Bounds on the Non-real Spectrum of a Singular Indefinite Sturm-Liouville Operator on \( \mathbb{R} \)

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A simple explicit bound on the absolute values of the non-real eigenvalues of a singular indefinite Sturm-Liouville operator on the real line with the weight function \( \text{sgn}(\cdot) \) and an integrable, continuous potential \( q \) is obtained.

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1 Introduction and main result

In this note we consider the indefinite Sturm-Liouville differential expression

\[ \tau = \text{sgn}(\cdot) \left( -\frac{d^2}{dx^2} + q \right) \]

on the real line for a continuous, real-valued potential \( q \in L^1(\mathbb{R}) \). The associated maximal operator is defined as

\[ (Af)(x) = \text{sgn}(x)(-f''(x) + q(x)f(x)), \quad x \in \mathbb{R}, \quad f \in D, \]

with domain \( D = \{ f \in L^2(\mathbb{R}) : f, f' \text{ are locally absolutely continuous and } \tau f \in L^2(\mathbb{R}) \} \). It is easy to see that \( A \) is neither symmetric nor self-adjoint with respect to the usual scalar product in \( L^2(\mathbb{R}) \), but \( A \) becomes symmetric and self-adjoint with respect to the indefinite inner product

\[ [f, g] := \int_{\mathbb{R}} \text{sgn}(x)f(x)\overline{g(x)}\,dx, \quad f, g \in L^2(\mathbb{R}). \]

Therefore it is not surprising that indefinite Sturm-Liouville operators of the form (1) may have non-real eigenvalues. The spectral properties of such differential operators have attracted interest for more than a century, see \([8, 11]\). For an overview we refer to \([13]\) and for recent results on the non-real spectrum see \([2–7, 10]\).

The main objective of this note is to proof an estimate on the absolute values of the non-real eigenvalues of the indefinite Sturm-Liouville operator \( A \) in (1) which depends only on the \( L^1 \)-norm of the continuous potential \( q \).

**Theorem 1.1** Every non-real eigenvalue \( \lambda \) of \( A \) satisfies the inequality

\[ |\lambda| \leq \frac{1}{C^2} \|q\|^2_1, \quad \text{where} \quad C = \ln \left( 1 + \frac{1}{1 + \sqrt{2}} \right). \]

For further estimates on the non-real spectrum of indefinite Sturm-Liouville operators in the singular case we refer to \([3]\), where bounds depending on the \( L^\infty \)-norm of the potential were obtained. Regarding the regular case, i.e. the Sturm-Liouville differential expression is defined on a finite interval with integrable coefficients, bounds in terms of the coefficients can be found in \([2, 7, 10]\); we also mention that the techniques in \([1, \text{Section 3}]\) may be used to prove related eigenvalue estimates.

2 Proof of Theorem 1.1

In the following we denote the restriction of a function \( f : \mathbb{R} \to \mathbb{C} \) to \( \mathbb{R}^\pm \) by \( f_\pm \). Observe that for a non-real eigenvalue \( \lambda \) of \( A \) and a corresponding eigenfunction \( f \in D \) the functions \( f_\pm \in L^2(\mathbb{R}^\pm) \) are nontrivial solutions of the differential equations

\[ f_+'' = -\lambda f_+ + q_+ f_+ \quad \text{on } \mathbb{R}^+ \quad \text{and} \quad f_-'' = \lambda f_- + q_- f_- \quad \text{on } \mathbb{R}^- \]

such that the matching condition

\[ \frac{f_+(0)}{f_-}(0) = \frac{f_+''(0)}{f_-'}(0) \]

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is satisfied; the values \( f_±(0) \) are non-zero since \( \lambda \) is assumed to be non-real. As the differential expression \( \tau \) is in the limit point case at \( \pm \infty \) the \( L^2 \)-solutions \( f_+ \) and \( f_- \) of (2) are unique up to a constant factor; cf. Lemma 9.37 and Theorem 9.9 in [12]. In this context we recall that a function \( g \) is called a solution of a second order differential equation on \( \mathbb{R}^\pm \) if \( g \) and \( g' \) are locally absolutely continuous on \( \mathbb{R}^\pm \) and \( g \) satisfies the equation almost everywhere in \( \mathbb{R}^\pm \).

The next lemma on the form and properties of solutions of the differential equations in (2) can be shown with the help of the Liouville-Green method in [9]. Here the square root \( \sqrt{\tau} \) is fixed by a cut along \( (-\infty, 0] \), so that \( \text{Re}\sqrt{\tau} > 0 \) for \( \mu \in \mathbb{C} \setminus \mathbb{R} \).

Lemma 2.1 For \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) there exist solutions \( f_± \) of the differential equations (2) of the form

\[
f_±(x) = \exp \left( \mp \sqrt{\frac{1}{\lambda}} x \right) (1 + R_±(x)), \quad x \in \mathbb{R}^\pm,
\]

where the functions \( R_± \) satisfy the estimates

\[
|R_±(x)| \leq \exp \left( \|q_±\|_1 |\lambda|^{-1/2} \right) - 1 \quad \text{and} \quad |R'_±(x)| \leq |\lambda|^{1/2} \left( \exp \left( \|q_±\|_1 |\lambda|^{-1/2} \right) - 1 \right), \quad x \in \mathbb{R}^±. \tag{5}
\]

The solutions \( f_± \) are (up to constant factor) the unique square-integrable solutions of (2).

The proof of Theorem 1.1 is now essentially a consequence of (2)–(3) together with the representation of \( f_± \) and estimates on \( R_± \) in Lemma 2.1.

**Proof of Theorem 1.1.** Assume that \( \lambda \) is a non-real eigenvalue of \( A \) such that

\[
|\lambda| > \|q\|_1^2 \left( \ln \left( 1 + \frac{1 + \sqrt{2}}{1 + \sqrt{2}} \right) \right)^{-2}
\]

and let \( \epsilon = \exp \left( \|q_±\|_1 |\lambda|^{-1/2} \right) - 1 \). Then

\[
\|q_±\|_1 |\lambda|^{-1/2} < \ln \left( 1 + \frac{1 + \sqrt{2}}{1 + \sqrt{2}} \right)
\]

and hence \( 0 < \epsilon < (1 + \sqrt{2})^{-1} - 1 < 1 \). For \( f_± \) and \( R_± \) in Lemma 2.1 we have \( |R_±(x)| \leq \epsilon \) and \( |R'_±(x)| \leq \epsilon |\lambda|^{1/2} \) for all \( x \in \mathbb{R}^± \). Moreover, (4) leads to

\[
f_±(0) = 1 + R_±(0) \quad \text{and} \quad f'_±(0) = \mp \sqrt{\frac{1}{\lambda}} \left( 1 + R_±(0) \right) + R'_±(0).
\]

The matching condition (3) can be rewritten in the form

\[
-\sqrt{\frac{1}{\lambda}} + \frac{R'_±(0)}{1 + R_±(0)} = \sqrt{\frac{1}{\lambda}} + \frac{R'_±(0)}{1 + R_±(0)}
\]

and together with the estimates for \( R_± \) we get

\[
\sqrt{2} = \frac{|\sqrt{\frac{1}{\lambda}} + \sqrt{\frac{1}{\lambda}}|}{\sqrt{|\lambda|}} \leq \frac{1}{|\lambda|} \left( \frac{|R'_±(0)|}{|1 + R_±(0)|} + \frac{|R'_±(0)|}{|1 + R_±(0)|} \right) \leq 2 \frac{\epsilon}{1 - \epsilon}.
\]

Rearranging the terms leads to \( (1 + \sqrt{2})^{-1} \leq \epsilon \); a contradiction. \( \square \)

**Acknowledgements** The authors wish to thank Gerald Teschl for very helpful comments. Jussi Behrndt gratefully acknowledges financial support by the Austrian Science Fund (FWF): Project P 25162-N26.

**References**


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