Bounds on the Non-real Spectrum of a Singular Indefinite Sturm-Liouville Operator on $\mathbb{R}$

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A simple explicit bound on the absolute values of the non-real eigenvalues of a singular indefinite Sturm-Liouville operator on the real line with the weight function $\operatorname{sgn}(\cdot)$ and an integrable, continuous potential q is obtained.

1 Introduction and main result

In this note we consider the indefinite Sturm-Liouville differential expression

$$\tau = \operatorname{sgn}(\cdot)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q\right)$$

on the real line for a continuous, real-valued potential $q \in L^1(\mathbb{R})$. The associated maximal operator is defined as

$$(Af)(x) = \operatorname{sgn}(x)\big(-f''(x) + q(x)f(x)\big), \quad x \in \mathbb{R}, \quad f \in \mathcal{D}, \tag{1}$$

with domain $\mathcal{D} = \{f \in L^2(\mathbb{R}) : f, f' \text{ are locally absolutely continuous and } \tau f \in L^2(\mathbb{R})\}$. It is easy to see that A is neither symmetric nor self-adjoint with respect to the usual scalar product in $L^2(\mathbb{R})$, but A becomes symmetric and self-adjoint with respect to the indefinite inner product

$$[f, g] := \int_{\mathbb{R}} \operatorname{sgn}(x) f(x)\overline{g(x)}\,\mathrm{d}x, \qquad f, g \in L^2(\mathbb{R}).$$

Therefore it is not surprising that indefinite Sturm-Liouville operators of the form (1) may have non-real eigenvalues. The spectral properties of such differential operators have attracted interest for more than a century, see [8, 11]. For an overview we refer to [13] and for recent results on the non-real spectrum see [2–7, 10].

The main objective of this note is to proof an estimate on the absolute values of the non-real eigenvalues of the indefinite Sturm-Liouville operator A in (1) which depends only on the L^1-norm of the continuous potential q.

Theorem 1.1 *Every non-real eigenvalue λ of A satisfies the inequality*

$$|\lambda| \le \frac{1}{C^2}\|q\|_1^2, \qquad \text{where } C = \ln\left(1 + \frac{1}{1+\sqrt{2}}\right).$$

For further estimates on the non-real spectrum of indefinite Sturm-Liouville operators in the singular case we refer to [3], where bounds depending on the L^∞-norm of the potential were obtained. Regarding the regular case, i.e. the Sturm-Liouville differential expression is defined on a finite interval with integrable coefficients, bounds in terms of the coefficients can be found in [2, 7, 10]; we also mention that the techniques in [1, Section 3] may be used to prove related eigenvalue estimates.

2 Proof of Theorem 1.1

In the following we denote the restriction of a function $f : \mathbb{R} \to \mathbb{C}$ to $\mathbb{R}^\pm$ by $f_\pm$. Observe that for a non-real eigenvalue λ of A and a corresponding eigenfunction $f \in \mathcal{D}$ the functions $f_\pm \in L^2(\mathbb{R}^\pm)$ are nontrivial solutions of the differential equations

$$f_+'' = -\lambda f_+ + q_+ f_+ \quad \text{on } \mathbb{R}^+ \qquad \text{and} \qquad f_-'' = \lambda f_- + q_- f_- \quad \text{on } \mathbb{R}^- \tag{2}$$

such that the matching condition

$$\frac{f_+'(0)}{f_+(0)} = \frac{f_-'(0)}{f_-(0)} \tag{3}$$

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is satisfied; the values $f_\pm(0)$ are non-zero since λ is assumed to be non-real. As the differential expression τ is in the limit point case at $\pm\infty$ the L^2-solutions f_+ and f_- of (2) are unique up to a constant factor; cf. Lemma 9.37 and Theorem 9.9 in [12]. In this context we recall that a function g is called a solution of a second order differential equation on $\mathbb{R}^\pm$ if g and g' are locally absolutely continuous on $\mathbb{R}^\pm$ and g satisfies the equation almost everywhere in $\mathbb{R}^\pm$.

The next lemma on the form and properties of solutions of the differential equations in (2) can be shown with the help of the Liouville-Green method in [9]. Here the square root $\sqrt{\cdot}$ is fixed by a cut along $(-\infty, 0]$, so that $\mathrm{Re}\sqrt{\mu} > 0$ for $\mu \in \mathbb{C}\setminus\mathbb{R}$.

Lemma 2.1 *For $\lambda \in \mathbb{C}\setminus\mathbb{R}$ there exist solutions $f_\pm$ of the differential equations* (2) *of the form*

$$f_\pm(x) = \exp\left(\mp\sqrt{\mp\lambda}x\right)\left(1 + R_\pm(x)\right), \qquad x \in \mathbb{R}^\pm, \tag{4}$$

where the functions $R_\pm$ satisfy the estimates

$$|R_\pm(x)| \le \exp\left(\|q_\pm\|_1|\lambda|^{-1/2}\right) - 1 \quad \textit{and} \quad |R'_\pm(x)| \le |\lambda|^{1/2}\left(\exp\left(\|q_\pm\|_1|\lambda|^{-1/2}\right) - 1\right), \quad x \in \mathbb{R}^\pm. \tag{5}$$

The solutions $f_\pm$ are (up to constant factor) the unique square-integrable solutions of (2).

The proof of Theorem 1.1 is now essentially a consequence of (2)–(3) together with the representation of $f_\pm$ and estimates on $R_\pm$ in Lemma 2.1.

Proof of Theorem 1.1. Assume that λ is a non-real eigenvalue of A such that

$$|\lambda| > \|q\|_1^2\left(\ln\left(1 + \frac{1}{1+\sqrt{2}}\right)\right)^{-2}$$

and let $\epsilon = \exp\left(\|q\|_1|\lambda|^{-1/2}\right) - 1$. Then

$$\|q\|_1|\lambda|^{-1/2} < \ln\left(1 + \frac{1}{1+\sqrt{2}}\right)$$

and hence $0 < \epsilon < (1+\sqrt{2})^{-1} < 1$. For $f_\pm$ and $R_\pm$ in Lemma 2.1 we have $|R_\pm(x)| \le \epsilon$ and $|R'_\pm(x)| \le \epsilon|\lambda|^{1/2}$ for all $x \in \mathbb{R}^\pm$. Moreover, (4) leads to

$$f_\pm(0) = 1 + R_\pm(0) \quad \text{and} \quad f'_\pm(0) = \mp\sqrt{\mp\lambda}\big(1 + R_\pm(0)\big) + R'_\pm(0).$$

The matching condition (3) can be rewritten in the form

$$-\sqrt{-\lambda} + \frac{R'_+(0)}{1 + R_+(0)} = \sqrt{\lambda} + \frac{R'_-(0)}{1 + R_-(0)}$$

and together with the estimates for $R_\pm$ we get

$$\sqrt{2} = \frac{\left|\sqrt{\lambda} + \sqrt{-\lambda}\right|}{\sqrt{|\lambda|}} \le \frac{1}{\sqrt{|\lambda|}}\left(\frac{|R'_+(0)|}{|1 + R_+(0)|} + \frac{|R'_-(0)|}{|1 + R_-(0)|}\right) \le 2\frac{\epsilon}{1-\epsilon}.$$

Rearranging the terms leads to $(1+\sqrt{2})^{-1} \le \epsilon$; a contradiction. □

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References

[1] A. A. Abramov, A. Aslanyan, and E. B. Davies, J. Phys. A **34**, 1113–1126 (2001).
[2] J. Behrndt, S. Chen, F. Philipp, J. Qi, Proc. Roy. Soc. Edinburgh Sect. A **144**, 1113–1126 (2014).
[3] J. Behrndt, F. Philipp, and C. Trunk, Math. Ann. **357**, 185–213 (2013).
[4] J. Behrndt, Q. Katatbeh, C. Trunk, J. Phys. A: Math. Theor **41**, 244003 (2008).
[5] J. Behrndt, Q. Katatbeh, C. Trunk, Proc. Amer. Math. Soc. **137**, 3787–3806 (2009).
[6] S. Chen, J. Qi, J. Spectr. Theory **4**, 53–63 (2014).
[7] S. Chen, J. Qi, B. Xie, Proc. Amer. Math. Soc. **144**, 547–559 (2016).
[8] O. Haupt., Math. Annalen **76**, 67–104 (1914).
[9] F. W. J. Olver, Asymptotics and Special Functions (A K Peter, New York, 1974), pp. 190–197.
[10] J. Qi, B. Xie, J. Differential Equations **255**, 2291–2301 (2013).
[11] R. G. D. Richardson, Am. J. Math. **40**, 283–316 (1918).
[12] G. Teschl, Mathematical Methods in Quantum Mechanics (Amer. Math. Soc., Providence, RI, 2009).
[13] A. Zettl, Sturm-Liouville Theory (Amer. Math. Soc., 2005).