symmetric with respect to the real axis, and in the case that A is an operator it follows that with the possible exception of at most 2κ nonreal eigenvalues with finite multiplicities $\sigma(A)$ is real. In particular, we have $\rho(A) \neq \varnothing$ in this case; in general, for a selfadjoint relation $\sigma(A) = \mathbb{C}$ is possible; cf. [28] and [6, Lemma 2.2] for more details. Finally, we recall that a selfadjoint operator or relation A in Π_κ with

$\rho(A) \neq \varnothing$ admits a spectral function with the usual properties; cf. [37, 39, 40] and [29].

3. Finite rank perturbations of selfadjoint operators in Pontryagin spaces

In this section we formulate and prove a Pontryagin space variant of a well known result on finite rank perturbations of selfadjoint operators in Hilbert spaces; cf. Corollary 3.2. For this, we need some preparation. Let $\mathcal{L} \subset \Pi_\kappa$ be a subspace of Π_κ which is also a Pontryagin space. Then we have $\mathcal{L} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-$, where $(\mathcal{L}_\pm, \pm[\cdot\,,\cdot])$ are Hilbert spaces and $\dim \mathcal{L}_- \leq \kappa$. Since the dimensions of $\mathcal{L}_+$ and $\mathcal{L}_-$ do not depend on the particular decomposition of $\mathcal{L}$ (see, e.g., [15]), there is no ambiguity in defining the numbers

$$\kappa_-(\mathcal{L}) := \dim \mathcal{L}_- \quad \text{and} \quad \kappa_+(\mathcal{L}) := \dim \mathcal{L}_+.$$

Of course, we have $\kappa_+(\mathcal{L}) = \infty$ if and only if $\mathcal{L}$ is infinite dimensional. Note furthermore that we always have $\kappa_-(\mathcal{L}) \leq \kappa$. The pair $\{\kappa_+(\mathcal{L}), \kappa_-(\mathcal{L})\}$ is called the *signature* of $\mathcal{L}$ (with respect to the inner product $[\cdot\,,\cdot]$).

Let A be a selfadjoint operator or a selfadjoint relation in the Pontryagin space Π_κ with $\rho(A) \neq \varnothing$, and let $\Delta \subset \mathbb{R}$ be an open interval such that $\Delta \subset \mathbb{R} \setminus \sigma_{\text{ess}}(A)$. Then the closed linear span $\mathcal{L}_\Delta(A)$ of all root subspaces corresponding to the eigenvalues of A in Δ is a Pontryagin space (see, e.g., [40]) and we call the integer

$$\operatorname{sig}(A, \Delta) := \kappa_+(\mathcal{L}_\Delta(A)) - \kappa_-(\mathcal{L}_\Delta(A)) \tag{3.1}$$

the *signature difference* of (the spectral subspace of) A corresponding to Δ. By $\operatorname{eig}(A, \Delta)$ we denote the dimension of $\mathcal{L}_\Delta(A)$, that is,

$$\operatorname{eig}(A, \Delta) := \kappa_+(\mathcal{L}_\Delta(A)) + \kappa_-(\mathcal{L}_\Delta(A)). \tag{3.2}$$

The main result in this section is the following theorem.

Theorem 3.1. *Let A_1 and A_2 be two selfadjoint operators or relations in a Pontryagin space Π_κ such that*

$$\dim \operatorname{ran} \big((A_1 - \lambda_0)^{-1} - (A_2 - \lambda_0)^{-1} \big) = n \tag{3.3}$$

for some (and hence for all) $\lambda_0 \in \rho(A_1) \cap \rho(A_2)$, and let $\Delta \subset \mathbb{R} \setminus \sigma_{\text{ess}}(A_1)$ (or, equivalently, $\Delta \subset \mathbb{R} \setminus \sigma_{\text{ess}}(A_2)$) be a nonempty open interval. Then the following assertions hold.

(i) $\operatorname{sig}(A_1, \Delta)$ *is finite if and only if* $\operatorname{sig}(A_2, \Delta)$ *is finite, and in this case*

$$\big| \operatorname{sig}(A_1, \Delta) - \operatorname{sig}(A_2, \Delta) \big| \leq n. \tag{3.4}$$

(ii) $\operatorname{eig}(A_1, \Delta)$ *is finite if and only if* $\operatorname{eig}(A_2, \Delta)$ *is finite, and in this case*

$$\big| \operatorname{eig}(A_1, \Delta) - \operatorname{eig}(A_2, \Delta) \big| \leq n + 2\kappa. \tag{3.5}$$

Both estimates (3.4) *and* (3.5) *are sharp and equality in* (3.5) *prevails if and only if equality prevails in* (3.4) *and either*

$$\kappa_-(\mathcal{L}_\Delta(A_1)) = \kappa \quad \textit{and} \quad \kappa_-(\mathcal{L}_\Delta(A_2)) = 0$$

or

$$\kappa_-(\mathcal{L}_\Delta(A_1)) = 0 \quad \textit{and} \quad \kappa_-(\mathcal{L}_\Delta(A_2)) = \kappa.$$

The following corollary for the case $\kappa = 0$ is well known in the perturbation theory of selfadjoint operators in Hilbert spaces, see, e.g., [14, Chapter 9.3, Theorem 3].

Corollary 3.2. *Let A_1 and A_2 be two selfadjoint operators in a Hilbert space such that*

$$\dim \operatorname{ran} \big((A_1 - \lambda_0)^{-1} - (A_2 - \lambda_0)^{-1}\big) = n$$

holds for some (and hence for all) $\lambda_0 \in \rho(A_1) \cap \rho(A_2)$, and let $\Delta \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A_1)$ (or, equivalently, $\Delta \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A_2)$) be a nonempty open interval. Then $\operatorname{eig}(A_1, \Delta)$ *is finite if and only if* $\operatorname{eig}(A_2, \Delta)$ *is finite, and in this case*

$$\big|\operatorname{eig}(A_1, \Delta) - \operatorname{eig}(A_2, \Delta)\big| \le n.$$

The proof of Theorem 3.1 makes use of Lemma 3.3 below, in which the following well known property of selfadjoint operators in Hilbert spaces is extended to the Pontryagin space setting: For a bounded selfadjoint operator T in a Hilbert space with scalar product $(\cdot\,,\cdot)$ we have $(a, b) \subset \rho(T)$ if and only if

$$((T - a)x, (T - b)x) \ge 0$$

for all x, and $\sigma(T) \subset [a, b]$ holds if and only if

$$((T - a)x, (T - b)x) \le 0.$$

for all x. This easily follows from the spectral theorem.

Lemma 3.3. *Let A be a bounded selfadjoint operator in a Pontryagin space Π_κ and let $a, b \in \mathbb{R}$, $a < b$. Then the following holds.*

(i) *If $[a, b] \subset \rho(A)$ then Π_κ admits a decomposition $\Pi_\kappa = \mathcal{M}_- \dot{+} \mathcal{M}_+$, such that* $\dim \mathcal{M}_- = \kappa$,

$$[(A - a)x, (A - b)x] < 0 \quad \textit{for } x \in \mathcal{M}_- \setminus \{0\},$$

and

$$[(A - a)x, (A - b)x] > 0 \quad \textit{for } x \in \mathcal{M}_+ \setminus \{0\}.$$

(ii) *If $\sigma(A) \subset (a, b)$ then Π_κ admits a decomposition $\Pi_\kappa = \mathcal{M}_- \dot{+} \mathcal{M}_+$, such that* $\dim \mathcal{M}_+ = \kappa$,

$$[(A - a)x, (A - b)x] < 0 \quad \textit{for } x \in \mathcal{M}_- \setminus \{0\},$$

and

$$[(A - a)x, (A - b)x] > 0 \quad \textit{for } x \in \mathcal{M}_+ \setminus \{0\}.$$

Proof. By a well known theorem of L.S. Pontryagin (see also [36, Theorem 12.1′]) there exists a κ-dimensional nonpositive subspace $\mathcal{L} \subset \Pi_\kappa$ which is A-invariant. Making use of [36, Theorem 3.3] we find a negative subspace $\mathcal{L}_- \subset \mathcal{L}$ such that $\mathcal{L} = \mathcal{L}_- \,[\dot{+}]\, \mathcal{L}^\circ$, where $\mathcal{L}^\circ = \mathcal{L} \cap \mathcal{L}^{[\perp]}$ denotes the isotropic part of $\mathcal{L}$. Evidently, $\mathcal{L}^\circ$ is A-invariant. By [15, Theorem IX.2.5] or [36, Theorem 3.4] there exist a subspace $\mathcal{P}_0 \subset \Pi_\kappa$ with $\dim \mathcal{P}_0 = \dim \mathcal{L}^\circ$ and a (uniformly) positive subspace $\mathcal{M}$ such that

$$\Pi_\kappa = \mathcal{L}_- \,[\dot{+}]\, (\mathcal{L}^\circ \dot{+} \mathcal{P}_0) \,[\dot{+}] \mathcal{M}. \tag{3.6}$$

Since $\mathcal{L}^\circ$, $\mathcal{L}$, and also $\mathcal{L}^{[\perp]} = \mathcal{L}^\circ \dot{+} \mathcal{M}$ are A-invariant, with respect to the decomposition

$$\Pi_\kappa = \mathcal{L}^\circ \dot{+} \mathcal{L}_- \dot{+} \mathcal{M} \dot{+} \mathcal{P}_0$$

the operator A admits the following operator matrix representation:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix}. \tag{3.7}$$

It is easily seen that the operator A_{33} is selfadjoint in the Hilbert space $(\mathcal{M}, [\cdot\,,\cdot])$. Let us now show that

$$\rho(A) \subset \rho(A_{22}) \cap \rho(A_{33}). \tag{3.8}$$

For this, denote by D the operator represented by the operator matrix in (3.7) with the off-diagonal entries replaced by zeros. Then D is a finite-dimensional perturbation of A. Hence, if $\lambda \in \rho(A)$, then $D - \lambda$ is a Fredholm operator with index zero. Since $A_{11} - \lambda$, $A_{22} - \lambda$, and $A_{44} - \lambda$ operate in finite-dimensional spaces, their Fredholm indices are zero, respectively, and hence so is that of $A_{33} - \lambda$. To prove (3.8), it thus remains to show that $A_{22} - \lambda$ and $A_{33} - \lambda$ are injective. First, we note that $A_{11} - \lambda$ is injective as $A_{11} = A \upharpoonright \mathcal{L}^\circ$. Assume that $(A_{22} - \lambda)x_2 = 0$ for some $x_2 \in \mathcal{L}_-$. Then, using (3.7), we see that

$$(A - \lambda) \begin{pmatrix} -(A_{11} - \lambda)^{-1} A_{12} x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} = 0,$$

which shows $x_2 = 0$. The fact that $A_{33} - \lambda$ is injective can be shown in a similar manner. This shows (3.8).

In both cases (i) and (ii) we have $a, b \in \rho(A)$. Therefore, the inner product

$$\langle x, y \rangle := [(A - a)x, (A - b)y], \quad x, y \in \Pi_\kappa,$$

defines a Krein space inner product on Π_κ. In the following, we shall restrict ourselves to the case (i). The proof of (ii) is similar. For $\mathfrak{m} \in \mathcal{M}$ we have

$$\begin{aligned} \langle \mathfrak{m}, \mathfrak{m} \rangle &= [A_{13}\mathfrak{m} + (A_{33} - a)\mathfrak{m}, A_{13}\mathfrak{m} + (A_{33} - b)\mathfrak{m}] \\ &= [(A_{33} - a)\mathfrak{m}, (A_{33} - b)\mathfrak{m}]. \end{aligned}$$

From $[a-\varepsilon, b+\varepsilon] \subset \rho(A) \subset \rho(A_{33})$ for some $\varepsilon > 0$ and the selfadjointness of A_{33} in the Hilbert space $(\mathcal{M}, [\cdot\,,\cdot])$ we conclude that

$$[(A_{33}-(a-\varepsilon))\mathfrak{m}, (A_{33}-(b+\varepsilon))\mathfrak{m}] \,\geq\, 0,$$

and hence

$$\langle \mathfrak{m}, \mathfrak{m}\rangle \,\geq\, \varepsilon(b-a+\varepsilon)[\mathfrak{m}, \mathfrak{m}],$$

which shows that $\mathcal{M}$ is uniformly $\langle\cdot\,,\cdot\rangle$-positive. Similarly, it is shown that $\mathcal{L}_-$ is $\langle\cdot\,,\cdot\rangle$-negative. Moreover, $\mathcal{L}^\circ$, $\mathcal{L}_-$ and $\mathcal{M}$ are mutually $\langle\cdot\,,\cdot\rangle$-orthogonal and $\mathcal{L}^\circ$ is $\langle\cdot\,,\cdot\rangle$-neutral. The application of [36, Lemma 3.1] to the space $\mathcal{L}^\circ$ (as a subspace of the Pontryagin space $(\mathcal{M}\langle\dot{+}\rangle\mathcal{L}_-)^{\langle\perp\rangle}$) yields the existence of a subspace $\mathcal{P}_1 \subset (\mathcal{M}\langle\dot{+}\rangle\mathcal{L}_-)^{\langle\perp\rangle}$ such that $\mathcal{P}_1$ and $\mathcal{L}^\circ$ are skewly linked; cf. [36, Definition 3.2]. In particular, the space $\mathcal{L}^\circ \dotplus \mathcal{P}_1$ is non-degenerate,

$$\dim\bigl(\mathcal{L}^\circ \dotplus \mathcal{P}_1\bigr) = \dim \mathcal{L}^\circ + \dim \mathcal{P}_1 = 2\dim \mathcal{L}^\circ,$$

and $\langle\cdot\,,\cdot\rangle$ has $\dim \mathcal{L}^\circ$ negative squares on $\mathcal{L}^\circ \dotplus \mathcal{P}_1$. According to (3.6) the codimension of $\mathcal{M}\langle\dot{+}\rangle\mathcal{L}_-$ in Π_κ is

$$\dim\bigl(\mathcal{L}^\circ \dotplus \mathcal{P}_0\bigr) = \dim \mathcal{L}^\circ + \dim \mathcal{P}_0 = 2\dim \mathcal{L}^\circ,$$

and hence we conclude

$$\Pi_\kappa = \mathcal{L}_-\langle\dot{+}\rangle(\mathcal{L}^\circ \dotplus \mathcal{P}_1)\langle\dot{+}\rangle\mathcal{M}.$$

From this decomposition we see that $(\Pi_\kappa, \langle\cdot\,,\cdot\rangle)$ is a Pontryagin space with $\kappa = \dim \mathcal{L}$ negative squares. Hence, there exists a decomposition $\Pi_\kappa = \mathcal{M}_-\langle\dot{+}\rangle\mathcal{M}_+$, where $\mathcal{M}_-$ is κ-dimensional and $\langle\cdot\,,\cdot\rangle$-negative and $\mathcal{M}_+$ is $\langle\cdot\,,\cdot\rangle$-positive. This implies (i). □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. (i) The proof of item (i) is divided into two steps. In Step 1 we verify the assertion for the case that A_1 and A_2 are bounded operators, Δ is bounded and $\overline{\Delta} \subset \mathbb{R}\setminus\sigma_{\rm ess}(A_1)$, which is equivalent to $\overline{\Delta} \subset \mathbb{R}\setminus\sigma_{\rm ess}(A_2)$, since by assumption (3.3) the resolvents of A_1 and A_2 differ by a finite rank operator. In the second step we show how the general case can be reduced to these assumptions.

Step 1. Let $\Delta = (a,b)$ be such that $\overline{\Delta} \subset \mathbb{R}\setminus\sigma_{\rm ess}(A_j)$, $j=1,2$. We may assume that $a,b \in \rho(A_1)\cap\rho(A_2)$. Denote by E_j be the spectral function of the selfadjoint operator A_j. According to Lemma 3.3, for $j=1,2$ we have the decompositions

$$(I-E_j(\Delta))\Pi_\kappa = \mathcal{M}^j_{+,\rm out} \dotplus \mathcal{M}^j_{-,\rm out} \quad\text{and}\quad E_j(\Delta)\Pi_\kappa = \mathcal{M}^j_{+,\rm in} \dotplus \mathcal{M}^j_{-,\rm in},$$

with

$$\dim \mathcal{M}^j_{-,\rm out} = \kappa_-((I-E_j(\Delta))\Pi_\kappa) \quad\text{and}\quad \dim \mathcal{M}^j_{+,\rm in} = \kappa_-(E_j(\Delta)\Pi_\kappa)$$

such that

$$[(A_j-a)x, (A_j-b)x] \,<\, 0 \quad \text{for } x \in \bigl(\mathcal{M}^j_{-,\rm out}[\dot{+}]\mathcal{M}^j_{-,\rm in}\bigr)\setminus\{0\},$$

and

$$[(A_j-a)x, (A_j-b)x] \,>\, 0 \quad \text{for } x \in \bigl(\mathcal{M}^j_{+,\rm out}[\dot{+}]\mathcal{M}^j_{+,\rm in}\bigr)\setminus\{0\}.$$

Evidently,

$$\Pi_\kappa = \left(\mathcal{M}^1_{+,\mathrm{out}} \dotplus \mathcal{M}^1_{-,\mathrm{out}}\right) \, [\dotplus] \, \left(\mathcal{M}^1_{+,\mathrm{in}} \dotplus \mathcal{M}^1_{-,\mathrm{in}}\right).$$

Let Q_1 be the projection onto $\mathcal{M}^1_{-,\mathrm{out}} \, [\dotplus] \, \mathcal{M}^1_{-,\mathrm{in}}$ with respect to this decomposition of Π_κ. Moreover, set

$$\mathcal{K} := \left(\mathcal{M}^2_{-,\mathrm{out}} \, [\dotplus] \, \mathcal{M}^2_{-,\mathrm{in}}\right) \cap \ker(A_1 - A_2).$$

We claim that the restriction $Q_1 \upharpoonright \mathcal{K}$ of Q_1 to $\mathcal{K}$ is one-to-one. Indeed, suppose that there exists $x \in \mathcal{K}$, $x \neq 0$, such that $Q_1 x = 0$. Then from $x \in (\mathcal{M}^2_{-,\mathrm{out}} \, [\dotplus] \, \mathcal{M}^2_{-,\mathrm{in}}) \cap \ker(A_1 - A_2)$ we deduce

$$[(A_1 - a)x, (A_1 - b)x] = [(A_2 - a)x, (A_2 - b)x] \, < \, 0.$$

But $Q_1 x = 0$ implies $x \in \mathcal{M}^1_{+,\mathrm{out}} \, [\dotplus] \, \mathcal{M}^1_{+,\mathrm{in}}$ and hence

$$[(A_1 - a)x, (A_1 - b)x] \, > \, 0;$$

a contradiction. Therefore, the restriction of the linear mapping Q_1 to $\mathcal{K}$ is one-to-one which yields $\dim \mathcal{K} \, \leq \, \dim \operatorname{ran} Q_1$, i.e.

$$\begin{aligned} \dim \mathcal{K} \, &\leq \, \dim \mathcal{M}^1_{-,\mathrm{out}} + \dim \mathcal{M}^1_{-,\mathrm{in}} \\ &= \kappa_-((I - E_1(\Delta))\Pi_\kappa) + \kappa_+(E_1(\Delta)\Pi_\kappa). \end{aligned}$$

This estimate and the fact that $E_1(\Delta)\Pi_\kappa$ is finite dimensional also implies that $\mathcal{K}$ is finite dimensional. On the other hand, as $\dim(\Pi_\kappa / \ker(A_1 - A_2)) = n$, it follows that

$$\begin{aligned} \dim \mathcal{K} \geq \, &\dim \mathcal{M}^2_{-,\mathrm{out}} + \dim \mathcal{M}^2_{-,\mathrm{in}} - n \\ &= \kappa_-((I - E_2(\Delta))\Pi_\kappa) + \kappa_+(E_2(\Delta)\Pi_\kappa) - n, \end{aligned}$$

and we obtain

$$\begin{aligned} &\kappa_+(E_2(\Delta)\Pi_\kappa) - \kappa_+(E_1(\Delta)\Pi_\kappa) \\ &\qquad \leq n + \kappa_-((I - E_1(\Delta))\Pi_\kappa) - \kappa_-((I - E_2(\Delta))\Pi_\kappa) \\ &\qquad = n + (\kappa - \kappa_-(E_1(\Delta)\Pi_\kappa)) - (\kappa - \kappa_-(E_2(\Delta)\Pi_\kappa)) \\ &\qquad = n + \kappa_-(E_2(\Delta)\Pi_\kappa) - \kappa_-(E_1(\Delta)\Pi_\kappa). \end{aligned}$$

This implies $\operatorname{sig}(A_2, \Delta) - \operatorname{sig}(A_1, \Delta) \, \leq \, n$. The same reasoning with A_1 and A_2 interchanged shows $\operatorname{sig}(A_1, \Delta) - \operatorname{sig}(A_2, \Delta) \, \leq \, n$ and hence (3.4) holds for the case of bounded operators and $\overline{\Delta} \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A_1)$.

Step 2. Let us now reduce the general case to that considered in Step 1. Assume that A_1 and A_2 are selfadjoint relations in Π_κ such that $\rho(A_1) \cap \rho(A_2) \neq \varnothing$ and (3.3) holds. We note that the theorem is true if it holds for bounded open intervals $\Delta = (a, b)$ such that $[a, b] \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A_1)$ (which is equivalent to $[a, b] \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A_2)$ by (3.3)), and hence we keep this assumption. Then we can choose a point $\lambda_0 > b$ such that $\lambda_0 \in \rho(A_1) \cap \rho(A_2)$, define the bounded selfadjoint operators

$$B_1 := (A_1 - \lambda_0)^{-1} \quad \text{and} \quad B_2 := (A_2 - \lambda_0)^{-1},$$

and put $\Delta' := ((b-\lambda_0)^{-1}, (a-\lambda_0)^{-1})$. From the identity (see, e.g., [29, Section 2])

$$(A_j - \eta)^{-1} = -\frac{1}{\eta-\lambda_0} - \frac{1}{(\eta-\lambda_0)^2}\left(B_j - \frac{1}{\eta-\lambda_0}\right)^{-1}, \quad \eta \neq \lambda_0,$$

we conclude $\{y,x\} \in A_j - \eta$ if and only if

$$\left\{-(\eta-\lambda_0)^2 y - (\eta-\lambda_0)x, x\right\} \in B_j - \frac{1}{\eta-\lambda_0}.$$

For $x=0$ this shows that y is an eigenvector corresponding to $\eta \in \sigma_{\mathrm{p}}(A_j)$ if and only if y is an eigenvector corresponding to $\frac{1}{\eta-\lambda_0} \in \sigma_{\mathrm{p}}(B_j)$, and for $x \neq 0$ this shows how Jordan chains of A_j corresponding to $\eta \in \sigma_{\mathrm{p}}(A_j)$ translate into Jordan chains corresponding to $\frac{1}{\eta-\lambda_0} \in \sigma_{\mathrm{p}}(B_j)$, and vice versa. These observations imply $\mathcal{L}_\Delta(A_j) = \mathcal{L}_{\Delta'}(B_j)$, $j=1,2$, and, in particular,

$$\operatorname{sig}(A_1,\Delta) = \operatorname{sig}(B_1,\Delta') \quad \text{and} \quad \operatorname{sig}(A_2,\Delta) = \operatorname{sig}(B_2,\Delta').$$

The functional calculus for selfadjoint relations from [29, Section 3] then yields $\overline{\Delta'} \subset \mathbb{R}\setminus\sigma_{\mathrm{ess}}(B_j)$, $j=1,2$, and hence the assertion follows from the above considerations and Step 1.

(ii) From (3.1) and (3.2) we see

$$\operatorname{eig}(A_j,\Delta) = \kappa_+(\mathcal{L}_\Delta(A_j)) + \kappa_-(\mathcal{L}_\Delta(A_j)) = \operatorname{sig}(A_j,\Delta) + 2\kappa_-(\mathcal{L}_\Delta(A_j))$$

for $j=1,2$. This and item (i) firstly imply that $\operatorname{eig}(A_1,\Delta)$ is finite if and only if $\operatorname{eig}(A_2,\Delta)$ is finite and secondly that

$$\begin{aligned}
&\big|\operatorname{eig}(A_1,\Delta) - \operatorname{eig}(A_2,\Delta)\big| \\
&\quad\leq \big|\operatorname{sig}(A_1,\Delta) - \operatorname{sig}(A_2,\Delta)\big| + 2\big|\kappa_-(\mathcal{L}_\Delta(A_1)) - \kappa_-(\mathcal{L}_\Delta(A_2))\big| \\
&\quad\leq n + 2\kappa.
\end{aligned}$$

This consideration also shows that $|\operatorname{eig}(A_1,\Delta) - \operatorname{eig}(A_2,\Delta)| = n+2\kappa$ if and only if $|\operatorname{sig}(A_1,\Delta) - \operatorname{sig}(A_2,\Delta)| = n$ and either $\kappa_-(\mathcal{L}_\Delta(A_1)) = \kappa$ and $\kappa_-(\mathcal{L}_\Delta(A_2)) = 0$ or $\kappa_-(\mathcal{L}_\Delta(A_1)) = 0$ and $\kappa_-(\mathcal{L}_\Delta(A_2)) = \kappa$. The fact that the estimates in (i) and (ii) are both sharp will be discussed independently in the next section. □

4. Explicit operator models for a class of scalar rational functions

The aim of this section is to provide an explicit and elementary operator model for special scalar rational functions of the form

$$M(\lambda) = \frac{\Pi_{i=1}^m(\lambda-\nu_i)}{\Pi_{i=1}^m(\lambda-\mu_i)}, \tag{4.1}$$

where it is assumed for simplicity that all zeros and poles are real, simple and distinct, that is, $\nu_i \neq \nu_j$ and $\mu_i \neq \mu_j$ for $i \neq j$, and $\nu_i \neq \mu_j$ for all

$1 \le i, j \le m$. It is no restriction to assume that the poles μ_i are numbered in such a way that

$$M(\lambda) = \sum_{i=1}^{\kappa} \frac{\alpha_i}{\lambda - \mu_i} + \sum_{i=\kappa+1}^{m} \frac{-\alpha_i}{\lambda - \mu_i} + 1 \tag{4.2}$$

holds with $\alpha_1, \ldots \alpha_m > 0$ and $1 \le \kappa \le m$. The model in Section 4.1 is convenient to show that the estimates in the previous section are sharp (see Section 4.2) and it has also been used in a similar context in [8] and [9]; we refer to [1, 2, 19, 24, 32, 33, 35, 38] for more general operator models for scalar, matrix and operator-valued rational functions, generalized Nevanlinna, definitizable and locally holomorphic functions.

4.1. The functions M and $-M^{-1}$ as Weyl functions

We construct a model for M via boundary triplets for non-densely defined symmetric operators in finite-dimensional Pontryagin spaces, and the key feature is that a boundary triple $\{\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ is provided such that

$$\widehat{A}_0 = \ker \widehat{\Gamma}_0 = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_m \end{pmatrix} \tag{4.3}$$

is selfadjoint in the Pontryagin space $\Pi_\kappa = (\mathbb{C}^m, [\cdot, \cdot])$, where

$$[x, y] = -\sum_{i=1}^{\kappa} x_i \bar{y}_i + \sum_{i=\kappa+1}^{m} x_i \bar{y}_i, \tag{4.4}$$

with $x = (x_1, \ldots, x_m)^\top$, $y = (y_1, \ldots, y_m)^\top \in \mathbb{C}^m$, and the corresponding Weyl function is M in (4.1)-(4.2). Since $\{\mathbb{C}, \Gamma_1, -\Gamma_0\}$ is a boundary triple with Weyl function $-M^{-1}$ this also yields that the eigenvalues of the selfadjoint matrix $\widehat{A}_1 = \ker \widehat{\Gamma}_1$ in Π_κ are the zeros of M, that is, $\sigma(\widehat{A}_1) = \{\nu_1, \ldots, \nu_m\}$, and it turns out that $\widehat{A}_1$ is the $m \times m$-matrix given by

$$\widehat{A}_1 = \begin{pmatrix} B_{\kappa,\kappa} & B_{\kappa,m-\kappa} \\ B_{m-\kappa,\kappa} & B_{m-\kappa,m-\kappa} \end{pmatrix} \tag{4.5}$$

where the $\kappa \times \kappa$-matrix $B_{\kappa,\kappa}$ is given by

$$B_{\kappa,\kappa} = \begin{pmatrix} -\alpha_1 + \mu_1 & -\sqrt{\alpha_1\alpha_2} & \cdots & -\sqrt{\alpha_1\alpha_\kappa} \\ -\sqrt{\alpha_2\alpha_1} & -\alpha_2 + \mu_2 & & -\sqrt{\alpha_2\alpha_\kappa} \\ \vdots & & \ddots & \vdots \\ -\sqrt{\alpha_\kappa\alpha_1} & -\sqrt{\alpha_\kappa\alpha_2} & \cdots & -\alpha_\kappa + \mu_\kappa \end{pmatrix}$$

and the $(m-\kappa) \times (m-\kappa)$-matrix $B_{m-\kappa,m-\kappa}$ is given by

$$B_{m-\kappa,m-\kappa} = \begin{pmatrix} \alpha_{\kappa+1} + \mu_{\kappa+1} & \sqrt{\alpha_{\kappa+1}\alpha_{\kappa+2}} & \cdots & \sqrt{\alpha_{\kappa+1}\alpha_m} \\ \sqrt{\alpha_{\kappa+2}\alpha_{\kappa+2}} & \alpha_{\kappa+2} + \mu_{\kappa+2} & & \sqrt{\alpha_{\kappa+2}\alpha_m} \\ \vdots & & \ddots & \vdots \\ \sqrt{\alpha_m\alpha_{\kappa+1}} & \sqrt{\alpha_\kappa\alpha_{\kappa+2}} & \cdots & \alpha_m + \mu_m, \end{pmatrix}$$

and the $\kappa \times (m-\kappa)$-matrix $B_{\kappa,m-\kappa}$ and the $(m-\kappa)\times\kappa$-matrix $B_{m-\kappa,\kappa}$ are

$$B_{\kappa,m-\kappa} = \begin{pmatrix} \sqrt{\alpha_1\alpha_{\kappa+1}} & \cdots & \sqrt{\alpha_1\alpha_m} \\ \sqrt{\alpha_2\alpha_{\kappa+1}} & \cdots & \sqrt{\alpha_2\alpha_m} \\ \vdots & & \vdots \\ \sqrt{\alpha_\kappa\alpha_{\kappa+1}} & \cdots & \sqrt{\alpha_\kappa\alpha_m} \end{pmatrix}$$

and

$$B_{m-\kappa,\kappa} = \begin{pmatrix} -\sqrt{\alpha_{\kappa+1}\alpha_1} & \cdots & -\sqrt{\alpha_{\kappa+1}\alpha_\kappa} \\ -\sqrt{\alpha_{\kappa+2}\alpha_1} & \cdots & -\sqrt{\alpha_{\kappa+2}\alpha_\kappa} \\ \vdots & & \vdots \\ -\sqrt{\alpha_m\alpha_1} & \cdots & -\sqrt{\alpha_m\alpha_\kappa}, \end{pmatrix}$$

respectively. The construction of the boundary triple $\{\mathbb{C},\widehat\Gamma_0,\widehat\Gamma_1\}$ below is based on the abstract coupling method in Proposition 6.2 and the following two elementary examples.

Example 4.1. Let $\alpha > 0$ and $\mu,\gamma\in\mathbb{R}$. Consider the trivial symmetric relation $S=\{\{0,0\}\}$ in the Hilbert space $\mathcal{H}=\mathbb{C}$ and its adjoint $S^*=\mathcal{H}\times\mathcal{H}$. Let

$$\Gamma_0\widehat f = \frac{1}{\sqrt{\alpha}}(f'-\mu f) \quad\text{and}\quad \Gamma_1\widehat f = -\sqrt{\alpha}f+\gamma\frac{1}{\sqrt{\alpha}}(f'-\mu f),$$

$\widehat f=\{f,f'\}\in S^*$, and note that

$$(f',g)-(f,g') = (\Gamma_1\widehat f,\Gamma_0\widehat g)-(\Gamma_0\widehat f,\Gamma_1\widehat g)$$

holds for all $\widehat f=\{f,f'\}$, $\widehat g=\{g,g'\}\in S^*$. It is not difficult to check that the mapping $(\Gamma_0,\Gamma_1)^\top : S^*\to\mathbb{C}^2$ is onto and hence $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ is a boundary triple for S^*. It follows that the selfadjoint relation $A_0=\ker\Gamma_0$ is given by

$$A_0=\bigl\{\{f,\mu f\}: f\in\mathcal{H}\bigr\}$$

and hence A_0 is the multiplication operator with the real constant μ in the Hilbert space $\mathcal{H}=\mathbb{C}$. Note also that $\sigma(A_0)=\{\mu\}$. From $\widehat{\mathfrak{N}}_\lambda(S^*)=\{\{f,\lambda f\}: f\in\mathcal{H}\}$ and the definition of the Weyl function corresponding to $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ it follows that

$$M(\lambda)=\frac{\Gamma_1\widehat f_\lambda}{\Gamma_0\widehat f_\lambda}=\frac{-\sqrt{\alpha}f+\gamma\frac{1}{\sqrt{\alpha}}(\lambda f-\mu f)}{\frac{1}{\sqrt{\alpha}}(\lambda f-\mu f)}=\frac{-\alpha}{\lambda-\mu}+\gamma, \qquad \lambda\neq\mu.$$

Example 4.2. Let $\alpha>0$ and $\mu,\gamma\in\mathbb{R}$. Consider the trivial symmetric relation $S=\{\{0,0\}\}$ in the Pontryagin space $\Pi_1=(\mathbb{C},[\cdot,\cdot])$, where $[x,y]:=-x\bar y$, $x,y\in\mathbb{C}$, and its adjoint $S^+=\mathcal{H}\times\mathcal{H}$. A similar computation as in the first example shows that

$$\Gamma_0\widehat f=\frac{1}{\sqrt{\alpha}}(f'-\mu f) \quad\text{and}\quad \Gamma_1\widehat f=\sqrt{\alpha}f+\gamma\frac{1}{\sqrt{\alpha}}(f'-\mu f),$$

$\widehat f=\{f,f'\}\in S^*$, is a boundary triple for S^+ with $A_0=\ker\Gamma_0$ as above and corresponding Weyl function

$$M(\lambda)=\frac{\alpha}{\lambda-\mu}+\gamma, \qquad \lambda\neq\mu.$$

In the following we apply Proposition 6.2 to realize M in (4.2) as a Weyl function. For this we make use of the symmetric relations and boundary triples in Examples 4.1 and 4.2, and we assume $\kappa < m$. More precisely, for $1 \le i \le \kappa$ we consider the boundary triples $\{\mathbb{C}, \Gamma_0^{(i)}, \Gamma_1^{(i)}\}$, where

$$\Gamma_0^{(i)} \widehat{f}_i = \frac{1}{\sqrt{\alpha_i}}(f_i' - \mu_i f_i) \quad \text{and} \quad \Gamma_1^{(i)} \widehat{f}_i = \sqrt{\alpha_i} f_i, \qquad \widehat{f}_i = \{f_i, f_i'\} \in \mathbb{C}^2,$$

and the relation $S_i = \{\{0,0\}\}$ is viewed as a symmetric relation in the Pontryagin space $\Pi_1 = (\mathbb{C}, [\cdot,\cdot])$. For $\kappa + 1 \le i \le m-1$ we define the boundary triples $\{\mathbb{C}, \Gamma_0^{(i)}, \Gamma_1^{(i)}\}$, where

$$\Gamma_0^{(i)} \widehat{f}_i = \frac{1}{\sqrt{\alpha_i}}(f_i' - \mu_i f_i) \quad \text{and} \quad \Gamma_1^{(i)} \widehat{f}_i = -\sqrt{\alpha_i} f_i, \qquad \widehat{f}_i = \{f_i, f_i'\} \in \mathbb{C}^2,$$

and the relation $S_i = \{\{0,0\}\}$ is viewed as a symmetric relation in the Hilbert space $\mathcal{H} = (\mathbb{C}, (\cdot,\cdot))$. For $i = m$ we use the boundary triple $\{\mathbb{C}, \Gamma_0^{(m)}, \Gamma_1^{(m)}\}$, where

$$\Gamma_0^{(m)} \widehat{f}_m = \frac{1}{\sqrt{\alpha_m}}(f_m' - \mu_m f_m), \qquad \widehat{f}_m = \{f_m, f_m'\} \in \mathbb{C}^2,$$

and

$$\Gamma_1^{(m)} \widehat{f}_m = -\sqrt{\alpha_m} f_m + \frac{1}{\sqrt{\alpha_m}}(f_m' - \mu_m f_m), \qquad \widehat{f}_m = \{f_m, f_m'\} \in \mathbb{C}^2,$$

and $S_m = \{\{0,0\}\}$ is symmetric in the Hilbert space $\mathcal{H} = (\mathbb{C}, (\cdot,\cdot))$. In the case $\kappa = m$ (which is not treated separately here) the minus sign in front of the term $\sqrt{\alpha_m} f_m$ in the definition of the boundary map $\Gamma_1^{(m)}$ has to be removed and the the relation $S_m = \{\{0,0\}\}$ should then be viewed as a symmetric relation in $\Pi_1 = (\mathbb{C}, [\cdot,\cdot])$.

In the present situation it is clear that the orthogonal sum $A_0^{(1)} \times \dots A_0^{(m)}$ is given by the diagonal matrix $\widehat{A}_0$ in (4.3), and the Pontryagin space from Proposition 6.2 is $\Pi_\kappa = (\mathbb{C}^m, [\cdot,\cdot])$, where the indefinite inner product $[\cdot,\cdot]$ is as in (4.4). The relation H in Proposition 6.2 is the restriction of the diagonal matrix $\widehat{A}_0$ to the subspace

$$\operatorname{dom} H = \left\{ f = (f_1, \dots, f_m)^\top \in \mathbb{C}^m : \sum_{i=1}^{\kappa} \sqrt{\alpha_i} f_i - \sum_{i=\kappa+1}^{m} \sqrt{\alpha_i} f_i = 0 \right\}$$

and it follows from Proposition 6.2 that $\{\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, where

$$\widehat{\Gamma}_0 \widehat{f} = \Gamma_0^{(i)} \widehat{f}_i = \frac{1}{\sqrt{\alpha_i}}(f_i' - \mu_i f_i), \qquad 1 \le i \le m,$$

and

$$\begin{aligned} \widehat{\Gamma}_1 \widehat{f} &= \sum_{i=1}^{m} \Gamma_1^{(i)} \widehat{f}_i \\ &= \sum_{i=1}^{\kappa} \sqrt{\alpha_i} f_i - \sum_{i=\kappa+1}^{m} \sqrt{\alpha_i} f_i + \frac{1}{\sqrt{\alpha_m}}(f_m' - \mu_m f_m) \end{aligned} \tag{4.6}$$

is a boundary triple for

$$H^+ = \left\{ \{\widehat{h}_1, \dots, \widehat{h}_m\} \in \mathbb{C}^2 \times \cdots \times \mathbb{C}^2 : \frac{h_1' - \mu_1 h_1}{\sqrt{\alpha_1}} = \cdots = \frac{h_m' - \mu_m h_m}{\sqrt{\alpha_m}} \right\}$$

such that the corresponding Weyl function is given by M in (4.2). It remains to compute the explicit form of $\ker \widehat{\Gamma}_1$ and to show that it coincides with the $m \times m$-matrix in (4.5). Note first that by (4.6) we have for $\widehat{f} \in \ker \widehat{\Gamma}_1$ that

$$\frac{1}{\sqrt{\alpha_m}}(f_m' - \mu_m f_m) = -\sum_{i=1}^{\kappa} \sqrt{\alpha_i} f_i + \sum_{i=\kappa+1}^{m} \sqrt{\alpha_i} f_i. \tag{4.7}$$

Since

$$\frac{1}{\sqrt{\alpha_j}}(f_j' - \mu_j f_j) = \frac{1}{\sqrt{\alpha_m}}(f_m' - \mu_m f_m), \qquad 1 \leq j \leq m,$$

we conclude together with (4.7) that

$$f_j' = \mu_j f_j - \sum_{i=1}^{\kappa} \sqrt{\alpha_j \alpha_i} f_i + \sum_{i=\kappa+1}^{m} \sqrt{\alpha_j \alpha_i} f_i, \qquad 1 \leq j \leq m,$$

and hence for $1 \leq j \leq \kappa$

$$f_j' = (-\alpha_j + \mu_j) f_j - \sum_{i=1, i \neq j}^{\kappa} \sqrt{\alpha_j \alpha_i} f_i + \sum_{i=\kappa+1}^{m} \sqrt{\alpha_j \alpha_i} f_i \tag{4.8}$$

and for $\kappa + 1 \leq j \leq m$

$$f_j' = (\alpha_j + \mu_j) f_j - \sum_{i=1}^{\kappa} \sqrt{\alpha_j \alpha_i} f_i + \sum_{i=\kappa+1, i \neq j}^{m} \sqrt{\alpha_j \alpha_i} f_i. \tag{4.9}$$

Now the first κ rows of the $m \times m$ matrix $\ker \widehat{\Gamma}_1$ can be read off from (4.8) and the the remaining last $m - \kappa$ rows are obtained from (4.9). It follows that $\widehat{A}_1$ in (4.5) coincides with $\ker \widehat{\Gamma}_1$.

Observe also that $\widehat{A}_0$ and $\widehat{A}_1$ are rank one perturbations of each other since both are one dimensional extensions of the nondensely defined symmetric matrix H and Krein's formula yields

$$(\widehat{A}_1 - \lambda)^{-1} = (\widehat{A}_0 - \lambda)^{-1} - \gamma(\lambda) M(\lambda)^{-1} \gamma(\bar{\lambda})^+, \qquad \lambda \in \rho(\widehat{A}_0) \cap \rho(\widehat{A}_1). \tag{4.10}$$

Since $\{\mathbb{C}, \widehat{\Gamma}_1, -\widehat{\Gamma}_0\}$ is a boundary triple for H^+ with corresponding Weyl function $-M^{-1}$ and each pole of $-M^{-1}$ is also a pole of the resolvent of $\widehat{A}_1 = \ker \widehat{\Gamma}_1$ from (6.3) it follows that that the m distinct poles of $-M^{-1}$, and hence the m distinct zeros of M, coincide with the eigenvalues of $\widehat{A}_1$. We also mention that the model constructed here satisfies the minimality condition

$$\mathbb{C}^m = \overline{\operatorname{span}} \left\{ \ker(H^+ - \lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\},$$

which can be checked by a direct computation.

4.2. Sharpness of the estimates in Theorem 3.1

The aim of this subsection is to show that both estimates in Theorem 3.1 are sharp in the following sense: For any $\kappa, n \in \mathbb{N}$ and any open interval $\Delta \subset \mathbb{R}$ there exist selfadjoint matrices A and B in a finite-dimensional Pontryagin space Π_κ such that

$$\dim \operatorname{ran}\big((A-\lambda)^{-1} - (B-\lambda)^{-1}\big) = n, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R};$$

and

$$\big|\operatorname{sig}(A,\Delta) - \operatorname{sig}(B,\Delta)\big| = n$$

and

$$\big|\operatorname{eig}(A,\Delta) - \operatorname{eig}(B,\Delta)\big| = n + 2\kappa$$

hold. For this, fix $\kappa \in \mathbb{N}$, $n \in \mathbb{N}$, and $a < b$, set $\Delta = (a,b)$, and choose real numbers $\mu_1, \dots, \mu_{2\kappa+1}$ such that

$$a < \mu_{\kappa+1} < \mu_1 < \mu_{\kappa+2} < \mu_2 < \cdots < \mu_{2\kappa} < \mu_\kappa < \mu_{2\kappa+1} < b$$

and real numbers $\nu_1, \dots, \nu_{2\kappa+1}$ such that

$$b < \nu_1 < \cdots < \nu_{2\kappa+1}.$$

Next consider the function

$$M(\lambda) = \frac{\Pi_{i=1}^m (\lambda - \nu_i)}{\Pi_{i=1}^m (\lambda - \mu_i)}$$

with $m = 2\kappa + 1$ as in (4.1). Here it follows that

$$M(\lambda) = \sum_{i=1}^{\kappa} \frac{\alpha_i}{\lambda - \mu_i} + \sum_{i=\kappa+1}^{2\kappa+1} \frac{-\alpha_i}{\lambda - \mu_i} + 1,$$

and by definition M has $2\kappa+1$ poles $\mu_1, \dots, \mu_{2\kappa+1}$ in the interval Δ and no zeros in Δ. Hence the $(2\kappa+1) \times (2\kappa+1)$-matrix $\widehat{A}_0$ in (4.3) is selfadjoint in the Pontryagin space $\Pi_\kappa = (\mathbb{C}^{2\kappa+1}, [\cdot,\cdot])$ (see (4.4)) and $\widehat{A}_0$ has $2\kappa+1$ distinct eigenvalues $\mu_1, \dots, \mu_{2\kappa+1}$ in Δ; more precisely, here

$$\operatorname{eig}(\widehat{A}_0, \Delta) = 2\kappa+1, \quad \kappa_+(\mathcal{L}_\Delta(\widehat{A}_0)) = \kappa+1, \quad \kappa_-(\mathcal{L}_\Delta(\widehat{A}_0)) = \kappa,$$

and hence

$$\operatorname{sig}(\widehat{A}_0, \Delta) = \kappa_+(\mathcal{L}_\Delta(\widehat{A}_0)) - \kappa_-(\mathcal{L}_\Delta(\widehat{A}_0)) = 1.$$

Since $-M^{-1}$ has no poles in Δ, the selfadjoint $(2\kappa+1)\times(2\kappa+1)$-matrix $\widehat{A}_1$ in (4.5) has no eigenvalues in Δ, so that,

$$\operatorname{eig}(\widehat{A}_1, \Delta) = \operatorname{sig}(\widehat{A}_1, \Delta) = \kappa_+(\mathcal{L}_\Delta(\widehat{A}_1)) = \kappa_-(\mathcal{L}_\Delta(\widehat{A}_1)) = 0.$$

Furthermore, by construction we have

$$\dim \operatorname{ran}\big((\widehat{A}_0-\lambda)^{-1} - (\widehat{A}_1-\lambda)^{-1}\big) = 1, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R};$$

cf. (4.10), and as

$$\big|\operatorname{sig}(\widehat{A}_0,\Delta) - \operatorname{sig}(\widehat{A}_1,\Delta)\big| = 1$$

and

$$\big|\operatorname{eig}(\widehat{A}_0,\Delta) - \operatorname{eig}(\widehat{A}_1,\Delta)\big| = 1 + 2\kappa,$$

it follows that the estimates in Theorem 3.1 (i) and (ii) are both sharp in the case $n = 1$. If $n > 1$ fix some points $\eta \in \Delta$, $\zeta \in \mathbb{R} \setminus \Delta$, and consider the matrices

$$\widehat{B}_0 = \begin{pmatrix} \widehat{A}_0 & 0 \\ 0 & \eta I_{n-1,n-1} \end{pmatrix} \quad \text{and} \quad \widehat{B}_1 = \begin{pmatrix} \widehat{A}_1 & 0 \\ 0 & \zeta I_{n-1,n-1} \end{pmatrix},$$

where $I_{n-1,n-1}$ is the $(n-1) \times (n-1)$-identity matrix in the Hilbert space $(\mathbb{C}^{n-1}, (\cdot,\cdot))$ and the $(2\kappa + n) \times (2\kappa + n)$-matrices $\widehat{B}_0$ and $\widehat{B}_1$ are viewed as selfadjoint matrices in the Pontryagin space $\Pi_\kappa = (\mathbb{C}^{2\kappa+n}, [\cdot,\cdot])$ with $[\cdot,\cdot]$ given by

$$[x, y] = -\sum_{i=1}^{\kappa} x_i \bar{y}_i + \sum_{i=\kappa+1}^{2\kappa+n} x_i \bar{y}_i, \qquad x, y \in \mathbb{C}^{2\kappa+n}.$$

Here we have

$$\operatorname{eig}(\widehat{B}_0, \Delta) = 2\kappa + n, \quad \kappa_+(\mathcal{L}_\Delta(\widehat{B}_0)) = \kappa + n, \quad \kappa_-(\mathcal{L}_\Delta(\widehat{B}_0)) = \kappa,$$

and $\operatorname{sig}(\widehat{B}_0, \Delta) = \kappa_+(\mathcal{L}_\Delta(\widehat{B}_0)) - \kappa_-(\mathcal{L}_\Delta(\widehat{B}_0)) = n$, and

$$\operatorname{eig}(\widehat{B}_1, \Delta) = \operatorname{sig}(\widehat{B}_1, \Delta) = \kappa_+(\mathcal{L}_\Delta(\widehat{B}_1)) = \kappa_-(\mathcal{L}_\Delta(\widehat{B}_1)) = 0,$$

and by construction

$$\dim \operatorname{ran}\big((\widehat{B}_0 - \lambda)^{-1} - (\widehat{B}_1 - \lambda)^{-1}\big) = n, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Therefore

$$\big|\operatorname{sig}(\widehat{B}_0, \Delta) - \operatorname{sig}(\widehat{B}_1, \Delta)\big| = n$$

and

$$\big|\operatorname{eig}(\widehat{B}_0, \Delta) - \operatorname{eig}(\widehat{B}_1, \Delta)\big| = n + 2\kappa,$$

and hence we have shown that the estimates in Theorem 3.1 (i) and (ii) are both sharp for any $\kappa \in \mathbb{N}$, $n \in \mathbb{N}$, and any open interval $\Delta = (a, b)$.

5. Sturm-Liouville problems with boundary condition depending rationally on the spectral parameter

In this section we illustrate the estimates in Section 3 and the operator model in Section 4 in the context of a singular Sturm-Liouville type spectral problem with λ-rational boundary conditions. Similar spectral problems were considered in various publications, see, e.g. [10, 11, 12, 13, 23, 25, 42, 43, 44] for a small selection and also [3, 17, 26, 27, 30, 41] for more abstract treatments of λ-dependent boundary value problems. The present construction is based on the coupling technique from [20], see also [5, 7].

Let $q \in L^1_{\text{loc}}((0,\infty))$ be a real valued function such that the differential expression $-\frac{d^2}{dx^2} + q$ is regular at zero and in the limit point case at ∞. We consider the λ-dependent boundary value problem

$$-f'' + qf = \lambda f, \qquad s(\lambda)f(0) + t(\lambda)f'(0) = 0, \qquad \lambda \in \mathbb{C}, \tag{5.1}$$

in $L^2((0,\infty))$, where it is assumed that s and t are rational functions such that

$$r(\lambda) = \frac{s(\lambda)}{t(\lambda)} = \frac{\Pi_{i=1}^m(\lambda - \nu_i)}{\Pi_{i=1}^m(\lambda - \mu_i)} \tag{5.2}$$

is of the form as in (4.1)-(4.2). In particular, it is assumed that s and t are such that the zeros and poles of r are real, simple and distinct. Then there exist $1 \le \kappa \le m$ and $\alpha_1, \dots, \alpha_m > 0$ such that

$$r(\lambda) = \sum_{i=1}^{\kappa} \frac{\alpha_i}{\lambda - \mu_i} + \sum_{i=\kappa+1}^{m} \frac{-\alpha_i}{\lambda - \mu_i} + 1. \tag{5.3}$$

If λ is a pole of s (of t) the boundary condition in (5.1) is understood as $f(0) = 0$ ($f'(0) = 0$, respectively). We shall say that $\lambda \in \mathbb{C}$ is an eigenvalue of (5.1) and f is a corresponding eigenfunction if $f \neq 0$ belongs to the maximal domain

$$\mathcal{D}_{\max} = \bigl\{ f \in L^2((0,\infty)) : f, f' \in \mathrm{AC}_{\mathrm{loc}}((0,\infty)),\ -f'' + qf \in L^2((0,\infty)) \bigr\}$$

and (5.1) is satisfied; here $\mathrm{AC}_{\mathrm{loc}}((0,\infty))$ denotes the space of all locally absolutely continuous functions on $(0,\infty)$.

Let S be the minimal operator associated with $-\frac{d^2}{dx^2} + q$ in $L^2((0,\infty))$, that is,

$$Sf = -f'' + qf, \qquad \mathrm{dom}\, S = \bigl\{ f \in \mathcal{D}_{\max} : f(0) = f'(0) = 0 \bigr\},$$

and recall that the adjoint of S is the maximal operator

$$S^* f = -f'' + qf, \qquad \mathrm{dom}\, S^* = \mathcal{D}_{\max}.$$

It is not difficult to check that $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ with $\Gamma_0 f = f(0)$ and $\Gamma_1 f = f'(0)$ is a boundary triple for S^* and $A_0 = S^* \upharpoonright \ker \Gamma_0$ corresponds to the Dirichlet boundary condition at the left endpoint 0. The Weyl function corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is given by

$$m(\lambda) = \frac{f'(0)}{f(0)}, \quad f \in \ker(S^* - \lambda), \quad \lambda \in \rho(A_0).$$

We recall that, since the symmetric operator S is simple, the function m cannot be analytically extended to a larger set than $\rho(A_0)$. In particular, if λ is an isolated eigenvalue of A_0 it is a pole of the resolvent of A_0 and therefore a pole of m. Conversely, if λ is a pole of m then it is an eigenvalue of A_0 (cf. (6.3)).

The next auxiliary lemma shows that the eigenvalues of the problem (5.1) in $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(A_0)$ coincide with the zeros of the function $m + r$.

Lemma 5.1. *For $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(A_0)$ the following are equivalent:*

(i) *λ is an eigenvalue of (5.1);*
(ii) *$m(\lambda) = -r(\lambda)$ (or λ is a pole of both m and r).*

In particular, the eigenvalues of (5.1) are discrete in $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(A_0)$.

Proof. (i)⇒(ii) Let λ be an eigenvalue of (5.1) with corresponding eigenfunction $f \neq 0$ and assume first that $\lambda \in \rho(A_0)$. Then we have $f(0) \neq 0$ for $f \in \ker(S^* - \lambda)$, $f \neq 0$, and hence r has no pole at λ. Therefore,

$$m(\lambda)f(0) = f'(0) = -r(\lambda)f(0)$$

and thus $m(\lambda)+r(\lambda) = 0$. If $\lambda \in \sigma(A_0)$ then λ is an isolated eigenvalue of A_0. Thus $f(0) = 0$, $f'(0) \neq 0$, and λ is a pole of m. From $r(\lambda)f(0) + f'(0) = 0$ it follows that λ is also a pole of r. This yields (ii).

(ii)⇒(i) Assume that $m(\lambda) = -r(\lambda)$. If $\lambda \in \rho(A_0)$ then λ is neither a pole of m nor of r, and for $f \in \ker(S^* - \lambda)$ we have

$$r(\lambda)f(0) + f'(0) = -m(\lambda)f(0) + f'(0) = 0.$$

If λ is an isolated eigenvalue of A_0 then λ is a pole of m and hence of r which implies that $f \in \ker(S^* - \lambda)$ satisfies the boundary condition in (5.1).

Suppose that $\lambda_0 \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(A_0)$ is an accumulation point of eigenvalues λ_n, $n \in \mathbb{N}$, of (5.1). Then $(m+r)(\lambda_n) = 0$ for all $n \in \mathbb{N}$ so that λ_0 cannot be a pole of $m + r$. Hence, either $\lambda_0 \in \rho(A_0)$ or it is a removable singularity of $m+r$. In both cases it follows that $m = -r$. Hence, the domain of definition of r coincides with $\rho(A_0)$ which implies that the spectrum of A_0 consists of a finite number of eigenvalues; a contradiction. □

The aim is now to construct a linearization $\widetilde{A}$ for the λ-dependent boundary value problem (5.1) with the help of the model discussed in Section 4. Let $\mu_1, \dots \mu_m$ be the poles of the rational function r in (5.2)-(5.3) and consider the selfadjoint diagonal matrix

$$H_0 = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_m \end{pmatrix} \tag{5.4}$$

in the Pontryagin space $\Pi_\kappa = (\mathbb{C}^m, [\cdot,\cdot])$, where $[\cdot,\cdot]$ is defined as in (4.4). Denote by H the restriction of H_0 to the subspace

$$\operatorname{dom} H = \left\{ h = (h_1, \dots, h_m)^\top \in \mathbb{C}^m : \sum_{i=1}^{\kappa} \sqrt{\alpha_i} h_i - \sum_{i=\kappa+1}^{m} \sqrt{\alpha_i} h_i = 0 \right\} \tag{5.5}$$

and recall from Section 4.1 that $\{\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ is a boundary triple for

$$H^+ = \left\{ \{\widehat{h}_1, \dots, \widehat{h}_m\} \in (\mathbb{C}^2)^m : \frac{h_1' - \mu_1 h_1}{\sqrt{\alpha_1}} = \cdots = \frac{h_m' - \mu_m h_m}{\sqrt{\alpha_m}} \right\},$$

where

$$\widehat{\Gamma}_0 \widehat{h} = \frac{1}{\sqrt{\alpha_i}}(h_i' - \mu_i h_i), \qquad 1 \leq i \leq m,$$

and

$$\widehat{\Gamma}_1 \widehat{h} = \sum_{i=1}^{\kappa} \sqrt{\alpha_i} h_i - \sum_{i=\kappa+1}^{m} \sqrt{\alpha_i} h_i + \frac{1}{\sqrt{\alpha_m}}(h_m' - \mu_m h_m), \tag{5.6}$$

such that the corresponding Weyl function is given by r in (5.2)-(5.3). Therefore, if $\lambda \in \mathbb{C}$ is not a pole of r and $\widehat{h} = \{h, \lambda h\} \in H^+$, then

$$\widehat{\Gamma}_1 \widehat{h} = r(\lambda) \widehat{\Gamma}_0 \widehat{h}.$$

We equip $\widetilde{\Pi}_\kappa = L^2((0,\infty)) \times \Pi_\kappa$ with the indefinite inner product

$$\left[\begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} g \\ h' \end{pmatrix} \right] := (f,g) + [h,h'], \quad f,g \in L^2((0,\infty)),\ h,h' \in \Pi_\kappa,$$

so that $\widetilde{\Pi}_\kappa$ is a Pontryagin space with κ negative squares.

The next theorem provides a selfadjoint operator $\widetilde{A}$ in $\widetilde{\Pi}_\kappa$ which can be viewed as a linearization or solution operator for the boundary value problem (5.1) in the sense that the eigenvalues of $\widetilde{A}$ coincide with the eigenvalues of the problem (5.1). The construction of $\widetilde{A}$ is based on the coupling method in [20] and was also used in [5, 7]. The new feature here is that $\widetilde{A}$ can be determined explicitely with the help of the model in Section 4 and that Theorem 3.1 yields information on the signature of the root subspaces corresponding to isolated eigenvalues.

Theorem 5.2. *Let* $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ *and* $\{\mathbb{C}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ *be the boundary triples for* S^* *and* H^+ *from above. Then*

$$\widetilde{A} := \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} S^* f \\ h' \end{pmatrix} \right\} \in S^* \times H^+ : \begin{matrix} \Gamma_0 f + \widehat{\Gamma}_0 \{h,h'\} = 0 \\ \Gamma_1 f - \widehat{\Gamma}_1 \{h,h'\} = 0 \end{matrix} \right\}$$

is a selfadjoint operator in $\widetilde{\Pi}_\kappa$*. The eigenvalues of* $\widetilde{A}$ *coincide with the eigenvalues of the* λ*-rational boundary value problem* (5.1)*, and for each eigenvalue* λ *of* $\widetilde{A}$ *one has*

$$\dim \ker(\widetilde{A} - \lambda) = 1. \tag{5.7}$$

If $\lambda \in \mathbb{R}$ *is an isolated eigenvalue of* $\widetilde{A}$ *then there exists an open interval* $\Delta \subset \mathbb{R}$ *such that* $\Delta \cap \sigma(\widetilde{A}) = \{\lambda\}$ *and* $\operatorname{sig}(\widetilde{A}, \Delta) \in \{-1, 0, 1\}$.

Proof. It follows in the same way as in [7, Proof of Theorem 4.1] that $\widetilde{A}$ is selfadjoint in $\widetilde{\Pi}_\kappa$ and it is straightforward to check that $\widetilde{A}$ is an operator. As $\widetilde{A} \cap (A_0 \times H_0)$ is a symmetric operator with defect one we have

$$\dim \operatorname{ran}\big((\widetilde{A} - \lambda_0)^{-1} - \big((A_0 \times H_0) - \lambda_0\big)^{-1}\big) = 1 \tag{5.8}$$

for all $\lambda_0 \in \rho(\widetilde{A}) \cap \rho(A_0 \times H_0)$. Furthermore, [5, Theorem 4.5 (iii)] implies that the geometric multiplicity of the eigenvalues of $\widetilde{A}$ is one and hence (5.7) is true.

In order to see that the eigenvalues of $\widetilde{A}$ coincide with the eigenvalues of the boundary value problem (5.1) assume first that λ is an eigenvalue of $\widetilde{A}$. Then

$$\left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} \lambda f \\ \lambda h \end{pmatrix} \right\} \in \widetilde{A}, \qquad \{f, \lambda f\} \in S^*,\ \widehat{h} = \{h, \lambda h\} \in H^+,$$

and $f \neq 0$ as otherwise $\widehat{\Gamma}_0\widehat{h} = -\Gamma_0 f = 0$ and $\widehat{\Gamma}_1\widehat{h} = \Gamma_1 f = 0$ imply $\widehat{h} = \{h, \lambda h\} \in H$ and H has no eigenvalues; cf. (5.4)-(5.5). Since $\{f, \lambda f\} \in S^*$, the differential equation

$$-f'' + qf = \lambda f$$

in (5.1) is satisfied. Moreover, if λ is not a pole of r then

$$r(\lambda)f(0) = r(\lambda)\Gamma_0 f = -r(\lambda)\widehat{\Gamma}_0\widehat{h} = -\widehat{\Gamma}_1\widehat{h} = -\Gamma_1 f = -f'(0),$$

and if λ is a pole of r then $\lambda \in \sigma_{\mathrm{p}}(H_0)$ and hence

$$f(0) = \Gamma_0 f = -\widehat{\Gamma}_0\{h, \lambda h\} = 0.$$

Thus λ is an eigenvalue of (5.1) with corresponding eigenvector f.

Conversely, if λ is an eigenvalue of (5.1) with corresponding eigenvector f and λ is not a pole of r then $\lambda \notin \sigma(H_0)$ and hence there exists $\widehat{h} = \{h, \lambda h\} \in H^+$ such that $\widehat{\Gamma}_0\widehat{h} = -\Gamma_0 f$. From

$$\widehat{\Gamma}_1\widehat{h} = r(\lambda)\widehat{\Gamma}_0\widehat{h} = -r(\lambda)\Gamma_0 f = -r(\lambda)f(0) = f'(0) = \Gamma_1 f$$

it follows that

$$\left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} \lambda f \\ \lambda h \end{pmatrix} \right\} \in \widetilde{A}. \tag{5.9}$$

In the case that λ is a pole of r the eigenvector f satisfies the boundary condition $f(0) = 0$ and hence $\{f, \lambda f\} \in A_0$. Note also that $\Gamma_1 f = f'(0) \neq 0$. Furthermore, λ is an eigenvalue of H_0 and hence $\widehat{h} = \{h, \lambda h\} \in H_0$ for some $h \neq 0$. Then $\Gamma_0 f = 0 = \widehat{\Gamma}_0\widehat{h}$ and as $\Gamma_1\widehat{h} \neq 0$ it is clear that h can be chosen such that $\widehat{\Gamma}_1\widehat{h} = \Gamma_1 f$. It follows that (5.9) holds and therefore λ is an eigenvalue of $\widetilde{A}$.

In order to show that real isolated eigenvalues of $\widetilde{A}$ satisfy $\operatorname{sig}(\widetilde{A}, \Delta) \in \{-1, 0, 1\}$ we first note that (5.8) and $\sigma_{\mathrm{ess}}(H_0) = \varnothing$ yield

$$\sigma_{\mathrm{ess}}(\widetilde{A}) = \sigma_{\mathrm{ess}}(A_0 \times H_0) = \sigma_{\mathrm{ess}}(A_0).$$

This implies, in particular, that there exists an open interval $\Delta \subset \mathbb{R}$ such that $\Delta \cap \sigma(\widetilde{A}) = \{\lambda\}$ and $\Delta \setminus \{\lambda\} \subset \rho(A_0 \times H_0)$. In the case that there is no Jordan chain of length > 1 of $\widetilde{A}$ at λ it follows from (5.7) that $\operatorname{sig}(\widetilde{A}, \Delta) \in \{-1, 0, 1\}$. In the following assume that there is a Jordan chain of length > 1 of $\widetilde{A}$ at λ and let $(f, h)^\top$ and $(g, k)^\top \in \operatorname{dom} \widetilde{A}$ such that

$$(\widetilde{A} - \lambda)\begin{pmatrix} f \\ h \end{pmatrix} = 0 \quad \text{and} \quad (\widetilde{A} - \lambda)\begin{pmatrix} g \\ k \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}.$$

Then $f \neq 0$ and $h \neq 0$, which is a consequence of the definition of $\widetilde{A}$ and the fact that S and H do not possess eigenvalues. It also follows that

$$\begin{aligned} \|f\|^2 + [h, h] &= \left[\begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f \\ h \end{pmatrix} \right] = \left[(\widetilde{A} - \lambda)\begin{pmatrix} g \\ k \end{pmatrix}, \begin{pmatrix} f \\ h \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} g \\ k \end{pmatrix}, (\widetilde{A} - \lambda)\begin{pmatrix} f \\ h \end{pmatrix} \right] = 0 \end{aligned}$$

and this implies $[h, h] < 0$.

We claim that in the case $\lambda \in \sigma_p(A_0 \times H_0)$ we have $\lambda \in \sigma_p(A_0) \cap \sigma_p(H_0)$ and $f \in \ker(A_0 - \lambda)$ and $h \in \ker(H_0 - \lambda)$. In fact, if $\lambda \in \sigma_p(A_0 \times H_0)$ then it is clear that $\lambda \in \sigma_p(A_0) \cup \sigma_p(H_0)$ and hence $\lambda \in \sigma_p(A_0)$ or $\lambda \in \sigma_p(H_0)$. If $\lambda \in \sigma_p(A_0)$ it follows from $\dim\ker(A_0 - \lambda) = 1$ and $f \in \ker(S^* - \lambda)$ that $f \in \ker(A_0 - \lambda)$ and this implies $0 = \Gamma_0 f = -\widehat{\Gamma}_0\{h, \lambda h\}$. Thus $\{h, \lambda h\} \in H_0$ and hence $h \in \ker(H_0 - \lambda)$ and $\lambda \in \sigma_p(H_0)$. The same argument shows that $\lambda \in \sigma_p(H_0)$ implies $h \in \ker(H_0 - \lambda)$ and $f \in \ker(A_0 - \lambda)$, so that $\lambda \in \sigma_p(A_0)$. Therefore,

$$\mathcal{L}_\Delta(A_0 \times H_0) = \operatorname{span}\left\{ \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ h \end{pmatrix} \right\}$$

and as $[h,h] < 0$ it follows that $\operatorname{sig}(A_0 \times H_0, \Delta) = 0$. Hence (5.8) and Theorem 3.1 (i) yield $\operatorname{sig}(\widetilde{A}, \Delta) \in \{-1, 0, 1\}$. In the case $\lambda \notin \sigma_p(A_0 \times H_0)$ we also have $\operatorname{sig}(A_0 \times H_0, \Delta) = 0$ and $\operatorname{sig}(\widetilde{A}, \Delta) \in \{-1, 0, 1\}$ follows again from (5.8) and Theorem 3.1 (i). □

Making use of the explicit form of H^+ and the boundary mappings $\widehat{\Gamma}_0$ and $\widehat{\Gamma}_1$ the linearization $\widetilde{A}$ in the previous theorem can be determined more explicitely. This is done in a similar way as in the end of Section 4.1. In fact, suppose that

$$\left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} S^* f \\ h' \end{pmatrix} \right\} \in \widetilde{A}$$

for some $f \in \operatorname{dom} S^*$ and $\{h, h'\} \in H^+$. The boundary condition $\Gamma_1 f - \widehat{\Gamma}_1\{h, h'\} = 0$ together with (5.6) yields

$$\frac{1}{\sqrt{\alpha_m}}(h_m' - \mu_m h_m) = \Gamma_1 f - \sum_{i=1}^{\kappa} \sqrt{\alpha_i} h_i + \sum_{i=\kappa+1}^{m} \sqrt{\alpha_i} h_i$$

and as

$$\frac{1}{\sqrt{\alpha_1}}(h_1' - \mu_1 h_1) = \cdots = \frac{1}{\sqrt{\alpha_m}}(h_m' - \mu_m h_m)$$

it follows for $1 \le j \le \kappa$ that

$$h_j' = \sqrt{\alpha_j} f'(0) + (\mu_j - \alpha_j) h_j - \sum_{i=1, i\neq j}^{\kappa} \sqrt{\alpha_j \alpha_i} h_i + \sum_{i=\kappa+1}^{m} \sqrt{\alpha_j \alpha_i} h_i \tag{5.10}$$

and for $\kappa + 1 \le j \le m$ that

$$h_j' = \sqrt{\alpha_j} f'(0) + (\mu_j + \alpha_j) h_j - \sum_{i=1}^{\kappa} \sqrt{\alpha_j \alpha_i} h_i + \sum_{i=\kappa+1, i\neq j}^{m} \sqrt{\alpha_j \alpha_i} h_i. \tag{5.11}$$

With the help of (5.10) and (5.11) the linearization $\widetilde{A}$ in Theorem 5.2 can be explicitely computed. This yields a form similar to (4.5).

6. Appendix

In this appendix we briefly review the notion of boundary triples and their Weyl functions for symmetric operators and relations in Pontryagin spaces; cf. [18, 21, 22, 31] for more details. In addition, we provide a construction of a boundary triple for a certain intermediate extension of a direct sum of symmetric relations such that the associated Weyl function is the sum of the Weyl functions associated to the symmetric relations; this result is used in Section 4.

Definition 6.1. Let S be a closed symmetric relation in a Pontryagin space Π_κ. A *boundary triple* for S^+ is a triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space $(\mathcal{G}, (\cdot,\cdot))$ and linear mappings $\Gamma_0, \Gamma_1 : S^+ \to \mathcal{G}$ such that the abstract Green's identity

$$[f', g] - [f, g'] = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}) - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})$$

holds for all $\widehat{f} = \{f, f'\}, \widehat{g} = \{g, g'\} \in S^+$ and $\Gamma = (\Gamma_0, \Gamma_1)^\top : S^+ \to \mathcal{G}^2$ is surjective.

In the case that S is a densely defined symmetric operator the adjoint S^+ is also an operator and instead of boundary mappings defined on the graph S^+ we shall use boundary mappings defined on $\operatorname{dom} S^+$, that is, we require $\Gamma_0, \Gamma_1 : \operatorname{dom} S^+ \to \mathcal{G}$ such that

$$[S^+ f, g] - [f, S^+ g] = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)$$

holds for all $f, g \in \operatorname{dom} S^+$ and $\Gamma = (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} S^+ \to \mathcal{G}^2$ is surjective. From the context it will always be clear if the boundary mappings are defined on the adjoint relation or on the domain of the adjoint operator.

Assume that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triple for S^+. Then the mapping

$$\Theta \mapsto A_\Theta = \left\{ \widehat{f} \in S^+ : \Gamma \widehat{f} = \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} \in \Theta \right\} \tag{6.1}$$

is a bijection between the space of closed linear relations in $\mathcal{G} \times \mathcal{G}$ and the set of closed extension $A_\Theta \subset S^+$ of S, and

$$(A_\Theta)^+ = A_{\Theta^*} \tag{6.2}$$

holds. In particular, A_Θ is a selfadjoint extension of S in the Pontryagin space Π_κ if and only if Θ is a selfadjoint relation in the Hilbert space $\mathcal{G}$. The selfadjoint extensions corresponding to the kernels of the boundary mappings Γ_0 and Γ_1 are denoted by

$$A_0 = \ker \Gamma_0 \qquad \text{and} \qquad A_1 = \ker \Gamma_1,$$

and we remark that the extension A_0 corresponds to the selfadjoint relation $\Theta_0 = \{\{0, g'\} : g' \in \mathcal{G}\}$ and A_1 corresponds to the zero operator $\Theta_1 = 0$ in $\mathcal{G}$. The extension A_0 will often play the role of a fixed selfadjoint extension, and it will usually be assumed that $\rho(A_0) \neq \varnothing$. This condition is automatically satisfied when A_0 is an operator.

In the following we use the notation

$$\mathfrak{N}_\lambda(S^+) = \ker(S^+ - \lambda) \quad \text{and} \quad \widehat{\mathfrak{N}}_\lambda(S^+) = \left\{ \{f, \lambda f\} : f \in \mathfrak{N}_\lambda(S^+) \right\}$$

for $\lambda \in \mathbb{C}$. Suppose that $\rho(A_0) \neq \varnothing$. Then we have the direct sum decomposition

$$S^+ = A_0 \dotplus \widehat{\mathfrak{N}}_\lambda(S^+) = \ker \Gamma_0 \dotplus \widehat{\mathfrak{N}}_\lambda(S^+), \qquad \lambda \in \rho(A_0),$$

and hence it follows that the boundary mapping Γ_0 restricted to $\widehat{\mathfrak{N}}_\lambda(S^+)$ is bijective. The *γ-field* and *Weyl function* corresponding to the boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are defined for $\lambda \in \rho(A_0)$ by

$$\gamma(\lambda) : \mathcal{G} \to \Pi_\kappa, \qquad \varphi \mapsto \gamma(\lambda)\varphi = \pi_1 \big(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^+)\big)^{-1} \varphi$$

and

$$M(\lambda) : \mathcal{G} \to \mathcal{G}, \qquad \varphi \mapsto M(\lambda)\varphi = \Gamma_1 \big(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^+)\big)^{-1} \varphi,$$

respectively; here $\pi_1 : \Pi_\kappa \times \Pi_\kappa \to \Pi_\kappa$ is the projection onto the first component. It can be shown that $\gamma(\lambda) \in \mathcal{B}(\mathcal{G}, \Pi_\kappa)$ and $M(\lambda) \in \mathcal{B}(\mathcal{G})$ for all $\lambda \in \rho(A_0)$ and both functions $\lambda \mapsto \gamma(\lambda)$ and $\lambda \mapsto M(\lambda)$ are analytic on $\rho(A_0)$. The γ-field and Weyl function satisfy the identities

$$\gamma(\lambda) = \big(I + (\lambda - \mu)(A_0 - \lambda)^{-1}\big)\gamma(\mu)$$

and

$$M(\lambda) - M(\mu)^* = (\lambda - \overline{\mu})\gamma(\mu)^+ \gamma(\lambda)$$

for all $\lambda, \mu \in \rho(A_0)$, and these identities also yield

$$M(\lambda) = M(\mu)^* + (\lambda - \overline{\mu})\gamma(\mu)^+ \big(I + (\lambda - \mu)(A_0 - \lambda)^{-1}\big)\gamma(\mu). \tag{6.3}$$

Assume now that Θ is a closed relation in $\mathcal{G}$ and consider the corresponding closed extension $A_\Theta \subset S^+$ of S in (6.1). Then for all $\lambda \in \rho(A_0)$ one has

$$\lambda \in \rho(A_\Theta) \quad \text{if and only if} \quad 0 \in \rho(\Theta - M(\lambda))$$

and

$$\lambda \in \sigma_i(A_\Theta) \quad \text{if and only if} \quad 0 \in \sigma_i(\Theta - M(\lambda)), \quad i = \mathrm{p}, \mathrm{c}, \mathrm{r}.$$

Furthermore, for all $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$ one has the following variant of Krein's resolvent formula for canonical extensions:

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\overline{\lambda})^+.$$

In the next proposition it shown how the sum of given Weyl functions can be realized as a Weyl function of a certain intermediate extension; cf. [20, Proposition 4.3]. Here it is assumed for convenience that the defect of the underlying symmetric relations is the same.

Proposition 6.2. *Let S_i, $i = 1, \dots, m$, be closed symmetric relations in the Pontryagin spaces $\mathcal{H}_i$ and let $\{\mathcal{G}, \Gamma_0^{(i)}, \Gamma_1^{(i)}\}$ be boundary triples for S_i^+ with $A_0^{(i)} = \ker \Gamma_0^{(i)}$ and corresponding Weyl functions M_i. Then*

$$H = \left\{ \{\widehat{f}_1, \dots, \widehat{f}_m\} \in S_1^+ \times \cdots \times S_m^+ : \begin{matrix} \Gamma_0^{(1)} \widehat{f}_1 = \cdots = \Gamma_0^{(m)} \widehat{f}_m = 0 \\ \Gamma_1^{(1)} \widehat{f}_1 + \cdots + \Gamma_1^{(m)} \widehat{f}_m = 0 \end{matrix} \right\}$$

is a closed symmetric relation in the Pontryagin space $\mathcal{H}_1 [\dot+] \dots [\dot+] \mathcal{H}_m$ and the adjoint relation H^+ is given by

$$H^+ = \left\{ \{\widehat{f}_1, \dots, \widehat{f}_m\} \in S_1^+ \times \cdots \times S_m^+ : \Gamma_0^{(1)} \widehat{f}_1 = \cdots = \Gamma_0^{(m)} \widehat{f}_m \right\}. \tag{6.4}$$

Then $\{\mathcal{G}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$*, where*

$$\widehat{\Gamma}_0 \widehat{f} := \Gamma_0^{(i)} \widehat{f}_i, \quad 1 \le i \le m, \qquad \widehat{f} = \{\widehat{f}_1, \dots, \widehat{f}_m\} \in H^+, \tag{6.5}$$

and

$$\widehat{\Gamma}_1 \widehat{f} := \sum_{i=1}^{m} \Gamma_1^{(i)} \widehat{f}_i, \qquad \widehat{f} = \{\widehat{f}_1, \dots, \widehat{f}_m\} \in H^+, \tag{6.6}$$

is a boundary triple for H^+ *with* $\widehat{A}_0 = \ker \widehat{\Gamma}_0 = A_0^{(1)} \times \cdots \times A_0^{(m)}$ *and corresponding Weyl function*

$$\lambda \mapsto \sum_{i=1}^{m} M_i(\lambda), \qquad \lambda \in \rho(\widehat{A}_0) = \bigcap_{i=1}^{m} \rho(A_0^{(i)}). \tag{6.7}$$

Proof. It can be easily verified that $\{\mathcal{G}^m, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where

$$\widetilde{\Gamma}_j \{\widehat{f}_1, \dots, \widehat{f}_m\} = \{\Gamma_j^{(1)} \widehat{f}_1, \dots, \Gamma_j^{(m)} \widehat{f}_m\}, \quad j = 0, 1, \quad \widehat{f}_i \in S_i^+, \ i = 1, \dots, m,$$

is a boundary triple for $S_1^+ \times \cdots \times S_m^+$ with $\widetilde{A}_0 = \ker \widetilde{\Gamma}_0 = A_0^{(1)} \times \cdots \times A_0^{(m)}$ and corresponding Weyl function

$$\lambda \mapsto \begin{pmatrix} M_1(\lambda) & & 0 \\ & \ddots & \\ 0 & & M_m(\lambda) \end{pmatrix}, \qquad \lambda \in \rho(\widetilde{A}_0) = \bigcap_{i=1}^{m} \rho(A_0^{(i)}).$$

Now consider the relation H above and note that

$$\begin{aligned} \widetilde{\Theta}_H &:= \widetilde{\Gamma} H = \left\{ \left\{ \begin{pmatrix} \Gamma_0^{(1)} \widehat{f}_1 \\ \Gamma_1^{(1)} \widehat{f}_1 \end{pmatrix}, \dots, \begin{pmatrix} \Gamma_0^{(m)} \widehat{f}_m \\ \Gamma_1^{(m)} \widehat{f}_m \end{pmatrix} \right\} : \{\widehat{f}_1, \dots, \widehat{f}_m\} \in H \right\} \\ &= \left\{ \left\{ \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ x_{m-1} \end{pmatrix}, \begin{pmatrix} 0 \\ -\sum_{i=1}^{m-1} x_i \end{pmatrix} \right\} : x_1, \dots, x_{m-1} \in \mathcal{G} \right\} \end{aligned}$$

is a closed symmetric relation in $\mathcal{G}^m$. The adjoint of $\widetilde{\Theta}_H$ in $\mathcal{G}^m$ is given by

$$\widetilde{\Theta}_H^* = \left\{ \left\{ \begin{pmatrix} y \\ x_1 \end{pmatrix}, \dots \begin{pmatrix} y \\ x_m \end{pmatrix} \right\} : y, x_1, \dots, x_m \in \mathcal{G} \right\}$$

and by (6.1)-(6.2) the preimage of $\widetilde{\Theta}_H^*$ under $\widetilde{\Gamma}$ is the adjoint of H. It is easy to see that this is the relation H^+ in (6.4).

Let now $\widehat{f} = \{f, f'\} = \{\widehat{f}_1, \dots, \widehat{f}_m\}$, $\widehat{g} = \{g, g'\} = \{\widehat{g}_1, \dots, \widehat{g}_m\} \in H^+$. Then we have

$$\begin{aligned} [f', g] - [f, g'] &= \big(\widetilde{\Gamma}_1 \widehat{f}, \widetilde{\Gamma}_0 \widehat{g}\big) - \big(\widetilde{\Gamma}_0 \widehat{f}, \widetilde{\Gamma}_1 \widehat{g}\big) \\ &= \sum_{l=1}^{m} \big(\Gamma_1^{(l)} \widehat{f}_l, \Gamma_0^{(l)} \widehat{g}_l\big) - \sum_{l=1}^{m} \big(\Gamma_0^{(l)} \widehat{f}_l, \Gamma_1^{(l)} \widehat{g}_l\big) \\ &= \left(\sum_{l=1}^{m} \Gamma_1^{(l)} \widehat{f}_l, \Gamma_0^{(1)} \widehat{g}_1 \right) - \left(\Gamma_0^{(1)} \widehat{f}_1, \sum_{l=1}^{m} \Gamma_1^{(l)} \widehat{g} \right) \\ &= \big(\widehat{\Gamma}_1 \widehat{f}, \widehat{\Gamma}_0 \widehat{g}\big) - \big(\widehat{\Gamma}_0 \widehat{f}, \widehat{\Gamma}_1 \widehat{g}\big) \end{aligned}$$

and the surjectivity of $(\widehat\Gamma_0, \widehat\Gamma_1)^\top : H^+ \to \mathcal{G} \times \mathcal{G}$ is obvious, hence (6.5)-(6.6) is a boundary triple for H^+ with

$$\widehat A_0 = \ker \widehat\Gamma_0 = A_0^{(1)} \times \cdots \times A_0^{(m)}.$$

Let now $\lambda \in \rho(\widehat A_0) = \bigcap_{i=1}^m \rho(A_0^{(i)})$ and consider

$$\widehat f_\lambda = \{\widehat f_{\lambda,1}, \dots, \widehat f_{\lambda,m}\} \in \widehat{\mathfrak{N}}_\lambda(H^+).$$

Then $\widehat f_{\lambda,i} \in \widehat{\mathfrak{N}}_\lambda(S_i^+)$ and hence $M_i(\lambda)\Gamma_0^{(i)}\widehat f_{\lambda,i} = \Gamma_1^{(i)}\widehat f_{\lambda,i}$ for $i = 1, \dots, m$. Therefore,

$$\sum_{i=1}^m M_i(\lambda)\widehat\Gamma_0\widehat f_\lambda = \sum_{i=1}^m M_i(\lambda)\Gamma_0^{(i)}\widehat f_{\lambda,i} = \sum_{i=1}^m \Gamma_1^{(i)}\widehat f_{\lambda,i} = \widehat\Gamma_1\widehat f_\lambda, \quad \lambda \in \rho(\widehat A_0),$$

shows that the Weyl function corresponding to the boundary triple (6.5) is given by (6.7). □

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Jussi Behrndt
Institut für Numerische Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
e-mail: `behrndt@tugraz.at`
URL: `http://www.math.tugraz.at/ behrndt/`

Friedrich Philipp
Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina
e-mail: `fmphilipp@dm.uba.ar`
URL: `http://cms.dm.uba.ar/Members/fmphilipp/`